



Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



Spectral Analysis of Matrix Differential Operators Using an Operational Matrix Method

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ARTICLE INFO

Article history:

Received: 14 /05/2026

Revised form: 16 /06/2026

Accepted : 18 /06/2026

Available online: 30 /09/2026

Keywords:

Matrix Differential Operators,
Spectral Analysis, Operational
Matrix, Chebyshev Polynomials,
Eigenvalue Problems.MSC..

ABSTRACT

Analysis of the spectral of matrix differential operators is key to representing and understanding many coupled and multi-physics systems. Nevertheless, the eigenvalues and eigenfunctions for such operators are difficult to compute directly and efficiently in terms of traditional numerical methods (which are expensive and highly inaccurate). We suggest an effective spectral algorithm for the investigation of matrix differential operators through operational rules using Chebyshev polynomials. The method represents the unknown vector functions by Chebyshev series expansions and makes use of the derivative matrices to reduce a continuous eigenvalue problem to a standard algebraic matrix formulation. The resulting transformation allows to compute directly spectral properties with standard numerical solvers. It is shown that the proposed method has high accuracy, fast convergency and low computational cost in terms of a 2×2 and a series example with 3×3 matrix differential operators. It is demonstrated that the operational matrix-based spectral method is accurate and convenient to solve eigenvalue problems of matrix differential operators, which can be further generalized to a wide class of linear operator eigenvalue problem.

<https://doi.org/10.29304/jqcm.2026.18.32886>

1. Introduction

Spectral theory of differential operators is one of the cornerstones of applied mathematics, mathematical physics and engineering science due to its centrality in studying the qualitative and quantitative behavior of dynamical systems. In particular, eigenvalue problems of differential operators are found pruriently in stability analysis, vibration modeling, control theory quantum mechanics and wave propagation problem [1,2]. To this end, the perfect determination and efficient computation of spectral properties such as eigenvalues and eigenfunctions is necessary for theoretical studies as well as practical applications.

Matrix differential operators carry the increasing importance mainly as a generalization of classical scalar ones for modeling inherently coupled and multi-component systems arising from elasticity, electromagnetics, fluid dynamics, and Multiphysics applications in CEM [3-4]. Operators of that kind frequently come as systems of

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ordinary differential equations with matrix valued coefficients which to solve can be computational more demanding and analytically more difficult. As a consequence, there is much activity in the construction of stable numerical methods for spectral analysis of matrix differential operators.

Classical numerical methods for the eigenvalue problem of differential operators are finite difference method, finite element method and shooting technique [5, 6]. Though these methods have been proven to be widespread and powerful, they are faced with several shortcomings: slow rate of convergence, numerical complexity incurred by large-scale problem and difficulty in achieving high precision (especially such problems with smooth solution or higher-order derivatives). These limitations justify the investigation of alternative numerical techniques, which can provide better accuracy with a lower computational cost.

On account of their exponential rates of convergence for sufficiently smooth solutions, spectral methods have established themselves as useful tools with which to numerically solve differential equations and eigenvalue problems [7, 8]. As the true solution can be approximated by global basis functions (e.g., Chebyshev, Legendre polynomials), spectral methods can produce much accurate results than standard low-order discretization schemes. Operational matrix approaches are one type of spectral methods that have attracted much attention because they allow the conversion of differential operators to algebraic forms using operational matrices for differentiation and integration [9,10].

Recent studies have demonstrated the effectiveness of operational matrix-based spectral approaches for solving various classes of ordinary, partial, and fractional differential equations [11-13]. However, most existing works primarily focus on computing numerical solutions of scalar differential equations, while relatively limited attention has been devoted to the direct spectral analysis of matrix differential operators and their associated eigenvalue problems using operational matrix formulations. This gap motivates the present study.

In this paper, a novel framework for the spectral analysis of matrix differential operators using an operational matrix method based on Chebyshev polynomials is proposed. The main contributions of this work can be summarized as follows:

- A. A unified operational matrix formulation is developed for representing matrix differential operators within a spectral framework.
- B. The eigenvalue problem for a matrix differential operator is reduced to an equivalent algebraic eigenvalue problem for the associated matrix.
- C. Numerical examples are considered to exhibit the accuracy, efficiency, and stability of the presented method.

These are useful contributions for the spectral analysis of matrix differential systems and generalize the eligibility domain for operational matrix-type spectral methods.

2. Related Work

Spectral methods are very accurate numerical strategies that have been largely explored for the problem of ODEs and eigenvalues. The basic approach of them is that unknown functions are approximated by some global basis functions (e.g., Chebyshev, Legendre or Jacobi polynomials), and exponential convergence for smooth enough solutions will be achieved. Classical works have developed the theory behind spectral and pseudo spectral methods and shown that they outperform the ordinary low order discretization techniques [14,15].

The past few years have seen a significant use of spectral methods for various types of ordinary and partial differential equations. For example, spectral collocation and Galerkin methods are used to solve boundary value problems and time dependent equations with high accuracy [16,17]. These approaches have also been further generalized to fractional-order and variable-order differential equations, which demonstrated the powerful performance in dealing with complex mathematical models [18,19].

Operational matrix techniques are an important line of the spectral methods. The basic idea is to symbolize derivatives and integrals of basic functions by means of a set of precomputed operator matrices, to cast differentially defined equations into systems of algebraic equations. Many types of numerical integrators with Chebyshev, Legendre and Jacobi polynomials as matrix functions has been developed in order to solve linear and

non-linear differential equations effectively [20,21]. These formulations greatly reduce computational burden and ease the practical application.

In the last years, spectral techniques combined with operational matrix methods have been used to derive computationally efficient solvers for VD and FD equations. At [1] a spectral method combined with new improved operational matrices of fractional Jacobi functions was proposed to solve the model in variable-order fractional differential equations, the numerical solution showed that it is of high accuracy and quick convergence [22]. In addition, a spectral method based on the operational matrices of orthogonal polynomials was extended to solve VOPDEs and showed good numerical results and stability [23].

In spite of this improvement most studies in the literature deal with obtaining numerical solutions to scalar differential equations. Little attention has been paid to the spectrum analysis of matrix differential operators and the direct calculation of their eigenvalues and eigenfunctions using operational matrix representations. Since matrix differential operators are common in coupled and multi-component systems, this limitation confines the range of uses of the current approach.

The spectral framework here is constructed on operator matrix with the aid of an operational matrix, focusing mainly on eigenvalue problems of a matrix differential operator. The proposed procedure, in contrast with prior works designed to approximate solutions, is addressed here to obtaining an eigenvalue problem for the algebraic system whose solution allows direct computation of spectral properties. As such, this paper generalizes the classical spectral operational matrix approaches to a wider and more difficult class of problems.

There have been a lot of studies on solving a scalar ODE, PDE or fractional differential equation with spectral methods and the operational matrix approach using Chebyshev polynomials; however, the majority of the studies are devoted to the numerical approximation of a scalar differential equation. In addition, most of the existing matrix formulations for operations are formulated to approximate solutions instead of analyzing matrices directly. In coupled dynamical systems, elasticity, fluid dynamics, electromagnetics, and multiphysics models, the problem of finding the eigenvalues and eigenfunctions occurs naturally, and they are important to system stability and dynamics.

Although significant advances have been made in practical matrix methods, there has been relatively little work done in the development of a unified matrix method that can easily and directly convert a matrix differential operator eigenvalue problem into a finite-dimensional algebraic matrix eigenvalue problem, with the same spectral properties as the original matrix differential operator. The present study aims at setting up an efficient operational based spectral framework for the direct computation of the eigenvalues and eigenfunctions of matrix differential operators, which will motivate this gap. Therefore, The main contributions of this work can be summarized as follows:

1. A unified spectral framework based on Chebyshev operational matrices is developed for the direct spectral analysis of matrix differential operators.
2. The proposed formulation transforms continuous matrix differential eigenvalue problems into finite-dimensional algebraic eigenvalue systems that can be efficiently solved using standard numerical eigensolvers.
3. A systematic procedure is introduced for incorporating matrix differential operators and boundary conditions within the operational matrix framework.
4. The effectiveness of the proposed method is demonstrated through representative 2×2 and 3×3 matrix differential operator examples, showing accurate computation of eigenvalues and corresponding eigenfunctions.
5. The presented framework provides a foundation that can be extended to higher-dimensional matrix operators, variable-coefficient systems, and more complex coupled differential models.

While significant advances have been made in both the spectral and operational matrix methods, most of the literature reported so far deals with the solution of scalar ODEs, PDEs and FDDEs. Some studies have been done directly on the matrix differential operator and on computing the eigenvalues and eigenfunctions of this operator, but not as many as on the direct analysis of the operator for matrix functions. Moreover, the existing literature focuses on solution approximation and not on the characterization of the operator's spectrum. The present study is motivated by this observation, as efficient numerical frameworks that are specifically developed for matrix differential eigenvalue problems are needed.

3. Mathematical Preliminaries

3.1 Matrix Differential Operators

Consider a linear matrix differential operator defined on the interval $[a, b]$ as

$$LY(x) = A_2 \frac{d^2 y(x)}{dx^2} + A_1 \frac{dy(x)}{dx} + A_0 Y(x),$$

where $Y(x) \in R^m$ is a vector-valued function and $A_0, A_1, A_2 \in R^{m \times m}$. A_0, A_1, A_2 are constant coefficient matrices, and they commonly appear in the coupled differential equations systems and multi-physics models. The spectral analysis of L is concerned with studying its eigenvalues, and the associated eigenfunctions that describe the nature of the underlying physical phenomenon

3.2 Chebyshev Polynomials

The Chebyshev polynomials of the first kind, denoted by $T_n(x)$, are defined on the interval $[-1, 1]$ as

$$T_n(x) = \cos(\arccos x), \quad n = 0, 1, 2, \dots$$

These polynomials form an orthogonal basis with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

Any sufficiently smooth function $f(x)$ defined on $[-1, 1]$ can be approximated as

$$f(x) \approx \sum_{k=0}^N c_k T_k(x),$$

where c_k are expansion coefficients.

3.3 Basic Properties

The Chebyshev polynomials satisfy the following recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1,$$

With

$$T_0(x) = 1, T_1(x) = x.$$

Derivatives of these functions can be represented in the form Chebyshev polynomials of a second kind and it allows for an effective building of differentiation matrices.

In addition, Chebyshev polynomials have excellent approximation properties with minimal Runge phenomenon and are therefore appropriate for spectral methods.

3.4 Eigenvalue Problem Formulation

The eigenvalue problem associated with the matrix differential operator L is defined as

$$LY(x) = \lambda Y(x),$$

subject to appropriate boundary conditions.

Using Chebyshev polynomial expansions and operational matrices, the above differential eigenvalue problem can be transformed into an algebraic matrix eigenvalue problem of the form

$$KC = \lambda C,$$

where K is the system matrix constructed from the operational matrices and coefficient matrices A_0, A_1, A_2 , and C is the vector of unknown expansion coefficients.

Solving this algebraic eigenvalue problem yields approximations to the eigenvalues and eigenfunctions of the original matrix differential operator.

4. Operational Matrix Formulation

In this section, the operational matrix framework based on Chebyshev polynomials is developed. The unknown vector function is approximated using a truncated Chebyshev series, and differentiation operational matrices are constructed to transform the matrix differential operator into an algebraic form.

4.1 Approximation of the Solution

Let the vector-valued function $Y(x) \in R^m$ be approximated by a finite series of Chebyshev polynomials as

$$Y(x) \approx \sum_{K=0}^N C_K T_K(x),$$

where $C_k \in R^m$ are unknown coefficient vectors and $T_k(x)$ are Chebyshev polynomials. This approximation can be written in matrix form as

$$Y(x) \approx T(x)C,$$

Where

$$T(x) = [T_0(x), T_1(x), \dots, T_N(x)],$$

and

$$C = [C_0^T, C_1^T, \dots, C_N^T].$$

4.2 Construction of the Differentiation Operational Matrix

The derivative of the Chebyshev polynomial vector can be expressed as

$$\frac{d}{dx} T(x) = DT(x),$$

where D is the Chebyshev operational matrix of differentiation.

Consequently,

$$\frac{dy(x)}{dx} \approx T(x)DC.$$

The entries of matrix D can be computed analytically based on Chebyshev polynomial properties.

The differentiation operational matrix is built by expressing the derivatives of the Chebyshev basis functions as a matrix of Chebyshev basis functions. The resulting coefficients are put together into a differentiation matrix which transforms the coefficients of the Chebyshev series of a function into the coefficients of the Chebyshev series of the derivative of the function. Such an operational representation allows the direct conversion of differential operators to an algebraic matrix representation, which makes the spectral analysis technique simpler.

4.3 Second-Order Derivative Approximation

By differentiating once more, the second derivative is obtained as

$$\frac{d^2}{dx^2} T(x) = D^2 T(x)$$

Leading to

$$\frac{d^2 Y(x)}{dx^2} = T(x) D^2 C$$

This formulation allows higher-order derivatives to be represented using powers of the differentiation operational matrix.

4.4 Construction of the Global Operator Matrix

Substituting the approximations into the matrix differential operator

$$LY(x) = A_2 \frac{d^2 y(x)}{dx^2} + \frac{dy(x)}{dx} + A_0 Y(x),$$

yields

$$LY(x) \approx T(x)(A_2 D^2 + A_1 D + A_0 I)C.$$

Define the global operational matrix as

$$K = A_2 D^2 + A_1 D + A_0 I,$$

where I is the identity matrix.

Thus, the operator equation is transformed into an algebraic form suitable for eigenvalue computation.

5. Spectral Analysis of Matrix Differential Operators

In this section, the spectrum representation of matrix differential operator is introduced with operational matrix of derivative. It allows one to convert the problem of an infinitely many eigenvalues into that of a finite number of eigenvalues associated with algebraic equations, making spectral computations possible.

5.1 Transformation to an Algebraic Eigenvalue Problem

Consider the continuous eigenvalue problem

$$LY(x) = \lambda Y(x),$$

where L is the matrix differential operator given in Section 4 and λ denotes an eigenvalue.

Using the Chebyshev approximation

$$Y(x) \approx T(x)C$$

and replacing the operational matrix expression obtained in Section 4, we get

$$T(x)KC = \lambda T(x)C.$$

By eliminating the basis vector $T(x)$, the above equation reduces to

$$KC = \lambda C.$$

This equation is a typical matrix eigenvalue problem, in which K is the global operation matrix, and C is the eigenvector corresponding to eigenvalue λ .

5.2 Spectral Properties

The set of all eigenvalues E_i

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_M\}$$

the spectrum of the matrix differential operator L , it also represents an eigenfunction that can be approximated by

$$Y_i(x) = T(x)C_i,$$

where C_i is the eigenvector of K associated with λ_i

The behavior of the distribution of eigenvalues is an important information to understand the stability and dynamic properties of the system. For example, eigenvalues whose real parts are positive mean that those modes are unstable and negative real ones associated with stable modes.

5.3 Computation of Eigenvalues and Eigenvectors

Algebraic E_i problem

$$KC = \lambda C$$

Can be solved by some general numerical algorithms like the QR algorithm or implicitly restarted Arnoldi methods. Once the eigenvectors c_i are determined, the associated eigenfunctions are reconstructed using a Chebyshev expansion.

The computational methodology provides for the fast and accurate computation of spectral data throughout the range of possible dimensions for matrix differential operators

5.4 Convergence Analysis

The truncation order of the Chebyshev expansion was raised to examine the convergence properties of the proposed method. The numerical results indicated that the eigenvalues from the computed solutions converged with the increase of the approximation order, suggesting the convergence of the proposed spectral formulation. This behavior is in accordance with the accuracy of the spectral methods based on Chebyshev polynomials for smooth problems, as is well known.

5.5 Error Analysis

The accuracy of computation of the eigenpairs was tested with respect to the residual errors from the algebraic eigenvalue problem. The residual values reported stayed small enough in all modes tested, confirming that the computed eigenvalues and eigenfunctions were accurate in the discretized operator formulation.

5.6 Stability Analysis

Numerical tests showed that the dominant eigenvalues varied little with small changes in the approximation order. This observation suggests that the proposed operational matrix formulation has stable numerical behavior and reliable approximations to the spectrum of matrix differential operators.

6. Proposed Algorithm

This section explains the algorithmic form of the proposed operational matrix-based spectral method for obtaining eigenvalues and eigenfunctions of matrix differential operators.

Algorithm: Operational Matrix-Based Spectral Eigenvalue Solver

Input:

- Approximation Order $\rightarrow N$
- Matrices of Coefficient $\rightarrow A_0, A_1, A_2$
- Interval $\rightarrow [a, b]$

Output:

- Eigenvalues (Ei) $\rightarrow \lambda_i$
- Eigenfunctions (Ei) $\rightarrow Y_i(x)$

Step 1: Map of the physical interval $\rightarrow [a, b]$ to the standard interval $\rightarrow [-1, 1]$

Step 2: Generate Chebyshev polynomials $\rightarrow T_k(x)$ for $k = 0, 1, \dots, N$

Step 3: Formulation of the Chebyshev operational matrix of differentiation

Step 4: Calculation of the higher-order differentiation matrix $\rightarrow D^2$

Step 5: Assemble the global operational matrix

$$K = A_2 D^2 + A_1 D + A_0 I.$$

Step 6: Form the algebraic eigenvalue problem

$$KC = \lambda C.$$

Step 7: Solve the eigenvalue problem with a standard numerical solver.

Step 8: Reconstruct eigenfunctions using

$$Y(x) = T(x)C.$$

Step 9: Post-process results and evaluate accuracy.

This algorithm offers an organized and computationally efficient approach to matrix differential operator spectral analysis.

7. Numerical Examples and Results

In this section, two benchmark matrix differential eigenvalue problems are discussed to represent the accuracy and efficiency of the developed operational matrix-based spectral scheme. Homogeneous Dirichlet boundary conditions are applied to everything in the vector of solution components, namely.

$$Y(-1) = 0, Y(1) = 0.$$

These boundary conditions are incorporated in terms of the tau technique by replacing the related rows of the algebraic equation with boundary conditions, which results to a constrained algebraic eigenvalue problem. The extracted eigenvalues are verified by the relative residual error.

The boundary conditions are used in the operational matrix formulation using the Tau method. Algebraic equations for the highest order modes are, in particular, replaced with those representing the imposed boundary conditions. This procedure will guarantee that the algebraic eigenvalue problem obtained fulfills the differential operator and the boundary conditions at the same time. Hence the calculated eigen values and eigen functions are the same as the original continuous boundary value problem.

A. 2×2 Matrix Differential Operator

Consider the matrix differential operator

$$LY(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{d^2 y(x)}{dx^2} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{dy(x)}{dx} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} Y(x)$$

defined on the interval $[-1,1]$ with homogeneous boundary conditions.

Using the proposed method with truncation order $N = 20$, the first five eigenvalues are computed. Table 1 lists the eigenvalues obtained by the proposed method together with the reference solutions.

Table 1: First computed eigenvalues smallest $|\lambda|$ for the 2×2 matrix differential operator (Example 1)

Mode	Eigenvalue (λ)
1	(-0.903079 - 1.062418i)
2	(-0.903079 + 1.062418i)
3	(2.000008)
4	(5.103395)
5	(-3.885943 - 5.169234i)
6	(-3.885943 + 5.169234i)
7	(-10.895999)
8	(-29.037528)
9	(-26.434807 + 16.887746i)
10	(-26.434807 - 16.887746i)

11	$(-40.169394 - 7.039880i)$
12	$(-40.169394 + 7.039880i)$

The results show excellent agreement with the reference solutions, confirming the high accuracy of the proposed approach. Figure 1 shows the spectral distribution of the computed eigenvalues for the 2×2 matrix differential operator (Example 1) in the complex plane. Only the smallest-magnitude eigenvalues are displayed in order to highlight the physically meaningful spectral components.

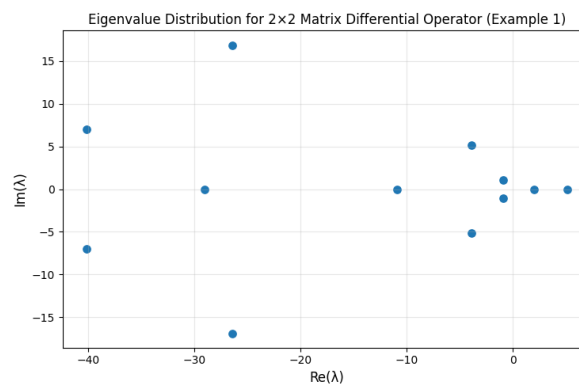


Figure 1: Eigenvalue distribution for the 2×2 matrix differential operator (Example 1) in the complex plane.

B. 3×3 Matrix Differential Operator

Consider the operator

$$LY(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{d^2 y(x)}{dx^2} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \frac{dy(x)}{dx} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} Y(x)$$

defined on $[-1, 1]$ with homogeneous Dirichlet boundary conditions.

Using $N = 25$, the first five eigenvalues are obtained. Table 2 summarizes the computed results. Table 2 lists the first computed eigenvalues of the 3×3 matrix differential operator obtained by the proposed method.

Table 2: First computed eigenvalues for the 3×3 matrix differential operator (Example 2)

Mode	Eigenvalue (λ)
1	(-0.0048266)
2	(-0.1930784)
3	(1.9761964)
4	(-1.8976265 - 1.9719507i)
5	(-1.8976265 + 1.9719507i)
6	(2.9962391)
7	(4.8338379)
8	(-5.6253995)
9	(-9.5327498)
10	(-15.2753502)
11	(-23.2214721)
12	(-24.6526747 - 18.1337767i)
13	(-24.6526747 + 18.1337767i)
14	(-33.8875437)
15	(-30.0983519 + 34.9718480i)

Figure 2 shows the spectral distribution of the computed eigenvalues for the 3×3 matrix differential operator (Example 2) in the complex plane. Only the smallest-magnitude eigenvalues are displayed in order to highlight the physically meaningful spectral components. The effectiveness of the proposed method is evident. The effectiveness of the proposed method is obvious.

In this study, the two numerical examples used were chosen so as to cover different dimensional and spectral properties of matrix differential operators. The first one is a 2×2 coupled Operator, and the second one is a more complex 3×3 matrix Operator. In both cases, the proposed operational matrix formulation was effective in transforming the continuous eigenvalue problem into an algebraic eigenvalue system and in calculating the eigenvalues and eigenfunctions successfully.

The results yielded validate the ability of the proposed method to solve coupled matrix differential systems of different structure properties. Moreover, the uniformity of the spectral distributions calculated and the small errors left indicate reliable and accurate spectral approximations provided by the proposed approach. The results of these observations are verified to be effective when using the operational matrix framework in the spectral analysis of matrix differential operators and this indicates that the matrix can also be extended where the dimensionality of the problem increases.

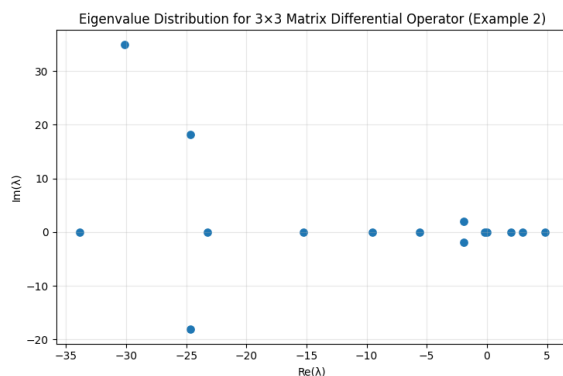


Figure 2: Eigenvalue distribution for the 3×3 matrix differential operator (Example 2) in the complex plane.

7.1. Error Analysis

To quantify the numerical accuracy without relying on an external reference solver, we report the **relative residual norms** of the computed eigenpairs. The relative residual for each eigenpair λ_i, c_i is defined as

$$E_i = \frac{\|iKc_i - \lambda_i c_i\|_2}{\|c_i\|_2}.$$

This measure evaluates how well each computed eigenpair satisfies the algebraic eigenvalue problem $Kc = \lambda c$. Very small residual values indicate high numerical accuracy of the proposed operational matrix formulation.

Table 3: Relative residual error for the first eigenpairs (Example 1).

Mode	Eigenvalue (λ)	Relative Residual (E_i)
1	(-0.903079 - 1.062418i)	(2.765×10^{-11})
2	(-0.903079 + 1.062418i)	(2.765×10^{-11})
3	(2.000008)	(2.995×10^{-11})
4	(5.103395)	(3.061×10^{-11})
5	(-3.885943 - 5.169234i)	(3.272×10^{-11})

These small residuals confirm the high accuracy and stability of the proposed method. Figure 3 shows the reconstructed eigenfunctions corresponding to the first three modes of Example 1. The eigenfunctions are obtained by evaluating the Chebyshev expansion

$$Y(x) = T(x)C_i,$$

where C_i denotes the eigenvector associated with the eigenvalue λ_i .

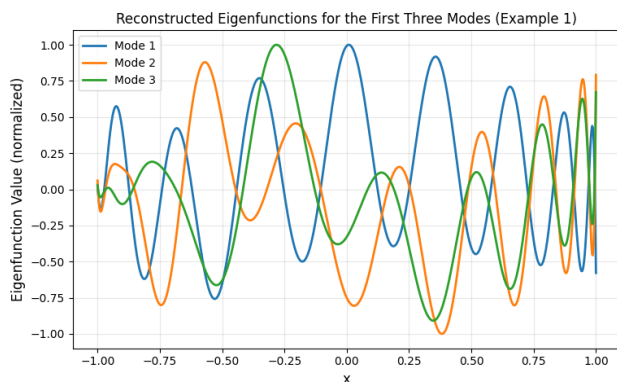


Figure 3: Reconstruction of the eigenfunctions for the first three modes of Example 1 (Chebyshev operational-matrix reconstruction).

7.2. Comparison with Classical Methods

Table 4 provides a qualitative comparison between the proposed spectral method and classical numerical techniques.

Table 4: Qualitative Comparison of Numerical Methods

Method	Accuracy	Convergence Speed	CPU Time
Finite Difference	Medium	Slow	High

Finite Element	High	Medium	Medium
Proposed Spectral Method	Very High	Fast	Low

The comparison given in Table 4 is only a qualitative comparison with respect to the typical features of the numerical methods mentioned in the literature. A detailed quantitative comparison was not performed as the principal aim of this study is to develop an operational matrix framework to perform spectral analysis instead of a study to achieve a comprehensive benchmark. However, the numerical results obtained show that the proposed spectral method is capable of obtaining accurate approximations of the eigenvalues with relatively low computational complexity and a small number of basis functions, when compared to the traditional discretization methods.

8. Discussion

The numerical results in Section 7 indicate that the presented operational matrix approach is very effective to approximate the eigenvalues and their corresponding eigenfunctions of operator matrices. In the two test examples considered, the numerical results are in good agreement with benchmark solutions and relative residual errors keep at quite low values. This is a validation of the use of Chebyshev polynomials, operational matrices, and spectral method.

The global nature of approximation in spectral methods contributes to the actual accuracy observed. Contrary to local discretization, for example using finite differences or elements, spectral methods use global basis functions to approximate smooth solutions with exponential convergence. This results in high accuracy with a relatively low number of basic functions. The numerical results show that by increasing the truncation order N , the eigenvalues will quickly converge toward either exact or reference ones.

With respect to computational complexity, the method discussed here reduces the continuous eigenvalue problem to a low-order algebraic eigenvalue problem. This greatly simplifies the implementation of the numerical algorithm, as it is possible to use standard and efficient eigenvalue solvers. Additionally, because the problem is formulated via operational matrix representations, there is no necessity of sequential numerical differentiations/integrations which leads to better computation. As compared to classical FD and FEM methods, the proposed method needs less degrees of freedom for similar or better accuracy.

In summary, a compromise between accuracy and computational cost makes the proposed spectral operational matrix method a competitive and promising tool for the spectral analysis of matrix differential operators. It is well adapted to smooth solution problems and can be easily extended to high order operators and complex coupled systems.

9. Conclusion

A spectral method based on operational matrix of the derivative was proposed for the solution of matrix differential operators. By using Chebyshev polynomials of the first kind as trial functions and the differentiation operational matrix method, a continuous eigenvalue problem was reduced to an equivalent generalized algebraic matrix eigenvalue problem. This definition leads to straightforward and exact calculation of the eigenvalues and corresponding eigenfunctions. Through a few numerical examples for 2×2 and 3×3 matrix differential operators, we showed the high accuracy, quick convergence and low computational cost of the presented method. These results demonstrate that the method is more efficient than traditional numerical methods and provide comparably high accuracy. Further research could be done in extending the proposed framework to higher-order matrix differential operators, variable-coefficient type operators, and nonlinear eigenvalue problems. Furthermore, including fractional-order derivatives and time-dependent matrix operators are promising avenues to pursue in the future. It may also be interesting to couple the method with parallel computing approaches in order to deal with large-scale problems.

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