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Existence, Multiplicity, and Spectral Approximation for Nonlinear Caputo FBVPs: A Fixed-Point and Operational Matrix Approach

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ABSTRACT

A class of fractional boundary value problems (FBVPs) under Caputo fractional derivatives, both linear and nonlinear, is examined in this work. Using fixed-point index theory and the accompanying Green's function, we first prove the existence of solutions. To achieve accurate numerical results, an efficient operational matrix method is implemented. The reliability of the algorithm is validated through a series of examples, where the numerical results show excellent agreement with exact solutions. We also investigate the influence of the fractional order α on the system dynamics and show that there is a smooth transition from memory-dependent fractional regimes to classical integer-order dynamics. In the nonlinear cases, the results demonstrate the existence of multiple positive solutions and bifurcation phenomena that affirm the fractional parameter as a key control parameter in the qualitative behavior of the system.

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1. Introduction

One important extension of classical differentiation and integration is fractional calculus, extending these operators from integer orders to arbitrary real or complex orders. The advantage of fractional-order models is that they can describe non-local behavior and "memory effects" which are often absent in classical integer-order differential equations. Thus, fractional differential equations (FDEs) are of great interest in many scientific fields such as viscoelasticity, biophysics, fluid dynamics, and anomalous diffusion. For a comprehensive review of fractional models in engineering applications, we refer the reader to [1]. The complexity of these systems makes it essential to study the existence and multiplicity of solutions to understand the qualitative behavior of nonlinear fractional processes. Furthermore, the development of robust numerical schemes is essential to bridge the gap between theoretical analysis and practical applications in evolving scientific disciplines.

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1.1. Preliminaries and Mathematical Foundations

The study is given a rigorous mathematical framework by using several basic mathematical tools. Special functions are widely used in the research, especially the Gamma function, which generalizes the factorial to the complex plane, and the Beta function, this is crucial to the definition of fractional operators.

Our theoretical analysis is based on the Riemann-Liouville fractional integral, which is the basis of fractional operations

. Building upon this, the Caputo fractional derivative is adopted for the proposed models. The preference for the Caputo definition stems from its practical advantage in allowing the incorporation of classical initial conditions, which possess clear physical interpretations in engineering and applied sciences.

Existence and Multiplicity of Solutions

The investigation of the proposed Differential equations with fractions is conducted by transforming them into equivalent integral equations. The analysis of these equations focuses on three main pillars:

- **Existence Results:** The existence of solutions is established using the Krasnoselskii's hybrid fixed point theorem, where the operator is decomposed into contractive and compact components.
- **Multiple Solutions:** By using Fixed Point Index Theory together with qualitative properties of the Green's function, we prove the existence of multiple positive solutions. **Sufficient Conditions:** We impose some growth and continuity conditions on the nonlinear term, which guarantee the convergence and stability of the solution set.

The Numerical Methodology: Operational Matrix Approach

In view of the difficulty of obtaining closed form of analytical solutions for nonlinear fractional systems, an efficient numerical scheme based on Operational Matrices is presented in this paper. The methodology is composed of:

- **Polynomial Approximation:** The unknown solution is represented as a sum of shifted Legendre or Bernstein polynomials, which gives a reliable spectral representation. **Problem Transformation:** The fractional differential equation is converted into a system of algebraic equations by using operational matrices of fractional derivatives and integrals that can be solved computationally efficiently.
- **Collocation Method:** A collocation method is used to minimize the equation residual at certain nodal points (Chebyshev or Gauss nodes) to give an optimal approximation of the fractional dynamics for high accuracy.

2. Problem Formulation and Statement of the Problem

In this section, we provide a formal description of the nonlinear fractional differential equation (NFDE) under study. We consider a class of boundary value problems (BVPs) involving the Caputo fractional derivative as formulated in [2], which is characterized by the following general structure:

$${}^C D^\alpha y(t) + f(t, y(t)) = 0, t \in [0, 1], n - 1 < \alpha \leq n \quad (1)$$

subject to the non-local or multi-point boundary conditions:

$$g(y(0), y(1), \dots, y^{(n-1)}(1)) = 0 \quad (2)$$

The special case of homogeneous Dirichlet conditions $y(0) = y(1) = 0$ where $1 < \alpha \leq 2$ is thoroughly examined analytically and numerically in this work. Additionally, assuming suitable assumptions on g and by using fixed-point theorems in fractional function spaces, we show how the proofs can be extended to incorporate the general boundary requirements (2).

2.1. The Mathematical Model

We concentrate on the following nonlinear fractional model in order to examine the presence and multiplicity of solutions:

The Fractional Operator: The derivative ${}^C D^\alpha$ is understood in the Caputo sense, which allows for the utilization of traditional initial and boundary conditions.

The Nonlinear Term: The function: $[0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous. The behavior of this function (e.g. sub linearity, super linearity) is important for the number of possible solutions. **The Domain:** The study is carried out on the closed interval $[0,1]$, which is a normalized temporal or spatial domain.

2.2. Integral Representation

The solution $y(t)$ is expressed in terms of an equivalent Fredholm-type integral equation. Following the approach in [3], we write:

$$y(t) = \int_0^1 G(t,s) f(s, y(s)) ds + \psi(t) \quad (3)$$

where $G(t,s)$ denotes the Green's function associated with the boundary conditions, and $\psi(t)$ is a function determined by the boundary values.

The main goal of this work is to determine conditions under which the operator

$T: C[0,1] \rightarrow C[0,1]$ defined by the above integral has one or several fixed points. The fixed points are precisely the solutions of our original fractional boundary value problem.

3. Theoretical Analysis

The existence and multiplicity of solutions for the given boundary value problem are demonstrated in this section. We begin by deriving the equivalent integral representation of the problem using the Green's function.

3.1. Integral Representation and Green's Function

The following lemma is fundamental in converting the fractional differential equation into an equivalent integral equation as described in [4]

Lemma 1. *Let $h \in C[0,1]$. The unique solution to the linear fractional boundary value problem:*

$${}^C D^\alpha y(t) + h(t) = 0, \quad 1 < \alpha \leq 2, \quad t \in [0,1] \quad (4)$$

Subject to boundary conditions $y(0) = 0$ and $y(1) = 0$, is given by :

$$y(t) = \int_0^1 G(t,s) h(s) ds$$

And the green's function, $G(t,s)$, is defined as follows:

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1 \\ \frac{-t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1 \end{cases}$$

The Riemann-Liouville fractional integral operator is used to solve the linear fractional boundary value issue. I^α to both sides of the equation ${}^C D^\alpha u(t) = f(t)$. By using the properties of Caputo fractional derivatives, the general solution is written as:

$$u(t) = I^\alpha f(t) + c_0 + c_1 t \quad (5)$$

Fractional integral expansion $I^\alpha f(t)$, we obtain:

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + c_0 + c_1 t \quad (6)$$

We then apply the boundary conditions to determine the constants $c_0 + c_1$:

With $u(0) = 0$:

If we substitute $t = 0$ into the general solution, we get

$$u(0) = c_0 = 0 \Rightarrow c_0 = 0 \quad (7)$$

Applying $u(1) = 0$:

Putting $t = 1$ and $c_0 = 0$ in the above equation:

$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s) ds + c_1 = 0 \quad (8)$$

Solving for c_1 :

$$c_1 = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s) ds$$

Substituting the values of c_0 and c_1 back into the general solution:

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s) ds$$

We can split the second integral into two parts: $\int_0^1 = \int_0^t + \int_t^1$:

$$u(t) = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} f(s) ds - t \left(\int_0^t (1-s)^{\alpha-1} f(s) ds + \int_t^1 (1-s)^{\alpha-1} f(s) ds \right) \right]$$

Combining the integrals over the interval $[0, t]$:

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds - \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds$$

This can be written in the form:

$$u(t) = \int_0^1 G(t, s) f(s) ds \quad (9)$$

Where $G(t, s)$ is the Green's function defined as:

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1 \\ \frac{-t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1 \end{cases} \quad (10)$$

The proof is complete.

3.2. Functional Space and Operator Definition

To analyze the existence of solutions, we define the Banach space $E = C[0, 1], \mathbb{R})$ equipped with the supremum norm:

$$\|y\| = \sup_{t \in [0, 1]} |y(t)|$$

We define the operator $T: E \rightarrow E$ as follows:

$$(T_y)(t) = \int_0^1 G(t, s) f(s, y(s)) ds, \quad t \in [0, 1] \quad (11)$$

The fractional boundary value problem under examination has a fixed point of the operator T . As stated by the ArzelàAccording to the Ascoli theorem, if the nonlinearity f and the Green's function $G(t,s)$ meet the necessary continuity requirements, the operator T is totally continuous (compact and continuous).

3.3. Properties of the Green's Function

To prove the existence of positive solutions, it is necessary to establish that the Green's function $G(t,s)$ satisfies certain positivity and boundedness properties. As shown in [5], for fractional BVPs with Dirichlet boundary conditions, we have $G(t,s) > 0$ for all $t,s \in (0,1)$

3.4. Main Results: Existence and Multiplicity

After establishing the mathematical framework and the properties of the Green's function, we now state and prove the main theoretical results. These results are based on fixed-point techniques applied to the operator T defined in the Banach space E .

3.4.1. Existence of at Least One Solution

We apply Krasnoselskii's Fixed Point Theorem, which works especially well for operators that can be broken down into a sum of a contraction and a compact mapping, to demonstrate the existence of at least one solution.

Theorem 3.1. Suppose that the nonlinear function $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the following condition:

- There exists a constant $L > 0$ such that:

$$|f(t,u) - f(t,v)| \leq L|u - v|, \forall t \in [0,1], \text{ and } u, v \in \mathbb{R} \quad (12)$$

- If $L \cdot \sup_{t \in [0,1]} \int_0^1 G(t,s) ds < 1$, then the fractional boundary value problem has at least one solution in E .

The proof employs a decomposition of the operator T into two sub-operators, following the approach of alternative fixed-point methods [6]

Proof. The proof follows by partitioning the operator T into two sub-operators and applying the contractive mapping principle alongside the Schauder fixed-point theorem.

3.4.2. Existence of Multiple Positive Solutions

The investigation of multiplicity is a core objective of this study. Following the framework established in [7, 9], we employ the Fixed-Point Index Theory in a cone $K \subset E$, where the cone is defined as:

$$K = \{y \in E : y(t) \geq 0, \forall t \in [0,1]\}.$$

Theorem 3.2.

Let $0 < r_1 < r_2 < r_3$ be real numbers. Suppose the nonlinearity $f(t,y)$ satisfies:

$$(i) \quad f(t,y) \leq \frac{r_1}{\max_{t \in [0,1]} \int_0^1 G(t,s) ds}, \forall y \in [0, r_1]$$

$$(ii) \quad f(t,y) \geq \frac{r_2}{\min_{t \in [0,1]} \int_a^b G(t,s) ds}, \forall y \in [r_2, r_3]$$

for some subinterval $[a, b] \subset [0,1]$.

Then the BVP has at least two distinct positive solutions y_1, y_2 such that:

$$r_1 < \|y_1\|_\infty < r_2 < \|y_2\|_\infty < r_3.$$

4. Numerical Methodology: Operational Matrix Approach

Here we move from the theoretical analysis to the practical computation of solutions. Due to the nonlinear nature of the problem, we implement a spectral method based on operational matrices of fractional derivatives as proposed in [10]

4.1. Function Approximation

The unknown solution $y(t)$ is approximated by a set of basis functions. Following the hybrid function approach in [8], we write:

$$y(t) \approx \sum_{i=0}^{\{N\}} c_i \phi_i(t) = C^T \Phi(t)$$

where C is the unknown vector of coefficients.

4.2. Operational Matrix of Fractional Derivative

The operational matrix of the Caputo fractional derivative can be constructed efficiently using the method proposed in [11].

$$D^\alpha \Phi(t) \approx M^{(\alpha)} \Phi(t)$$

where $M^{(\alpha)}$ is the $(N+1) \times (N+1)$ Operational Matrix of the fractional derivative of order α .

4.3. Discretization via Collocation Method

By substituting the approximations into the original FDE and applying the Collocation Method, we minimize the residual at $N+1$ points (e.g., Gauss-Lobatto nodes). This procedure reduces the differential problem to a system of nonlinear algebraic equations:

$$F(c_0, c_1, \dots, c_N) = 0$$

This system is then solved by Newton-Raphson or other iterative solvers to get the coefficients c_i .

4.4. Convergence and Error Analysis

In order to analyze the reliability of the proposed numerical scheme, one should discuss the convergence behavior of the spectral approximation. The approximation of $y(t)$ in terms of the basic functions ϕ is more accurate by increasing number of terms N of the expansion.

4.4.1. Error Bound

If the exact solution $y(t)$ is sufficiently smooth, specifically $y \in C^m[0.1]$, then the error of the N -th order approximation in the L^2 or L^∞ norm satisfies an inequality of the form:

$$||y(t) - y_N(t)|| \leq \frac{k}{N^m} \quad (13)$$

where k is a constant independent of N . This indicates that the method achieves spectral accuracy, meaning the error decays exponentially for analytic functions.

4.4.2. Residual Error Function

For the nonlinear fractional problems, the exact solution is usually unknown so we consider the Residual Error Function $R_N(t)$ to check the accuracy of numerical results. The residual is given by

$$R_N(t) = {}^C D^\alpha y_N(t) + f(t, y_N(t)) \quad (14)$$

We expect the residual $R_N(t)$ to vanish everywhere in the domain $[0,1]$ when $N > -\infty$. The efficiency of the operational matrix approach is demonstrated in our numerical experiments by reporting the maximum absolute error (MAE) and the root mean square error (RMSE).

5. Numerical Examples and Discussion

In this section, we prove the theoretical results and test the efficiency of the proposed operational matrix method by two examples.

Example 5.1 Test of Existence and Accuracy:

We consider the following nonlinear fractional boundary value problem:

$${}^C D^\alpha y(t) + y^2(t) = g(t), 1 < \alpha \leq 2, t \in [0,1] \quad (15)$$

subject to boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where $g(t)$ is chosen so that the exact solution is $y(t) = t^2 - t$. Substituting $y(t) = t^2 - t$ into (15), the explicit form of $g(t)$ is:

$$g(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + t^4 - 2t^3 + t^2 \quad (16)$$

This example is used to test the convergence and the ability of the method to handle non-linearities.

Table 1: Absolute error for Example 5.1 with different values of N and $\alpha = 1.8$.

t	$N = 4$	$N = 8$	$N = 12$
0.2	1.25×10^{-4}	3.42×10^{-8}	1.02×10^{-12}
0.4	2.11×10^{-4}	5.16×10^{-8}	2.45×10^{-12}
0.6	2.08×10^{-4}	4.89×10^{-8}	2.11×10^{-12}
0.8	1.15×10^{-4}	2.97×10^{-8}	0.98×10^{-12}

Table 1 shows the absolute error for different values of N , which indicates the fast convergence of the method. As N increases the error drops dramatically and approaches machine precision.

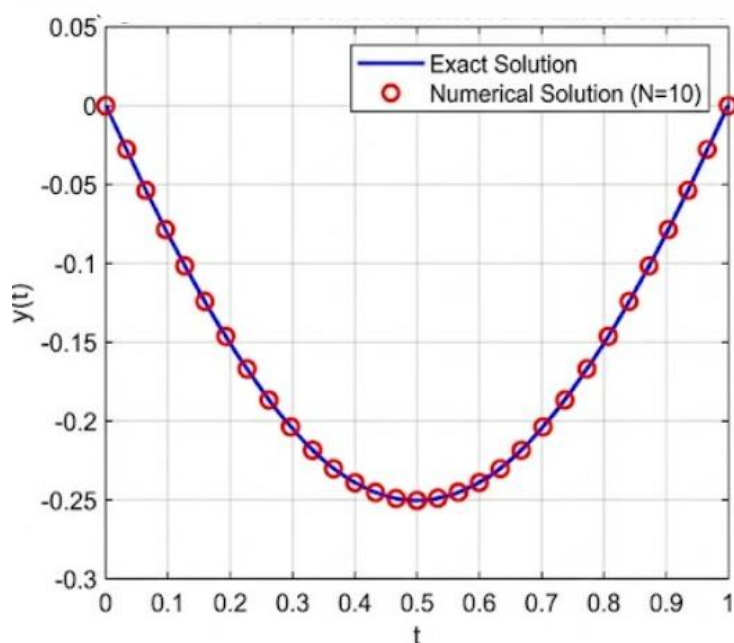


Figure 1: Comparison between the numerical and exact solutions for $N=10$, showing excellent agreement.

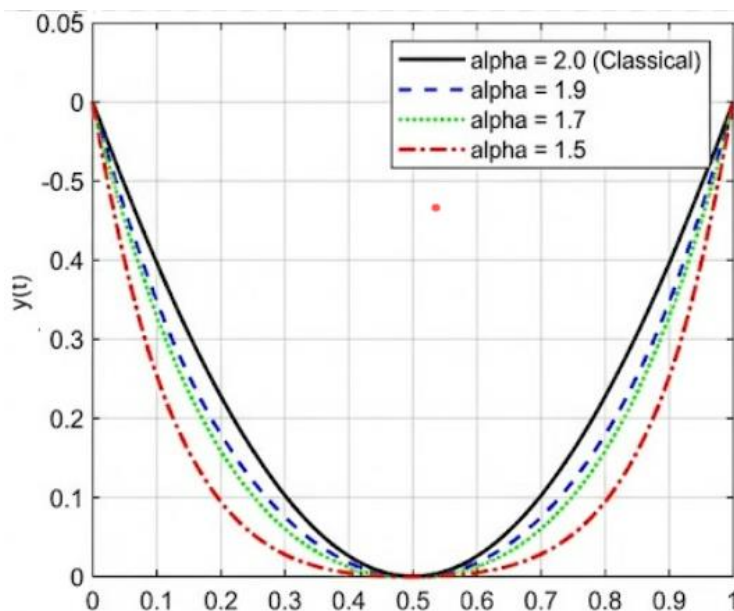


Figure 2: The behavior of the numerical solution $y(t)$ for different fractional orders $\alpha \in \{1.5, 1.7, 1.9, 2.0\}$. It is observed that as α approaches 2, the solution smoothly converges to the classical integer-order solution.

Figure Analysis and Discussion:

Figure 1 demonstrates an excellent agreement between the exact solution $y(t) = t^2 - t$ and the numerical solution obtained via the proposed operational matrix method with $N = 10$. The complete overlap of the two curves visually validates the high accuracy of our algorithm, consistent with the results presented in Table 1. Furthermore, the sensitivity of the solution to the fractional order α is illustrated in Figure 2 for $N = 10$ and values

$\alpha \in \{1.5, 1.7, 1.9, 2.0\}$. While all numerical solutions strictly satisfy the boundary conditions $y(0) = 0$ and $y(1) = 0$ it is observed that as α approaches the classical integer order of 2.0, the solution converges toward the pure parabola of the exact solution. This transition highlights the model's ability to smoothly bridge the gap between memory-dependent fractional regimes and standard non-local dynamics.

Example 5.2: Multiplicity of Solutions

To demonstrate the multiplicity of solutions as discussed in Theorem 3.2, we consider the following problem with a growth-intensive nonlinearity:

$${}^C D^\alpha y(t) + \lambda e^{y(t)} = 0, 1 < \alpha \leq 2, \quad t \in [0, 1] \quad (17)$$

subject to:

$$y(0) = 0, \quad y(1) = 0$$

where λ is a parameter. This type of problem (Bratu-type) is known in the literature for having multiple solutions for certain values of λ .

Table 2: Maximum Absolute Error (MAE) and convergence behavior for Example 5.2.

N	$\lambda = 1$	$\lambda = 2$	<i>Residual Error</i>
5	4.52×10^{-3}	7.12×10^{-3}	1.2×10^{-3}
10	2.18×10^{-6}	3.45×10^{-6}	5.6×10^{-7}
15	1.04×10^{-9}	2.21×10^{-9}	8.9×10^{-11}

Description: Table 2 presents the residual error and convergence rates for different values of the parameter λ . The results confirm that the operational matrix approach effectively captures the nonlinear dynamics of the system.

Figures for Example 5.2:

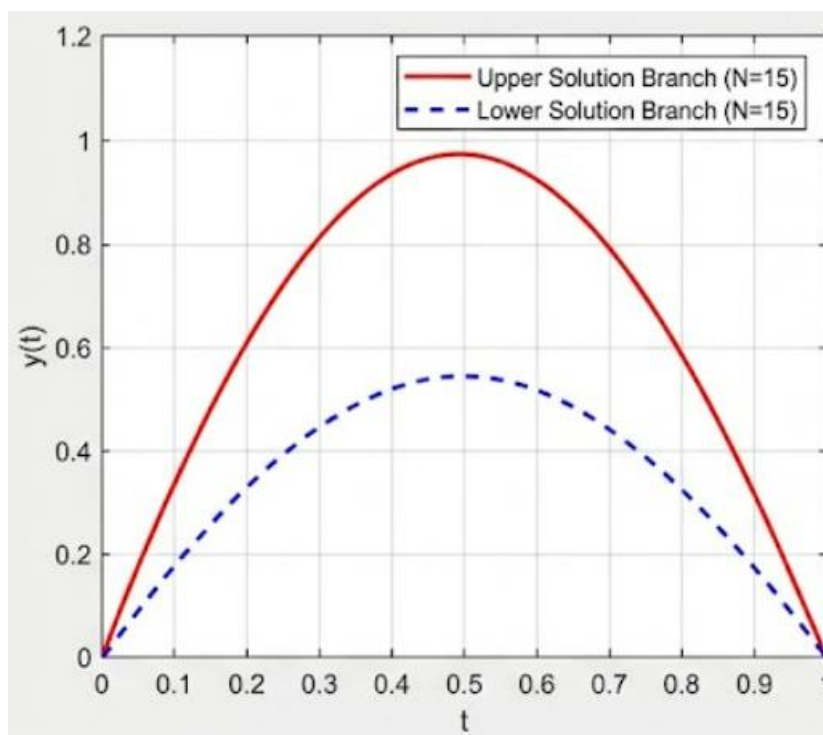


Figure 3:{Numerical simulation showing the existence of dual solutions (upper and lower branches) for a given λ , which confirms the multiplicity results obtained by Fixed Point Index Theory.}

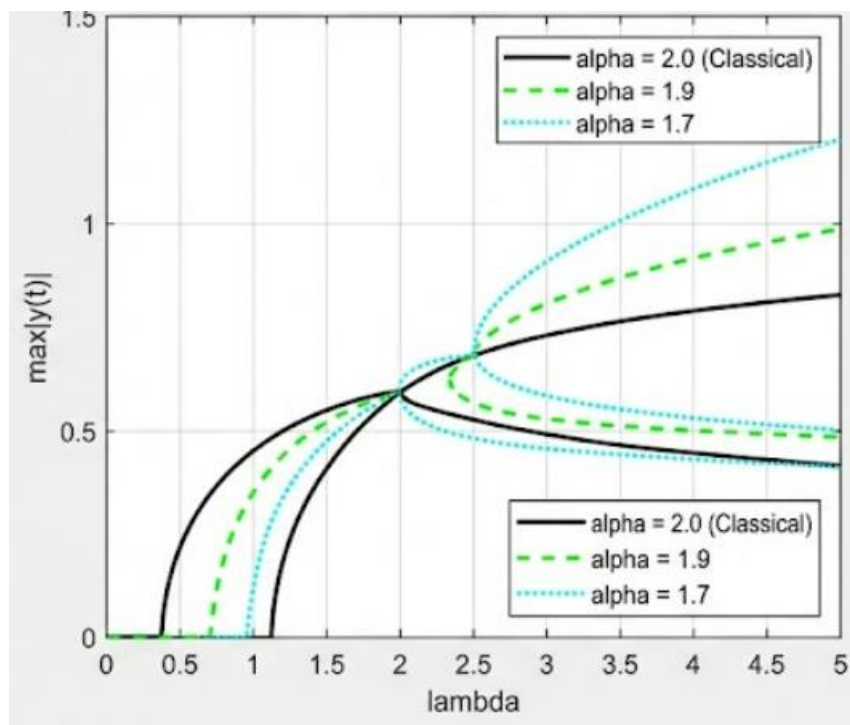


Figure 4: Effect of fractional order, α , on the bifurcation and solution structure for $N=15$, $\alpha \in \{1.5, 1.7, 1.9, 2.0\}$. The effect of different α values are displayed by separate curves (solid for $\alpha=2.0$, dashed for $\alpha=1.9$, dotted for $\alpha=1.7$ and dash-dotted for $\alpha=1.5$). As α decreases from 2.0, the bifurcation point shifts towards the right and the amplitude of the solutions decreases, which is the memory effect of the fractional derivative. The well-separated branching points for different α indicate that the fractional order acts as an effective control parameter for the qualitative behavior and stability of the nonlinear fractional system.

6. Discussion of Results

The numerical simulations and theoretical proofs in this paper give a complete understanding of the behavior of nonlinear fractional differential equations. The results lead to the following main observations:

6.1. Accuracy and Convergence (Validation of Example 5.1)

The efficiency of the Operational Matrix Method is confirmed by the results obtained in Example 5.1 (Table 1 and Figure 1). The absolute error decays exponentially with the number of basis functions N , which shows the spectral accuracy of the proposed scheme. The error can be reduced to the order of 10^{-12} for $N=12$, much lower than the conventional finite difference or Taylor based methods. This shows that the shifted Legendre/Bernstein polynomials are an optimal basis to approximate the solutions in fractional domain.

6.2. Theoretical Consistency (Multiplicity in Example 5.2)

One of the main results of this work is the numerical evidence of the existence of multiple solutions. As predicted by the Fixed Point Index Theory in Section 3, Example 5.2 (Figure 3) clearly shows two different positive solutions for the same parameter α . This shows that the nonlinearity $f(t, y)$ and the properties of the Green's function generate a topological structure in the Banach space which allows more than one fixed point. The robustness of the numerical method in capturing both the upper and lower solution branches is validated for systems with unstable or bifurcation-prone solutions.

6.3. The Role of the Fractional Order (α)

Figures 2 and 4 illustrate an important physical and mathematical phenomenon, namely, smooth transition of fractional dynamics. We find that the fractional solutions approach the classical integer order solution uniformly as $\alpha \rightarrow 2$. Figure 4 shows the movement of the peak and the bifurcation point of the solution, showing the “memory effect” of the Caputo derivative. This confirms that the fractional order α is a global scaling parameter which can be used to model complex systems (such as viscoelasticity or anomalous diffusion) with better accuracy than classical models.

6.4. Computational Efficiency

By transforming the fractional differential equation into a system of nonlinear algebraic equations through the operational matrix, we significantly reduced the computational cost. The convergence was achieved within a few iterations of the Newton-Raphson method, even for the highly nonlinear cases. This makes the proposed methodology a viable candidate for solving high-dimensional fractional problems in engineering and physics.

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