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Subclass of harmonic meromorphic functions with fixed residue ζ

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Abstract
By using the linear operator $\mathfrak{I}^n(\nu, l)f(z)$, we introduce a new subclass of meromorphic harmonic functions with fixed residue ζ in $\mathcal{U}_w$, and we investigate several convolution properties, coefficient inequalities, distortion theorem and extreme points for this class.

Keywords: Harmonic Function, Meromorphic Function, Linear Operator**,** Convolution Product.
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1.Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Such functions can be expressed as

$$f = h + \overline{g}, \qquad (1.1)$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be lacally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D (see [4]).

There are many papers on harmonic functions defined on the unit disk $\mathcal{U} = \{z: |z| < 1\}$ [2], [9], [10], [12].

For $0 \le w \le 1$, we let $S_H(w)$ denote the class of functions harmonic univalent, orientation preserving and meromorphic in $\mathcal{U}$, with $\lim_{z \to w} f(z) = \infty$ which are the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z - w| \qquad (1.2)$$

$$h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} c_k z^k, \text{ and } g(z) = \sum_{k=1}^{\infty} d_k z^k, \qquad (1.3)$$

and $\zeta = Res(f, w)$ with $0 < \zeta \le 1, z \in \mathcal{U}\setminus\{w\}$ or we may set for $z \in \mathcal{U}_w = \{z: 0 < |z - w| < 1 - w\}$

$$h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k, \text{ and } g(z) = \sum_{k=1}^{\infty} b_k (z-w)^k, \qquad (1.4)$$

We further remove the logarithmic singularity by letting $A = 0$ and focus the subclass $S_H(w)$ of all harmonic, orientation preserving, and meromorphic mapping which have the development

$$f(z) = h(z) + \overline{g(z)}, \qquad (1.5)$$

where

$$h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} c_k z^k, \text{ and } g(z) = \sum_{k=1}^{\infty} d_k z^k, \quad c_k, d_k \ge 0;\ z \in \mathcal{U}\setminus\{w\} \qquad (1.6)$$

or we may set for $z \in \mathcal{U}_w = \{z: 0 < |z - w| < 1 - w\}$

$$h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k, \text{ and } g(z) = \sum_{k=1}^{\infty} b_k (z-w)^k, \quad a_k, b_k \ge 0 \qquad (1.7)$$

where $h(z)$ has a simple pole at the point w with residue ζ. For $\zeta = 1$ and $w = 0$ the function f was studied by Bostanci, Yalcin and $\ddot{O}zt\ddot{u}$rk [3].

For the function f in the class $S_H(w)$, we define the following $I^n(\nu, l)$ operator, for $\nu \geq 0, l > 0,$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$I^n(\nu,l)f(z) = I^n(\nu,l)h(z) + \overline{I^n(\nu,l)g(z)}$$

where

$$I^n(\nu,l)h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\left(\frac{l+\nu k}{l}\right)^n a_k(z-w)^k \quad \text{and}$$

$$I^n(\nu,l)g(z) = \sum_{k=1}^{\infty}\left(\frac{l+\nu k}{l}\right)^n b_k(z-w)^k, \ a_k, b_k \geq 0. \qquad (1.8)$$

We note that $I^0(\nu,l)f(z) = f(z)$ and $I^1(1,\ 1)f(z) = \frac{\left((z-w)^2 f(z)\right)'}{(z-w)} = 2f(z) + (z-w)f'(z)$, and by specializing the parameters ν, l and n, we obtain the following operators studied by various authors:

(1) $I^n(1,l)f(z) = D_l^n f(z)$, (see Cho et al. [5], [6]);

(2) $I^n(\nu,1)f(z) = D_\nu^n f(z)$, (see Al-Oboudi and Al-Zkeri [1]);

(3) $I^n(1,1)f(z) = I^n f(z)$, (see Uralegaddi and Somanatha [13]).

For $\nu \geq 0, l > 0,$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,, we define the dual operator $\mathfrak{I}^n(\nu,l): S_H(w) \to S_H(w)$ by

$$\mathfrak{I}^n(\nu,l)f(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\left(\frac{l}{l+\nu k}\right)^n a_k(z-w)^k$$

El-Ashwah and Aouf [7, 8] stuied the linear operators for functions which are analytic in the punctured unit disk $\mathcal{U}\backslash\{0\}$.

Denoting by $\mathcal{J}^n(\nu,l)f(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\left(\frac{l}{l+\nu k}\right)^n (z-w)^k$, it is easy to verify that

$$\mathfrak{I}^n(\nu,l)f(z) = \mathcal{J}^n(\nu,l)f(z) * f(z),$$

$$\nu z \mathfrak{I}^{n+1}(\nu,l)f(z) = l\mathfrak{I}^n(\nu,l)f(z) - (1+l)\mathfrak{I}^{n+1}(\nu,l)f(z),$$

And

$$\mathfrak{I}^n(\nu,l)f(z) = \underbrace{\mathfrak{I}^1(\nu,l)\left(\frac{1}{(z-w)(1-(z-w))}\right) * \dots * \mathfrak{I}^1(\nu,l)\left(\frac{1}{(z-w)(1-(z-w))}\right)}_{n\ times} * f(z).$$

We note that $\mathfrak{I}^n(1,\ \beta)f(z) = P_\beta^\alpha f(z), \alpha > 0, \beta > 0$ (see Lashin [11]).

$$\mathfrak{I}^n(\nu,l)f(z) = \mathfrak{I}^n(\nu,l)h(z) + \overline{\mathfrak{I}^n(\nu,l)g(z)},$$

where

$$\mathfrak{I}^n(\nu,l)h(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\left(\frac{l}{l+\nu k}\right)^n a_k(z-w)^k \quad \text{and}$$

$$\mathfrak{I}^n(\nu,l)g(z) = \sum_{k=1}^{\infty}\left(\frac{l}{l+\nu k}\right)^n b_k(z-w)^k, \ a_k, b_k \geq 0. \qquad (1.9)$$

For $0 \leq \rho < 1$; $\frac{1}{2} \leq \sigma \leq 1$; $0 < \kappa \leq 1$; $\nu \geq 0, l > 0,$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 \leq w < 1$ where $\zeta = Res(f, w)$ with $0 < \zeta \leq 1$, $z \in \mathcal{U}\backslash\{w\}$, we let $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l)$ denote the harmonic functions of the form (1.1) such that

$$\left|\frac{(z-w)^2(\mathfrak{I}^n(\nu,l)f(z))' + 1}{(2\sigma-1)(z-w)^2\left(\mathfrak{I}^n(\nu,l)f(z)\right)' + (2\sigma\rho-1)}\right| < \kappa. \qquad (1.10)$$

Let us write

$$\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,\upsilon,l] = \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l) \cap S_H[w], \qquad (1.11)$$

Where $S_H[w]$ is the class of functions of the form (1.5) and (1.6) that are meromorphic and harmonic in $\mathcal{U}_w$.

2. MAIN RESULT:

In our first theorem, we introduce a sufficient conditiond for harmonic functions in $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l)$.

Theorem 2.1: Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.3). Then $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l)$ if

$$\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)(|a_k|+|b_k|)\leq 2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa), \qquad (2.1)$$

for $0\leq\rho<1;\ \frac{1}{2}\leq\sigma\ \leq 1; 0<\kappa\ \leq 1;\ \nu\geq 0, l>0$, , and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $0\leq w<1$ where $\zeta=Res(f,w)$ with $0<\zeta\leq 1, z\in\ \mathcal{U}\backslash\{w\}$.

Proof: Suppose (2.1) holds. Then we find from definition (1.10) that

$$|(z-w)^2(\mathfrak{I}^n(\nu,l)f(z))'+1|-\kappa\left|(2\sigma-1)(z-w)^2\left(\mathfrak{I}^n(\nu,l)f(z)\right)'+(2\sigma\rho-1)\right|<0,$$

Provided

$$\left|(1-\zeta)+\sum_{k=1}^{\infty}k\left(\frac{l}{l+\nu k}\right)^{n}(a_k+b_k)(z-w)^{k+1}\right|-\kappa\left|-\zeta(2\sigma-1)+(2\sigma\rho-1)+\sum_{k=1}^{\infty}k(2\sigma-1)\left(\frac{l}{l+\nu k}\right)^{n}(a_k+b_k)(z-w)^{k+1}\right|<0,$$

For $|z-w|=r<1-w$

$<\quad (1-\zeta)+\sum_{k=1}^{\infty}k\left(\frac{l}{l+\nu k}\right)^{n}(|a_k|+|b_k|)r^{k+1}-2\zeta\kappa\sigma+\zeta\kappa+2\kappa\sigma\rho-\kappa+\kappa\sum_{k=1}^{\infty}k(2\sigma-1)\left(\frac{l}{l+\nu k}\right)^{n}(|a_k|+|b_k|)r^{k+1}$

$$=\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)(|a_k|+|b_k|)r^{k+1}-2\kappa\sigma(\zeta-\rho)+(1-\zeta)(1-\kappa)\leq 0. \qquad (2.2)$$

The inequality in (2.2) holds true for all $|z-w|=r<1-w<1$. Therefore, letting $r\to 1$ in (2.2), we obtain

$$\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)(|a_k|+|b_k|)\leq 2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa).$$

Hence

$$f\in\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l).$$

The harmonic mappings

$$f(z)=z+\sum_{k=1}^{\infty}\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)}x_k(z-w)^k+\sum_{k=1}^{\infty}\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)}\overline{y_k(z-w)^k} \qquad (2.3)$$

Where $\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,\upsilon,l)$ because

$$\sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}|a_k|+\sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+\nu k}\right)^{n}(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}|b_k|$$

$$\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f(z)=h(z)+\overline{g(z)}$ where h and g are of the form (1.7).

Theorem 2.2: Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.7). Then $f\in\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,\upsilon,l]$ if and only if

$$\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^n (1+2\kappa\sigma-\kappa)(a_k+b_k) \leq 2\kappa\sigma(\zeta-\rho) - (1-\zeta)(1-\kappa), \qquad (2.4)$$

for $0 \leq \rho < 1;\ \frac{1}{2} \leq \sigma \leq 1; 0 < \kappa \leq 1;\ \nu \geq 0, l > 0,$, and $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $0 \leq w < 1$ where $\zeta = Res(f, w)$ with $0 < \zeta \leq 1,\ z \in \mathcal{U}\backslash\{w\}$.

Proof: Since $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l] \subset \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}(w,k,\rho,v,l)$, we only need to prove the 'only if ' part of the theorem. To this end, for functions f of the form (1.7), we notice that the condition

$$\left|\frac{(z-w)^2(\mathfrak{I}^n(v,l)f(z))' + 1}{(2\sigma-1)(z-w)^2(\mathfrak{I}^n(v,l)f(z))' + (2\sigma\rho-1)}\right| < \kappa,$$

Is equivalent to

$$Re\left\{\frac{(1-\zeta)+\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k)(z-w)^{k+1}}{2\sigma(\zeta-\rho)+(1-\zeta)-\sum_{k=1}^{\infty} k(2\sigma-1)\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k)(z-w)^{k+1}}\right\} < \kappa.$$

If we choose the values of z on the real axis and $(z-w) \to 1^-$ we get

$$\frac{(1-\zeta)+\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k)}{2\sigma(\zeta-\rho)+(1-\zeta)-\sum_{k=1}^{\infty} k(2\sigma-1)\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k)} < \kappa,$$

whence

$$(1-\zeta)+\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k) < \kappa 2\sigma(\zeta-\rho)+\kappa(1-\zeta) - \sum_{k=1}^{\infty} k(2\sigma-1)\left(\frac{l}{l+\nu k}\right)^n (a_k+b_k),$$

and so

$$\sum_{k=1}^{\infty} k\left(\frac{l}{l+\nu k}\right)^n (1+2\kappa\sigma-\kappa)(a_k+b_k) \leq 2\kappa\sigma(\zeta-\rho) - (1-\zeta)(1-\kappa),$$

which is equivalent to (2.4).

The following theorem gives the distortion bounds for functions in $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$.

Theorem 2.3: Let $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa\zeta}[w,k,\rho,v,l]$. Then for $|z-w| = r < 1-w$ we have

$$\frac{\zeta}{r} - \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k(1+2\kappa\sigma-\kappa)} r^2 \leq |f(z)| \leq \frac{\zeta}{r} + \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k(1+2\kappa\sigma-\kappa)} r^2. \qquad (2.5)$$

Proof: We only prove the left hand inequality. The proof of the right hand inequality is similar and will be omitted.

Let $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$. Taking the absolute value of f we have

$$|f(z)| = \left|\frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k(z-w)^k + \overline{\sum_{k=1}^{\infty} b_k(z-w)^k}\right|$$

$$\geq \frac{1}{z-w}\left[\zeta - |z-w|\sum_{k=1}^{\infty}(a_k+b_k)|z-w|^k\right]$$

$$\geq \frac{1}{r}\left[\zeta - r^2\sum_{k=1}^{\infty}(a_k+b_k)\right]$$

$$\geq \frac{\zeta}{r} - \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k(1+2\kappa\sigma-\kappa)}\sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+\nu k}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}(a_k+b_k)r^2.$$

$$\geq \frac{\zeta}{r} - \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k(1+2\kappa\sigma-\kappa)} r^2.$$

Now we show that $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ is closed under convex combination of its members.

Theorem 2.4: The class $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ is closed under convex combination.

Proof: For $i = 1, 2, \ldots$ let $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$, where f is given by

$$f(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k(z-w)^k + \overline{\sum_{k=1}^{\infty} b_k(z-w)^k},$$

Then by (2.4),

$$\sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+\nu k}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} a_k + \sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+\nu k}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} b_k \leq 1. \qquad (2.6)$$

For $\sum_{i=1}^{\infty} t_i, 0 \leq t_i \leq 1$, the convex combination of f may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right)(z-w)^k + \overline{\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)(z-w)^k}.$$

Then by (2.6),

$$\sum_{k=1}^{\infty}\left(\frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}\sum_{i=1}^{\infty} t_i a_{k_i} + \frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}\sum_{i=1}^{\infty} t_i b_{k_i}\right)$$

$$= \sum_{i=1}^{\infty} t_i\left\{\sum_{k=1}^{\infty}\left[\frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} a_{k_i} + \frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} b_{k_i}\right]\right\}$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by (2.4) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$.

Next, we determine the extreme points of the closed convex hulls of $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ denoted by $clco\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$.

Theorem 2.5: Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.7). Then $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ if and only if

$$f(z) = \sum_{k=0}^{\infty}(x_k h_k(z) + y_k g_k(z)), \quad (2.7)$$

where

$$h_0(z) = \frac{\zeta}{z-w}, \quad g_0(z) = 0,$$

$$h_k(z) = \frac{\zeta}{z-w} - \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}(z-w)^k, \quad (k = 1,2,3,\ldots)$$

and

$$g_k(z) = \frac{2\kappa\sigma(\zeta-\rho)-(1-\rho)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}(z-w)^k, \quad (k = 1,2,3,\ldots)$$

$\sum_{k=1}^{\infty}(x_k + y_k) = 1$, $x_k \geq 0$ and

$$y_k \geq 0.$$

In particular, the extreme points of $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ are $\{h_k\}$ and $\{g_k\}$.

Proof: For the functions $f(z)$ of the form (2.7), we have

$$f(z) = \sum_{k=0}^{\infty}(x_k h_k(z) + y_k g_k(z)),$$

$$= \sum_{k=0}^{\infty} x_k \frac{\zeta}{z-w} + \sum_{k=1}^{\infty}\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)} x_k (z-w)^k + \sum_{k=0}^{\infty}\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)} y_k \overline{(z-w)^k}.$$

Then

$$\sum_{k=1}^{\infty}\frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}\left(\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)} x_k\right) + \sum_{k=0}^{\infty}\frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}\left(\frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)} x_k\right)$$

$$\sum_{k=0}^{\infty}(x_k + y_k) - x_0 = 1 - x_0 \leq 1,$$

And so $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$, conversely, if $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$, then

$$a_k \leq \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)},$$

And

$$b_k \leq \frac{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)}{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}.$$

Set

$$x_k \leq \frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} a_k, \quad (k = 1,2,\ldots)$$

$$y_k \leq \frac{k\left(\frac{l}{l+vk}\right)^n(1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho)-(1-\zeta)(1-\kappa)} b_k, \quad (k = 0,2,\ldots)$$

$0 \leq x_k \leq 1, (k = 1,2,\ldots)$ and $0 \leq y_k \leq 1, (k = 0,1,\ldots)$.

We define $x_0 = 1 - \sum_{k=1}^{\infty} x_k - \sum_{k=0}^{\infty} y_k$

and note that by Theorem 2.2. $x_1 \geq 0$.
Conversely, we obtain

$$f(z) = \sum_{k=0}^{\infty} (x_k h_k(z) + y_k g_k(z)),$$

And hence this completes the proof of Theorem 2.5.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k + \sum_{k=1}^{\infty} b_k \overline{(z-w)^k},$$

and

$$F(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} A_k (z-w)^k + \sum_{k=1}^{\infty} B_k \overline{(z-w)^k},$$

We define the convolution of two harmonic functions $f(z)$ and $F(z)$ as

$$f(z) * F(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k A_k (z-w)^k + \sum_{k=1}^{\infty} b_k B_k \overline{(z-w)^k}. \quad (2.8)$$

Using this definition, we show that the class $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho,v,l]$ is closed under convolution.

Theorem 2.6: For $0 \leq \rho_1 \leq \rho_2 < 1$, let $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_2,v,l]$ and $F \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_1,v,l]$. Then $f * F \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_2,v,l] \subset \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_1,v,l]$.

Proof: Let
$f(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k + \sum_{k=1}^{\infty} b_k \overline{(z-w)^k}$ be in $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_2,v,l]$
and
$F(z) = \frac{\zeta}{z-w} + \sum_{k=1}^{\infty} A_k (z-w)^k + \sum_{k=1}^{\infty} B_k \overline{(z-w)^k}$ be in $\mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_1,v,l]$.

For $F \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_1,v,l]$, we note that $A_k \leq 1$ and $B_k \leq 1$.

Now, for the convolution function $(f * F)$, we obtain

$$\sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_1)-(1-\zeta)(1-\kappa)} a_k A_k + \sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_1)-(1-\zeta)(1-\kappa)} b_k B_k$$

$$\leq \sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_1)-(1-\zeta)(1-\kappa)} a_k + \sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_1)-(1-\zeta)(1-\kappa)} b_k$$

$$\leq \sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_2)-(1-\zeta)(1-\kappa)} a_k + \sum_{k=1}^{\infty} \frac{k\left(\frac{l}{l+vk}\right)^n (1+2\kappa\sigma-\kappa)}{2\kappa\sigma(\zeta-\rho_2)-(1-\zeta)(1-\kappa)} b_k \leq 1.$$

Since $0 \leq \rho_1 \leq \rho_2 < 1$, and $f \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_2,v,l]$.

Therefore

$f * F \in \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_2,v,l] \subset \mathfrak{S}\mathfrak{R}_H^{n,\sigma,\kappa,\zeta}[w,k,\rho_1,v,l]$

REFERENCES

[1] F. M. Al-Oboudi and H. A. Al-Zkeri, Applications of Briot-Bouquet differential subordination to certain subclasses of meromorphic functions, Arab J. Math. Sci., 12(2005), no. 1, 1-14.

[2] Y. Avci and E. Zlotkiewicz, On harmonic univalent mapping, Ann. Univ. Marie Curei-Skolodowska Sect. A.44,(1990), 1-7.

[3] H. Bostanci, S. Yalcin and M. Öztürk, On meromorphically harmonic starlike functions with respect to symmetric conjugate points, J. Math. Anal. Appl. 328 (2007), 370-379.

[4] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. Al. Math., 9 (1984), 3-25.

[5] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations, J. Math. Anal. Appl., 300(2004), 505-520.

[6] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations, Integral Transformations Special Functions, 16(2005), no. 18, 647-659.

[7] R. M.El-Ashwah and M. K. Aouf, Some properties of certain subclasses of maromorphiclly p-valent functions involving extended multiplier transformations, Comput. Math. Appl., 59(2010), 2111-2120.

[8] R. M. El-Ashwah and M. K. Aouf, Differential subordination and superordination on p-valent meromorphic function defined by extended multiplier transformations, Europ. J. Pure Appl. Math., 3(2010), no. 6, 1070-1085.

[9] J. M. Jahangiri and H. Silverman, Harmonic close-to-convex mappings, J. Appl. Math. Stchast. Anal., 15(2002), 23-28.

[10] J. M. Jahangiri, Y. C. Kim and H. M. Srivastava, Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform, Spec. Funct., 14(2003), 237-242.

[11] A. Y. Lashin, On certain subclass of meromorphic functions associated with certain integral operators, Comput. Math. Appl., 59(2010), no. 1, 524-531.

[12] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl., 220(1998), 283-289.

[13] B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, Bull. Austral. Math. Soc., 43(1991), 137-140.

صنف من الدوال التوافقية الميرومورفية براسب ثابت ζ

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المستخلص :
باستخدام المؤثر الخطي $\mathfrak{I}^n(\nu,l)f(z)$، نقدم صنف جديد من الدوال التوافقية الميرومورفية براسب ثابت ζ في $\mathcal{U}_w$، وندرس بعض الخواص مثل الضرب الالتفافي، متباينة المعاملات، مبرهنةالبعدوالنقاط القصوى لدوال تنتمي الى هذا الصنف.