

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



S-Maximal Prime Submodules

Hasan Shakir Wali^{1,*}, Haithab Abood Shahad²

¹Department of Mathematics College of Education for Pure Sciences, Wasit University, Wasit, Iraq, std.2024205.h.wali@uowasit.edu.iq

²Department of Mathematics College of Education for Pure Sciences, Wasit University, Wasit, Iraq, hshahad@uowasit.edu.iq

ARTICLE INFO

Article history:

Received: 21 /05/2026

Revised form: 05/06/2026

Accepted : 09/06/2026

Available online: 30/09/2026

Keywords:

Prime submodules, S-maximal prime submodules, S-maximal semiprime submodules, S-Jacobson radical, Multiplication modules, Compressible factor modules. Annihilator ideals.

ABSTRACT

In this work, we propose a novel extension of the notion of prime submodules for unital modules over commutative rings with identity, called S-maximal prime submodules. A proper submodule N of an R -module M is called S-maximal prime if whenever $rm \in N$, where $r \in R$ and $m \in M$, then either $m \in N + SJ(N)$ or $rM \subseteq N + SJ(N)$, where $SJ(N)$ denotes the S-Jacobson radical of N . This concept extends the classical notion of prime submodules. In particular, We prove that prime submodules are necessarily S-maximal prime submodules, while the converse fails in general. Moreover, if an ideal I of a ring R is an S-maximal prime submodule of the R -module M , then I is called an S-maximal prime ideal of R . Several fundamental properties and characterizations of S-maximal prime submodules are investigated. Furthermore, we introduce the notion of the S-maximal prime radical of submodules and establish some of its basic properties. Special attention is given to multiplication modules, where strong connections between S-maximal prime submodules and compressible factor modules are obtained.

MSC..

<https://doi.org/10.29304/jqcm.2026.18.32919>

1. Introduction

In the present article, we adopt the convention that every ring is a commutative ring possessing an identity element, and every module is a unital R -module. A proper submodule N of an R -module M is defined to be prime if the condition $ry \in N$ (for some $r \in R$ and $y \in M$) implies that either $y \in N$ or $r \in [N : M]$, where the ideal colon is given by $[N : M] = \{ r \in R \mid rM \subseteq N \}$.

This idea serves as a natural extension of prime ideals from classical ring theory. Initially defined and thoroughly analyzed by Dauns [1978], the notion of prime submodules has subsequently drawn considerable interest from various researchers (e.g., [2-8]). Over time, mathematicians have introduced multiple extensions of this concept, including essential prime submodules and e-small prime submodules.

Before proceeding to our main results, it is necessary to establish several core definitions. We denote the Jacobson radical of a module M as $SJ(M)$, which represents the intersection of all maximal submodules of M . Furthermore, a non-zero submodule K of an R -module M is considered essential provided that its intersection with any non-zero submodule S of M is non-trivial (i.e., $K \cap S \neq (0)$). Additionally, we classify an element $y \in M$ as a torsion element whenever $ry = 0$ for some non-zero scalar $r \in R$. An R -module M is identified as a multiplication module if any

*Corresponding author: Hasan Shakir Wali

Email addresses: std.2024205.h.wali@uowasit.edu.iq

Communicated by 'sub etitor'

given submodule $S \subseteq M$ can be expressed as $S = IM$ for a specific ideal I of R , which is equivalent to the formulation $S = [S : M]M$

This manuscript introduces the "S-maximal prime submodule" by incorporating the S-Jacobson radical, $SJ(N)$, into the prime condition. We prove this concept is a strict generalization of classical prime submodules, offering a broader framework for studying modules.

2. S-Maximal Prime Submodules

In this part of the study, we introduce a generalization of prime submodules, termed the S-Maximal Prime Submodules (S-max-PSs). We explore its fundamental properties and provide illustrative characteristics.

Definition 1 For a given ring R and an R -module M , we call a proper submodule $N \subset M$ an S-maximal submodule if it is not strictly contained within any proper semi-essential submodule of M . In other words, any semi-essential submodule W of M that properly contains N (i.e., $N \subsetneq W \subseteq M$) is necessarily equal to M

Definition 2. Suppose let $N \subset M$ is a proper submodule of an R -module M . We term N an S-maximal prime submodule (or simply S-max-PS) whenever the inclusion $rm \in N$, where $r \in R$ and $m \in M$ implies that $m \in N + SJ(N)$ or $rM \subseteq N + SJ(N)$. Herein, $SJ(N)$ represents the intersection over all S-maximal submodules K of M that satisfy $N \subseteq K$. Notably, in the special instance where M is replaced by the ring R itself, N acts as an S-maximal prime ideal of R .

The following findings establish the main characterizations of S-max-PSs.

Proposition 1. A proper submodule N of an R -module M is considered an S-max-PS if and only if the following holds true: for any submodule D of M and any ideal I of the ring R such that $ID \subseteq N$, it must follow that either $D \subseteq N + SJ(N)$ or $I \subseteq [N + SJ(N) : M]$.

Proof. ($\Rightarrow$) Suppose $I \subseteq R$ be an ideal and $D \subseteq M$ be a submodule fulfilling $ID \subseteq N$. We proceed by assuming that $D \not\subseteq N + SJ(N)$. Consequently, an element $d \in D$ exists wherein $d \notin N + SJ(N)$. Knowing that $ID \subseteq N$, it implies $xd \in N$ for each element $x \in I$. Given that N acts as an S-max-PS of M and d is not in $N + SJ(N)$, it forces the condition $x \in [N + SJ(N) : M]$. Therefore, we get $I \subseteq [N + SJ(N) : M]$.

($\Leftarrow$) Conversely, assume there is an element $a \in R$ and an element $m \in M$ with $am \in N$. This implies for the generated principal ideal (a) and the cyclic submodule (m) that $(a)(m) \subseteq N$. Based on our initial hypothesis, we have either $(m) \subseteq N + SJ(N)$ or $(a) \subseteq [N + SJ(N) : M]$. Consequently, it directly follows that $m \in N + SJ(N)$ or $a \in [N + SJ(N) : M]$, finalizing the proof. $\square$

As an immediate consequence of Proposition 1, we obtain the subsequent corollary.

Corollary 1. Let M be an R -module and $N \leq M$. Then N is an S-maximal prime submodule of M if and only if for every $a \in R$ and every submodule $D \leq M$ satisfying $aD \subseteq N$, one has either $D \subseteq N + SJ(N)$ or $a \in [N + SJ(N) : M]$.

Remark 1. Note that All prime submodule of an R -module K is an S-maximal pSs. However, the converse does not necessarily hold, as illustrated in the examples.

Example 1. Suppose $M = \mathbb{Z}_{18}$ as a $\mathbb{Z}$ -module, let $N = \langle 3 \rangle$. Since the set of maximal submodules of $\mathbb{Z}_{18}$ is $\{\langle 2 \rangle, \langle 3 \rangle\}$, we obtain $\{SJ(N) \text{ represents the intersection over all S-maximal submodules of } \mathbb{Z}_{18}, N \subseteq K\} = \langle 6 \rangle$. Now, for any $a \in \mathbb{Z}$ and $y \in \mathbb{Z}_{18}$ with $ay \in N$, it follows that either $y \in N + SJ(N)$ or $a \in [N + SJ(N) : M]$. Thus, N is an S-max-PS of $\mathbb{Z}_{18}$. Furthermore, It follows that N is not a classical prime submodule, as there are elements $r \in \mathbb{Z}$ and $m \in M$ with $rm \in N$, yet $m \notin N$ and $rM \not\subseteq N$. This strictly separates the two concept

Example 2. Suppose $M = \mathbb{Z}_4 \times \mathbb{Z}_2$ and take $N = \mathbb{Z}_4 \times \{0\}$. Then N is a proper submodule of M and satisfies $SJ(N) = N$. For any $a \in \mathbb{Z}$ and $(h, k) \in M$, if $a(h, k) \in N$, then either $(x, y) \in N + SJ(N)$ or $aM \subseteq N$. Thus, N is an S-max-PS of M .

Proposition 2. Assume that M is an R -module and $N \leq M$ with $SJ(N) \subseteq N$. Then N is a prime submodule of M precisely when N is an S-max-PS submodule of M .

Proof. Since $SJ(N) \subseteq N$, Thus $N + SJ(N) = N$. Hence all $m \in M$, $r \in R$,

$$r \cdot m \in N \text{ Then } m \in N \text{ or } rM \subseteq N,$$

which is equivalent to

$$r \cdot m \in N \text{ Then } r \in [N:M] \text{ or } m \in N$$

Thus N is prime if and only if N is an S -max-PS. $\square$

The proposition directly leads to the upcoming corollaries.

Corollary 2. Assume M is an R -module and N is a submodule of M . If the S -Jacobson radical of N is trivial $SJ(N) = 0$, then N qualifies as a PS of M iff it is an S -max-PS of M .

Proposition 3. Let N represent an S -max-PS within an R -module M . Provided that $SJ(N) \subseteq N$, it logically follows that the residual ideal $[N:M]$ functions as an S -max-PI of the ring R .

Proof. This is a direct consequence of Proposition 2. $\square$

The upcoming case illustrates that the reverse implication of Proposition 3 is not universally valid.

Example 3. Let $R = \mathbb{Z}$ and the R -module $M = \mathbb{Z} \oplus \mathbb{Z}$. Take $N = 2\mathbb{Z} \oplus 0$ of M (N is submodule) We then have: $[N:M] = \{r \in \mathbb{Z} \mid rM \subseteq N\} = 0$, which constitutes a prime ideal of R , and is therefore an S -max-PI. The $SJ(N)$ can be expressed as: $SJ(N) = 2\mathbb{Z} \oplus \mathbb{Z}$, leading to the conclusion: $N + SJ(N) = 2\mathbb{Z} \oplus \mathbb{Z}$. By selecting $r = 2$ and $m = (1, 0)$, we find that $r \cdot m = (2, 0) \in N$. Nevertheless, $m \notin N + SJ(N)$ and $r \cdot M = 2\mathbb{Z} \oplus 2\mathbb{Z} \not\subseteq N + SJ(N)$. As a result, N fails to be an S -max-PS of M . This demonstrates that the converse statement of Proposition 3 is not valid in general

Proposition 4. Suppose M is a multiplication non-singular R -module, and let N be a Pr- S satisfying $SJ(N) \subseteq N$. Under these conditions, N is an S -max-PS of M is equivalent to the ideal $[N:M]$ is an S -max-PI from R .

Proof. $(\Rightarrow)$ This direction is an immediate consequence of Proposition 3.

$(\Leftarrow)$ Assume there exist $r \in R$ and $m \in M$ satisfying $rm \in N$. This implies that $r(m) \subseteq N$. Given that M is a multiplication module, we can find an ideal I in R yielding $(m) = I \cdot M$. Consequently, $rIM \subseteq N$, which leads to $rI \subseteq [N:M]$.

Knowing that $[N:M]$ serves as an S -max-PI of the ring R , we deduce that either $I \subseteq [N:M]$ or $r \in [N:M]$. As a result, we have two possibilities:

$$IM \subseteq N \subseteq N + SJ(N)$$

or

$$rM \subseteq N \subseteq N + SJ(N).$$

Therefore, it must be the case Thus $m \in N + SJ(N)$ or $r \in [N + SJ(N):M]$. This confirms that N is indeed an S -max-PS of M . $\square$

Proposition 5. Assume M is an R -module containing a submodule N . If the residual ideal $[N + SJ(N):M]$ forms a maximal ideal within the ring R , then N is classified as an S -max-PS of M .

Proof. Assume there exist $s \in R$ and $m \in M$ such that $sm \in N$, while $s \notin [N + SJ(N):M]$. By the hypothesis that $[N + SJ(N):M]$ is maximal in R , we can express the ring as $R = \langle s \rangle + [N + SJ(N):M]$, where $\langle s \rangle$ denotes the principal ideal generated by s . Therefore, we can find elements $r \in R$ and $x \in [N + SJ(N):M]$ satisfying the equation $1 = rs + x$. Multiplying by m yields $m = rsm + xm$. Since $sm \in N \subseteq N + SJ(N)$ and $xm \in N + SJ(N)$, we conclude that $m \in N + SJ(N)$. This verifies that N is an S -max-PS of M . $\square$

Remark 2. It is a recognized mathematical fact that if we take a non-trivial multiplication R -module M and a prime submodule N within it, then any submodule D of M that strictly contains N ($N \subsetneq D$) will satisfy the condition $[D:M] \not\subseteq [N:M]$.

Corollary 3. Consider a multiplication module M over a ring R , and let N act as a Pr- S of M . Assume that the residual $[N + SJ(N):M]$ constitutes a prime ideal in R , and furthermore, that $N + SJ(N) \subseteq D$ holds for all submodule D of M . Under these conditions, N serves as an S -max-PS of M .

Proposition 6. Suppose M is an R -module containing submodules N and D such that $N \subseteq D$. Provided that N acts as an S -max-PS of M and the relation $SJ_D(N) = SJ_M(N)$ holds, it implies that N is equivalently an S -max-PS of the module D .

Proof. Assume $ad \in N$ for some element $a \in R$ and $d \in D$. Because N is established as an S -max-PS of M , we must have either $d \in N + SJ_M(N)$ or $aM \subseteq N + SJ_M(N)$. If the first condition holds, meaning $d \in N + SJ_M(N) = N + SJ_D(N)$, the proof is complete. In the alternative case, we observe that

$$aD \subseteq aM \subseteq N + SJ_M(N) = N + SJ_D(N).$$

Thus, N is indeed an S -max-PS of D . $\square$

Proposition 7 . Let M be an R -module containing a submodule N for which $[N + SJ(N):M]$ constitutes a prime ideal in the ring R . Under these circumstances, N is considered an S -max-PS from M if and only if the equality $T^{-1}N = T^{-1}(N + SJ(N))$ is valid for any multiplicatively closed subset T of R satisfying $T \cap [N + SJ(N):M] = \emptyset$.

Proof. $(\Rightarrow)$ Let T be a multiplicatively closed subset of R such that $T \cap [N + SJ(N):M] = \emptyset$, and let $m \in M$ with $\frac{m}{1} \in T^{-1}M$. If $\frac{m}{1} \in T^{-1}N$, thus $tm \in N$ for a number $t \in T$. because N is an S -max-PS of M and $t \notin [N + SJ(N):M]$, it follows that $m \in N + SJ(N)$. Hence $\frac{m}{1} \in T^{-1}(N + SJ(N))$.

$(\Leftarrow)$ Suppose that $rm \in N$ with $r \notin [N + SJ(N):M]$. Let $T = \{1, r, r^2, \dots\}$, which is multiplicatively closed and disjoint from $[N + SJ(N):M]$. Thus $\frac{m}{1} \in T^{-1}N = T^{-1}(N + SJ(N))$, and $m \in N + SJ(N)$. This N is an S -max-PS of M . $\square$

Proposition 8. Given all finitely generated non-singular multiplication R -module M with submodules Y and N , if I is a max-PI of R satisfying $\text{ann}(Y) \subseteq I$, then IY is necessarily an S -max-PS of M .

Proof. Assume $am \in IY$ there exists scalar $a \in R$ and element $m \in M$. Given the premise this M is a multiplication module, we can identify an ideal J within R that satisfies $(m) = JY$. Consequently, we have

$$aJY \subseteq IY,$$

leading to the relation

$$aJ \subseteq I + \text{ann}(Y) = I.$$

Knowing that I is a maximal prime ideal of R , it must be true this either $J \subseteq I$ or $a \in I$

If $a \in I$, it follows that $aM \subseteq IY \subseteq IY + SJ(IY)$.

Alternatively, if $J \subseteq I$, we obtain $m \in JY \subseteq IY \subseteq IY + SJ(IY)$.

In both scenarios, we conclude that either $m \in IY + SJ(IY)$ or $aM \subseteq IY + SJ(IY)$. Thus, IY is proven to be an S -max-PS of M . $\square$

Corollary 4. Consider a multiplication R -module M , let N be a Pr- S of M satisfying $SJ(N)$ subset of N . The submodule N acts as an S -max-PS of M if and only if the quotient module M/N is a compressible R -module.

Proposition 9. Assume M is an R -module, and let N be a Pr- S of M . Suppose that for any submodule D of M strictly containing N ($N \subsetneq D$), the equality $[N + SJ(N):M] = [N + SJ(N):D]$ holds, along with $N + SJ(N) \subseteq N$. Then N is classified as an S -max-PS of M .

Proof. Assume there exists an element $ay \in N$, with $a \in R$ and $m \in M$, with $y \notin N + SJ(N)$. Let us define a new submodule $D = N + SJ(N) + \langle y \rangle$. It is clear that $N + SJ(N)$ is strictly contained in $N + SJ(N) + \langle y \rangle$, which ensures $y \in D$.

Furthermore, we can deduce that $a \in [N:N + SJ(N) + \langle y \rangle]$. This subset is naturally contained within $[N + SJ(N):N + SJ(N) + \langle m \rangle]$, which by our hypothesis equals $[N + SJ(N):M]$.

Consequently, we have $a \in [N + SJ(N):M]$. This confirms that N satisfies the condition to be an S -max-PS of M . $\square$

Remark 3. The intersection of two S -max-PSs of an R -Mod M need not be a S -max-PS of M , as demonstrated by the upcoming example.

Example 4. Let $M = \mathbb{Z}$, $R = \mathbb{Z}$, $N = 2\mathbb{Z}$ and $D = 3\mathbb{Z}$. Thus N and D are S -max-PSs of M , but $N \cap D = 6\mathbb{Z}$ is not a S -max-PS of M because $2 \cdot 3 \in 6\mathbb{Z}$, $2 \in R$, $3 \in \mathbb{Z}$, but $3 \notin (N \cap D) + SJ(N)$ and $2 \notin [(N \cap D) + SJ(N):M] = 6\mathbb{Z}$.

Proposition 10. Let M be an R -module, and let N and D be two S -max-PSs of M with $SJ(N) \subseteq D$ and $D \subseteq SJ(N)$. Then $N \cap D$ is a S -max-PS of M .

Proof. Suppose $ay \in N \cap D$, with $m \in M$, $a \in R$. Thus $am \in N$ and $am \in D$. Since N and D are S -max-PSs of M , either $m \in N + SJ(N)$ or $aM \subseteq N + SJ(N)$, and either $m \in D + SJ(N)$ or $aM \subseteq D + SJ(N)$. But $N \subseteq SJ(N)$ and $D \subseteq SJ(N)$, so either $m \in SJ(N)$ or $aM \subseteq SJ(N)$ and either $m \in SJ(D)$ or $aM \subseteq SJ(D)$. Hence either $m \in (N \cap D) + SJ(N)$ or $aM \subseteq (N \cap D) + SJ(N \cap D)$. Therefore $N \cap D$ is a S -max-PS of M . $\square$

Proposition 11. Assume M be an R -module and Suppose N, D are submodules from M with $SJ(N) \subseteq D$ and $D \not\subseteq N$. If N is an S -max-PS of M , then $N \cap D$ is an S -max-PS from D .

Proof. Since $D \not\subseteq N$, it follows that $N \cap D$ is a proper submodule of D . Let $a \in R$ and $y \in D$ such that $ay \in N \cap D$. Then $ay \in N$ and $ay \in D$.

because N is an S -max-PS of M , we conclude either

$$y \in N + SJ(N) \quad \text{or} \quad aM \subseteq N + SJ(N).$$

Since $y \in D$ and $ay \in D$, it follows that either

$$y \in (N + SJ(N)) \cap D \quad \text{or} \quad aD \subseteq (N + SJ(N)) \cap D.$$

As $SJ(N) \subseteq D$, by the modular law we obtain

$$D \cap (N + SJ(N)) = (SJ(N) \cap D) + (N \cap D).$$

By [13, Cor. 9.9], we have $SJ(D) = D \cap SJ(N)$. Hence either

$$y \in (D \cap N) + SJ(D) \quad \text{or} \quad aD \subseteq (D \cap N) + SJ(D).$$

Therefore $N \cap D$ is an S -max-PS of D . $\square$

Example 5. Assume $R = \mathbb{Z}$ and $M = \mathbb{Z}$ as an R -module. Consider the prime submodules $D_1 = 2\mathbb{Z}$ and $D_2 = 3\mathbb{Z}$ of M . Then $N = D_1 \cap D_2 = 6\mathbb{Z}$. Take $a = 2$ and $y = 3$. Then $ay = 6 \in N$, but $y \notin N$ and $aM = 2\mathbb{Z} \not\subseteq N$. Since $SJ(N) = 0$, we have $N + SJ(N) = N$. Hence N is not an S -max-PS of M . This shows that the intersection of prime submodules need not be an S -max-PS.

Proposition 12 Assume $f \in \text{Hom}(M, M')$ be an onto R -morphism, Assume N be an S -max-PS from M' . Thus $f^{-1}(N)$ is an S -max-PS of M .

Proof. First, we show that $f^{-1}(N)$ is a proper submodule of M . Since N is proper in M' and f is onto, and from this $f^{-1}(N) \neq M$.

Now Assume $m \in M$ and $a \in R$ with $am \in f^{-1}(N)$. Then $f(am) = af(m) \in N$. because N is an S -max-PS of M' , we have either

$$f(m) \in SJ(N) + N \quad \text{or} \quad aM' \subseteq SJ(N) + N$$

If $f(m) \in N + SJ(N)$, then

$$m \in f^{-1}(N) + f^{-1}(SJ(N)).$$

By [15, Thm. (1.4)(a)], we have

$$f^{-1}(SJ(N)) \subseteq SJ(f^{-1}(N)).$$

Hence

$$m \in f^{-1}(N) + SJ(f^{-1}(N)).$$

If $aM' \subseteq N + SJ(N)$, then since f is onto, we obtain

$$af(M) \subseteq N + SJ(N),$$

and hence

$$aM \subseteq f^{-1}(N) + f^{-1}(SJ(N)) \subseteq f^{-1}(N) + SJ(f^{-1}(N)).$$

Therefore $f^{-1}(N)$ is an S -max-PS of M . $\square$

Proposition 13. Assume we have an epimorphism f from an R -module M onto an R -module M' . If N serves as an S -max-PS within M and contains the kernel of f ($\ker f \subseteq N$), then its image $f(N)$ constitutes an S -max-PS in M' .

Proof. To begin, we must verify that $f(N)$ strictly sits inside M' . Assume the opposite, namely $f(N) = M'$. For any element $y \in M$, its image $f(y)$ belongs to $M' = f(N)$, meaning we can find some $x \in N$ satisfying $f(y) = f(x)$. This gives $f(y - x) = 0$, leading to $y - x \in \ker f \subseteq N$. Consequently, $y \in N$, which forces $M = N$. This contradicts the assumption that N is a proper submodule, confirming that $f(N) \subsetneq M'$.

Next, suppose $am' \in f(N)$ for some scalar $a \in R$ and $m' \in M'$. Because f is surjective, choose $y \in M$ with $f(y) = m'$. It follows that $am' = af(y) = f(ay) \in f(N)$. Thus, an element $x \in N$ exists where $f(x) = f(ay)$, yielding $f(x - ay) = 0$. This implies $x - ay \in \ker f \subseteq N$, from which we deduce $ay \in N$.

Knowing that N is an S -max-PS of M , we face two possibilities: either $y \in N + SJ(N)$ or $aM \subseteq N + SJ(N)$. Mapping these conditions through f , we get:

$$f(y) \in f(SJ(N)) + f(N) \quad \text{or} \quad aM' \subseteq f(SJ(N)) + f(N).$$

According to [15, Thm. (1.4)(a)], the inclusion $f(SJ(N)) \subseteq SJ(f(N))$ holds. Therefore, the previous relations become:

$$m' \in f(N) + SJ(f(N)) \quad \text{or} \quad aM' \subseteq f(N) + SJ(f(N)).$$

This concludes the proof that $f(N)$ is an S -max-PS of M' . $\square$

Corollary 5. Consider an R -module M equipped with submodules D and N satisfying $D \subseteq N \subsetneq M$. Provided that $D \subseteq SJ(N)$, the submodule N qualifies as an S -max-PS of M exactly when the quotient N/D is an S -max-PS of the factor module M/D .

Proposition 14. Assume M be an R -module, and let N, D , and F be submodules of M such that F is an S -max-PS of M . If $N \cap D \subseteq F$ and the residual condition $[N:M] \not\subseteq [F + SJ(N):M]$ holds, then $D \subseteq F + SJ(N)$.

Proof. Given the non-inclusion $[N:M] \not\subseteq [F + SJ(N):M]$, we can pick an element $a \in [N:M]$ that does not belong to $[F + SJ(N):M]$. For any arbitrary $x \in D$, we clearly have $ax \in D$. Furthermore, since $a \in [N:M]$, it implies $aM \subseteq N$, resulting in $ax \in N$. Combining these facts yields $ax \in N \cap D \subseteq F$.

Knowing that F is an S -max-PS of M and $a \notin [F + SJ(N):M]$ (which means $aM \not\subseteq F + SJ(N)$), it necessitates that $x \in F + SJ(N)$. Consequently, we conclude that $D \subseteq F + SJ(N)$. $\square$

Conclusion

In this research, the S -Maximal PS is studied and presented as a new generalization of PSs. The following is a summary of the study's primary findings:

In the present study, we introduced and systematically investigated The notion of an S -maximal prime submodule (S -max-PS) as a substantial generalization of the classical notion of a prime submodule. By incorporating the S -Jacobson radical $SJ(N)$ into the defining condition, the new concept captures a broader class of submodules while preserving a structural essence similar to that of prime submodules. Our study established fundamental characterizations and properties of S -max-PSs. Specifically, we proved equivalent conditions involving ideals and submodules (Proposition 1 and its corollary), clarified the relationship between prime submodules and S -max-PSs (Proposition 2), and examined the behavior of the residual $[N:M]$ (Propositions 3, 4, and 5). Furthermore, we provided several Conditions guaranteeing a submodule to be an S -max-PS, particularly in the context of multiplication modules (Propositions 5, 6, and 8). The interaction of S -max-PSs with standard module operations—such as intersection (Propositions 10, and 11), pre-images, and homomorphic images (Propositions 13 and 14)—was also analyzed, revealing both preserving and non-preserving behaviors, as illustrated by explicit counterexamples. The results demonstrate that the theory of S -maximal prime submodules enriches the structure theory of modules, offering a more flexible framework that applies to a wider range of contexts, especially when the

Jacobson radical plays a non-trivial role. Future research may focus on further exploring the properties of S -max-PSs in specific classes of modules (such as injective, projective, or Artinian modules), investigating their associated spectral topologies, and developing localization techniques tailored to this new concept. Additionally, potential applications in related areas, such as module theory over non-commutative rings or in homological algebra, present promising avenues for further exploration.

References

- [1] Dauns, J. "Prime Modules." *Journal für die reine und angewandte Mathematik*, 2 (1978): 156-181.
- [2] Lu, C. P. "Prime submodules of Modules." *Commentarii Mathematici Universitatis Sancti Pauli*, 33 (1984): 61-69.
- [3] Shabad, A. H., and Al-Mothafar, N. S. "E-small prime sub-modules and e-small prime modules." *AIP Conference Proceedings*, 2023.
- [4] Shabad, A. H., and Al-Mothafar, N. S. "On essential-prime sub-modules." *Journal of Discrete Mathematical Sciences and Cryptography*, 24.7 (2021): 1973-1977.
- [5] McCasland, R. L., and Smith, P. F. "Prime submodules of Noetherian Modules." *Rocky Mountain Journal of Mathematics*, 23.3 (1993): 1041-1062.
- [6] McCasland, R. L., Moore, M. C., and Smith, P. F. "On spectrum of Modules over Commutative rings." *Communications in Algebra*, 25.1 (1997): 79-103.
- [7] Moore, M. E., and Smith, S. J. "Prime and Radical submodules over Commutative rings." *Communications in Algebra*, 30.10 (2002): 5037-5064.
- [8] Ebrahimi, S. A., and Esmaceli, K. S. "On McCasland submodules." *International Mathematical Forum*, 2.46 (2007): 2255-2260.
- [9] Hussin, S. A. "WE-Prime submodules and WE-semi-prime submodules." *Ibn AL-Haitham Journal for Pure and Applied Science*, 3.3 (2018): 109-117.
- [10] Hussin, W. A. "WN-2-Absorbing and WES-2-Absorbing submodules." *Ibn AL-Haitham Journal for Pure and Applied Science*, 31.3 (2018): 118-125.
- [11] Goodearl, K. R. *Ring Theory: Nonsingular Rings and Modules*. Marcel Dekker, New York, 1976.
- [12] Abd El-Bast, Z., and Smith, P. F. "Multiplication Modules." *Communications in Algebra*, 16.4 (1988): 755-779.
- [13] Larsen, M. D., and McCarthy, P. J. *Multiplicative Theory of Ideals*. Academic Press, New York and London, 1971.
- [14] Zelmanowitz, J. "Dense Rings of Linear transformations." In *Ring Theory II*, Marcel Dekker, New York, 1997.
- [15] Anderson, F. W., and Fuller, K. R. *Rings and Categories of Modules*. Springer-Verlag, New York, 1992.
- [16] Athab, E. A. *Prime and Semi prime Submodules*. M.Sc. Thesis, College of Science, University of Baghdad, 1996.
- [17] Kasch, F. *Modules and Rings*. London Mathematical Society Monographs, Academic Press, New York, 1982.