

A Stochastic Cancer Model for Investigating the Dynamical Effects of a Viral Injection Therapy

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ABSTRACT

This paper presents a stochastic model for tumor-viral-immune interactions, focusing on uninfected tumor cells, virus-infected cells, and cytotoxic T lymphocytes (CTLs) under environmental uncertainties. It incorporates logistic tumor growth, viral infection saturation, and immune-mediated clearance through stochastic differential equations, ensuring positive solutions and trajectory non-explosion. Lyapunov functions confirm boundedness and biological feasibility. The model highlights the importance of the interaction between viral infection and immune activation in tumor control, where effective immune stimulation limits tumor growth, while decay mechanisms promote persistence. Randomness affects population dynamics and therapeutic outcomes, reinforcing the model's relevance in cancer virotherapy and paving the way for future studies on stability, extinction criteria, and treatment strategies.

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1. Introduction

Viruses have been explored as anticancer agents since the late nineteenth century, with historical reports in oncolytic virotherapy indicating that some cancer patients experienced tumor regression following viral infections [1]. Viral therapy has garnered significant interest for cancer treatment, as researchers seek to enhance the use of disease-causing viruses as medications. These viruses can effectively target and eliminate tumor tissues, which are more resilient to damage compared to normal tissues. Despite their toxicity, advancements in genetic engineering now allow for the modification of many viruses to reduce pathogenicity while maintaining their ability to destroy cancer cells [2]. Understanding the molecular mechanics and kinetics of oncolytic viral cytotoxicity has received a lot of attention lately. These programs provided a promising alternative therapy approach to help cancer patients recover. However, both the immune response and the virus-cancer interaction have complex effects on virotherapy outcomes [3-7]. Most of the existing cancer medicines were developed through empirical research [7]. Several mathematical models have recently been developed to describe the outcome of these interactions. [9-15]. Additional models and techniques are being developed to study the dynamics of virotherapy [16-22].

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Population growth is influenced by spatial components and local relationships. Traditional mathematical frameworks like diffusion-reaction and Lotka-Volterra models are commonly employed in virotherapy computational models, typically assuming closely mixed populations. However, experimental evidence supporting these modeling techniques is lacking [12]. To address this problem, an in vitro computer model has been created that may clearly illustrate the interactions between viral and tumor populations [3,10]. The behavior of the solution, the stability of rest points, and the qualitative features of the model from [3,10] are analyzed in [23]. From a mathematical and clinical perspective, the model is noteworthy due to its significant properties. In [24], a stable tumor-free equilibrium state is revealed under specific conditions by the study's stochastic model for the growth of malignant malignancies with targeted treatment.

This work addresses the lack of stochastic models in tumor-virus-immune interaction literature, specifically those that include saturated viral infection, CTL-mediated responses, and density-dependent fluctuations. Current studies mainly focus on local stability, neglecting a comprehensive analysis of global existence and dynamics. This research develops a stochastic tumor-virus-immune model to fill these gaps and enhance the understanding of cancer virotherapy under uncertainty.

2. PERLIMINARIES

This section of the research is dedicated to mentioning some basic concepts related to stochastic differential equations, which lay the foundation for the objectives of this research.

Assume that a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is filtered by $\{\mathcal{F}_t\}_{t \geq 0}$, and $W(t) = (W_1(t), \dots, W_m(t))^T$, $t \geq 0$ be an m –dimensional Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f: \mathbb{R}^d \times [t_0, T] \rightarrow \mathbb{R}^d$ and $g: \mathbb{R}^d \times [t_0, T] \rightarrow \mathbb{R}^{d \times m}$ be two Boral measurable functions. Let

$$dx(t) = f(x(t), t)dt + g(x(t), t)dW(t) \text{ on } t_0 \leq t \leq T \quad (1)$$

be the d –dimensional Itô stochastic differential equation with an initial value $x(t_0) = x_0$, where $x_0: \Omega \rightarrow \mathbb{R}^d$ be an $\mathcal{F}_{t_0}$ -measurable with $\mathbb{E}|x_0|^2 < \infty$. Equation (1) is equivalent to the equation:

$$x(t) = x_0 + \int_{t_0}^t f(x(\tau), \tau)d\tau + \int_{t_0}^t g(x(\tau), \tau)dW(\tau) \text{ on } t_0 \leq t \leq T \quad (2)$$

See [25].

Definition 2.1 [25]: The stochastic process $\{x(t)\}_{t_0 \leq t \leq T}$, where $x(t) \in \mathbb{R}^d$ is called the solution of the equation (1) if

- (a) $\{x(t)\}$ is $\mathcal{F}_t$ - adapted and continuous;
- (b) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{d \times m})$;
- (c) equation (2) holds for every $t \in [t_0, T]$ with full measure.

Proposition 2.2 [25,26]: The formula $x_t = \frac{\exp\left\{\left(rK - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}}{x^{-1} + r \int_0^t \exp\left\{\left(rK - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau\right\}d\tau}$; $t \geq 0$. form a solution of the SDE

$$dx_t = rx_t(K - x_t)dt + \sigma x_t dW_t.$$

Definition 2.3[25] Let $\mathcal{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$ be the class of all functions $\mathcal{V}: \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ possessing the characteristic of being continuously twice differentiable in x and once in t . For $\mathcal{V} \in \mathcal{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$, define

$$\mathcal{V}_t = \frac{\partial \mathcal{V}}{\partial t}, \quad \mathcal{V}_t = \left(\frac{\partial \mathcal{V}}{\partial x_1}, \quad \dots, \quad \frac{\partial \mathcal{V}}{\partial x_d} \right), \quad \mathcal{V}_{xx} = \left(\frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j} \right)_{d \times d} = \begin{pmatrix} \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{V}}{\partial x_d \partial x_1} & \dots & \frac{\partial^2 \mathcal{V}}{\partial x_d \partial x_d} \end{pmatrix}.$$

Let $\mathcal{V} \in \mathcal{C}^{2,1}(\mathbb{S}_h \times \mathbb{R}^+; \mathbb{R}^+)$, where $0 < h \leq \infty$. Define the differential operator

$$\mathcal{L} := \frac{\partial}{\partial t} + \sum_{i=1}^d \tilde{f}_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d [g(x, t)g^T(x, t)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad (3)$$

associated with (2). If $\mathcal{L}$ acts on $\mathcal{V} \in \mathcal{C}^{2,1}(\mathbb{S}_h \times \mathbb{R}^+; \mathbb{R})$, then

$$\mathcal{L}\mathcal{V} = \mathcal{V}_t(x, t) + \mathcal{V}_x(x, t)f(x, t) + \frac{1}{2} \text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)]. \quad (4)$$

Theorem 2.4 [25]: If there is $\mathcal{V}(x, t) \in \mathcal{C}(\mathbb{S}_h \times [t, \infty); \mathbb{R}^+)$, with $\mathcal{V}(x, t) > 0$ and $\mathcal{L}\mathcal{V}(x, t) \leq 0$, for all $(x, t) \in \mathbb{S}_h \times [t, \infty)$, then the trivial solution of (2) is stochastic stable.

Lemma 2.5 [27, 28]: If $x: [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ is a stochastic process fulfill

$$\mathbb{E}|x(t) - x(s)|^{\alpha_1} \leq c|t - s|^{1+\alpha_2}, \quad 0 \leq s, t < \infty, \quad (5)$$

where $\alpha_1, \alpha_2, c > 0$ and there is a continuous modification $\tilde{x}(t)$ of $x(t)$ so that, $\forall v \in (0, \alpha_2/\alpha_1), \exists \psi: \Omega \rightarrow \mathbb{R}^+/\{0\}$ which is measurable with

$$\mathbb{P} \left\{ \omega: \sup_{0 < |t-s| < \psi(\omega), 0 \leq s, t < \infty} \frac{|\tilde{x}(t, \omega) - \tilde{x}(s, \omega)|}{|t-s|^v} \leq \frac{2}{1-2^{-v}} \right\} = 1. \quad (6)$$

Definition 2.6 [29]: The solution $x(t)$ of (1) has **stochastically ultimately bounded** (SUB) property, if, for every $\varepsilon \in (0, 1)$, there exists $\delta = \delta(\varepsilon) > 0$, so that for every $x_0 \in \mathbb{R}_+^3$, we have

$$\lim_{t \rightarrow \infty} \sup \mathbb{P}\{|x(t)| > \delta\} < \varepsilon. \quad (7)$$

Definition 2.7 [29] The solution $x(t)$ of (1) has stochastic permanent property if there is a couple of real numbers $\varphi(v) > 0$ and $X = X(v) > 0, v \in (0, 1)$ so that for any $X_0 \in \mathbb{R}_+^3$, we have

$$\lim_{t \rightarrow \infty} \inf \mathbb{P}\{|x(t)| \geq \varphi\} \geq 1 - v, \quad \lim_{t \rightarrow \infty} \inf \mathbb{P}\{|x(t)| \leq \chi\} \geq 1 - v. \quad (8)$$

Theorem 2.8 (Burkholder-Davis-Gundy Inequality) [31]: Let M be a continuous local martingale with $M_0 = 0$. Then for every stopping time T and $p > 0$, there exists constants c_p and C_p such that

$$c_p \cdot \mathbb{E} \left(\langle M \rangle_T^{\frac{p}{2}} \right) \leq \mathbb{E}(\sup_{t \leq T} |M_t|^p) \leq C_p \cdot \mathbb{E} \left(\langle M \rangle_T^{\frac{p}{2}} \right).$$

3. Cancer Stochastic Model

The model consists of three variables: virus-free tumor cells biomass as x , virus-infected tumour cells biomass as y and virus-specific CTL biomass as z . The virus-free tumour cells grow logistically at a rate of r and δ_x is the die a rate. The tumour is permitted to enlarge up to the maximum space k , known as the carrying capacity. The injected virus into the patient's body spreads in the tumor cells at a rate of β , which can be viewed as summarizing the virus replication rate. The injected virus eliminates the virus-infected tumour cells at a rate of a , which can be viewed as the viral cytotoxicity of the proposed cancer model. Also, the virus-infected tumor cells are growing logistically at a rate of s . It assumes that both the daughter cells are carrying the virus upon the division of virus-infected tumour cells. Certainly, this would be the case for a virus that mixes into the tumor cell genome. But for a non-mixing virus, there should be a high possibility of virus transmission to the daughter cells upon cell division to justify the assumption. The CTL kills virus-infected tumor cells at a rate of γ . The virus-specific CTL increases in reaction to antigen at a rate of δ and decays at a rate of b . Then the proposed system is specified by the following nonlinear differential equations:

$$\frac{dx}{dt} = rx \left(1 - \frac{x+y}{k}\right) - \beta xy - \delta_x x, \quad \frac{dy}{dt} = ry \left(1 - \frac{x+y}{k}\right) - \beta xy - ay - \gamma yz, \quad \frac{dz}{dt} = \delta yz - bz, \quad (9)$$

with initial conditions $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. The same model is studied in [23] with a different eco-epidemiological point of view. Here, we have analyzed the model using completely different mathematical approaches with some new eco-epidemiological outcomes, which may help the recent cancer model research.

We will construct the stochastic model (as we do in [30]) by making significant modifications to the model mentioned in [23] through we use the term saturated infection $\frac{\beta xy}{1+\eta y}$ since this gives crowding effect, viral saturation and therapy limitation and it is much more realistic. Thus, the deterministic model (9) become

$$\begin{aligned} dx &= \left[rx \left(1 - \frac{x+y}{k}\right) - \frac{\beta xy}{1+\eta y} - \delta_x x \right] dt + \sigma_1 x \left(1 - \frac{x+y}{k}\right) dW_1(t) \\ dy &= \left[sy \left(1 - \frac{x+y}{k}\right) + \frac{\beta xy}{1+\eta y} - ay - \gamma yz \right] dt + \sigma_2 y \left(1 - \frac{x+y}{k}\right) dW_2(t) \\ dz &= [\delta yz - bz] dt + \sigma_3 z dW_3(t) \end{aligned} \quad (10)$$

where η represent recovery/ removal and the initial conditions are: $x(0) = x_0 > 0, y(0) = y_0 > 0$ and $z(0) = z_0 > 0$.

3.1 The Biological Interpretation of The Model Variables

a. Uninfected tumor cells $x(t)$: The biomass of healthy cancer cells not infected by the virus. The biological behavior can be described as follows: It grows logistically at a rate of $rx(1 - (x + y)/k)$. This means that the tumor increases when it is small, but the growth slows down as the total size $(x + y)$ approaches the carrying capacity k . They die naturally at a rate of δx . It represents natural cell death or the effect of other treatments. You get infected with the virus at a rate of $-\beta xy/(1 + \eta y)$, where: β is the efficiency of virus transmission and η the

removal/recovery coefficient reduces the effectiveness of the infection when the number of infected cells increases. Biologically, this threshold say that the viral infection becomes "saturated" when the number of infected cells increases.

b. Infected tumor cells $y(t)$: They represent the cancerous cells that have been infected by the therapeutic virus. The biological behavior can be described as follows: It also grows logistically $sy(1 - (x + y)/k)$. That is, the infected cells can divide and survive temporarily. It increases due to the damage to healthy cells: $+\beta xy/(1 + \eta y)$ killed by the virus itself $-ay$ where a represents viral cytotoxicity. It is killed by immune cells CTLs $-\gamma y$, and this represents the immune response against infected cells.

c. Cytotoxic T Lymphocytes CTLs $z(t)$: Cytotoxic T Lymphocytes (CTLs) represent the specialized T cells that fight the virus and infected cells. The biological behavior can be described as follows It activates in response to antigens present in infected cells δyz . This means that the immune system takes time to recognize the infection and activate the response.

d. The Biological Interpretation of Random Noise: The terms $\sigma_1 dW_1(t)$, $\sigma_2 dW_2(t)$, $\sigma_3 dW_3(t)$ represent the random biological fluctuations, for example: Differences in immune response among patients. Irregular spread of the virus. Tumor microenvironment change. Genetic mutations. Fluctuation in cytokines and immunity.

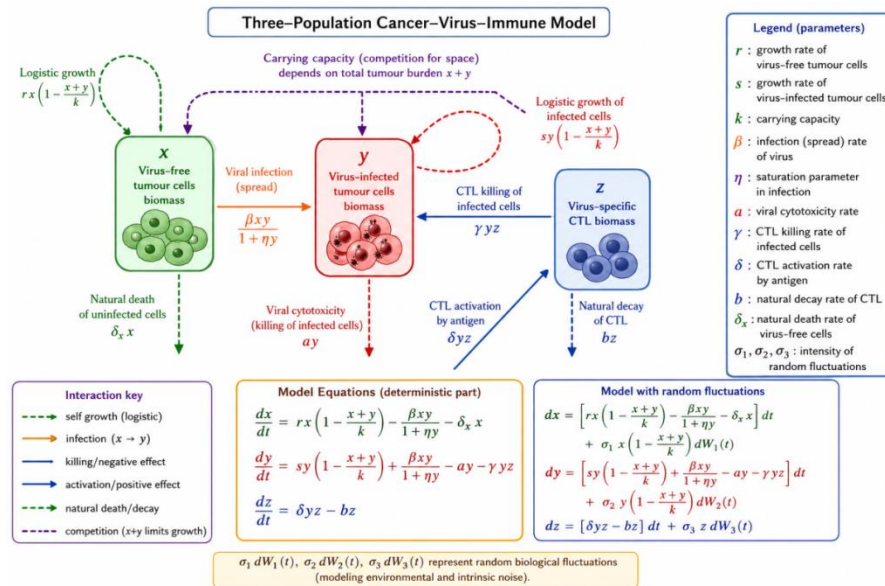
Biologically, cancer is not a completely deterministic system, but rather a highly heterogeneous stochastic system.

e. The general dynamic interpretation of the system: The system describes three simultaneous conflicts:

- i. Tumor growth.
- ii. The spread of the virus within the tumor.
- iii. Immune response.

Complex behaviors may emerge, such as: Tumor Eradication, Immune Escape, Viral Persistence, Oscillatory Tumor Dynamics, Noise-induced Remission and Metastable Immune states.

The biological diagram of the system interaction is shown below:



4. The Biological Feasibility of Proposed Cancer Model

This section focuses on studying the existence, uniqueness, positivity and boundedness of solutions of the proposed cancer model.

Let $X = (x, y, z)^T$ and $\mathbf{f} = (f_1, f_2, f_3)^T$, then the system (9) can be rewritten as $\dot{X} = \mathbf{f}(X)$. Here $f_1 = rx \left(1 - \frac{x+y}{k}\right) - \frac{\beta xy}{1+\eta y} - \delta_x x$, $f_2 = ry \left(1 - \frac{x+y}{k}\right) + \frac{\beta xy}{1+\eta y} - ay - \gamma yz$ and $f_3 = \delta yz - bz$.

Clearly the vector function $\mathbf{f}$ is a smooth function of the variables (x, y, z) in the positive octant $\mathbb{R}_+^3 = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$. So, the local existence and uniqueness of the solution of (9) hold. Also, using the initial conditions and integrating with respect to t , it can be easily verified that all three system variables remain always positive.

Theorem 4.1 (Positivity and Ultimate Boundedness) Consider the stochastic system (10)

with nonnegative initial data $(x(0), y(0), z(0)) \in \mathbb{R}_+^3$. Assume that $r > \frac{\sigma_1^2}{2}$, $s > \frac{\sigma_2^2}{2}$ and

$\delta k \left(1 - \frac{\sigma_2^2}{2s}\right) < b$. Then

1. The system admits a unique global positive solution $(x(t), y(t), z(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$ a.s.

2. The solution is ultimately bounded almost surely, and satisfies

$$\lim_{t \rightarrow \infty} \sup x(t) \leq k \left(1 - \frac{\sigma_1^2}{2r}\right) \text{ a.s., and } \lim_{t \rightarrow \infty} \sup y(t) \leq k \left(1 - \frac{\sigma_2^2}{2s}\right) \text{ a.s.}$$

3. Moreover, $\lim_{t \rightarrow \infty} \sup z(t) < \infty$ a.s.

Consequently, the region $\Delta := \{(x, y, z) \in \mathbb{R}_+^3 : x \leq \lambda_1, y \leq \lambda_2\}$, where $\lambda_1 := k \left(1 - \frac{\sigma_1^2}{2r}\right)$ and $\lambda_2 := k \left(1 - \frac{\sigma_2^2}{2s}\right)$, is an absorbing region of the stochastic system.

Proof First, we need to show the positivity of solutions. Consider the stochastic system (10). The drift and diffusion coefficients are locally Lipschitz in $\mathbb{R}_+^3$, hence there exists a unique local solution up to the explosion time τ . We show that none of the components can become negative.

To show the positivity of $x(t)$, define the stopping time $\tau_x := \inf\{t > 0 : x(t) = 0\}$.

For, $t < \tau_x$, $x(t) > 0$. Applying Itô's formula to $V_1(x) = \ln x$ gives

$$d(\ln x) = \frac{1}{x} dx - \frac{1}{2} \frac{1}{x^2} (dx)^2.$$

Substituting,

$$\begin{aligned} d(\ln x) = & \left[r \left(1 - \frac{x+y}{k}\right) - \frac{\beta y}{1+\eta y} - \delta_x - \frac{1}{2} \sigma_1^2 \left(1 - \frac{x+y}{k}\right)^2 \right] dt \\ & + \sigma_1 \left(1 - \frac{x+y}{k}\right) dW_1(t). \end{aligned}$$

Integrating from 0 to $t < \tau_x$,

$$\ln x(t) = \ln x_0 + \int_0^t G_1(\tau) d\tau + \int_0^t \sigma_1 \left(1 - \frac{x+y}{k}\right) dW_1(\tau),$$

where, G_1 is finite on every bounded interval. Hence

$$\ln x(t) > -\infty, \quad (t < \tau_x).$$

Therefore, $x(t) > 0$, $t < \tau_x$. If $\tau_x < \infty$, then $x(\tau_x) = 0$, which implies $\ln x(\tau_x) = -\infty$, but the previous representation shows that $x(\tau_x)$ remains finite almost surely. This contradiction yields $\tau_x = \infty$, a.s. Therefore $x(t) > 0$ for all $t \geq 0$ a.s.

To show the positivity of $y(t)$, define $\tau_y := \inf\{t > 0 : y(t) = 0\}$. Applying Itô's formula to $V_2(y) = \ln y$, one obtains

$$\begin{aligned} d(\ln y) = & \left[s \left(1 - \frac{x+y}{k}\right) + \frac{\beta x}{1+\eta y} - a - \gamma z - \frac{1}{2} \sigma_2^2 \left(1 - \frac{x+y}{k}\right)^2 \right] dt \\ & + \sigma_2 \left(1 - \frac{x+y}{k}\right) dW_2(t). \end{aligned}$$

Integrating over $[0, t]$,

$$\ln y(t) = \ln y_0 + \int_0^t G_2(\tau) d\tau + \int_0^t \sigma_2 \left(1 - \frac{x+y}{k}\right) dW_2(\tau),$$

where G_2 is finite. Consequently $\ln y$ cannot reach $-\infty$ in finite time. Hence $\tau_y = \infty$ a.s. and therefore $y(t) > 0$ for all $t \geq 0$.

To show the positivity of $z(t)$, define $\tau_z := \inf\{t > 0: z(t) = 0\}$. Applying Itô's formula to $V(z) = \ln z$, yields

$$d(\ln z) = \left[\delta y - b - \frac{1}{2} \sigma_3^2 \right] dt + \sigma_3 dW_3(t).$$

Hence

$$\ln z(t) = \ln z(0) + \int_0^t \left[\delta y(\tau) - b - \frac{1}{2} \sigma_3^2 \right] d\tau + \sigma_3 W_3(t).$$

The right-hand side is finite for every finite t . Therefore $\ln z(t)$ never reaches $-\infty$ in finite time, implying $\tau_z = \infty$ a.s. Thus $z(t) > 0$ for all $t \geq 0$.

Combining the previous three steps, we obtain

$$x(t) > 0, y(t) > 0, z(t) > 0, \text{ for all } t \geq 0, \text{ a.s.}$$

whenever $x_0 > 0, y_0 > 0, z_0 > 0$. Hence $\mathbb{R}_+^3$ is positively invariant under the stochastic flow generated by system (9).

To find the upper estimate for $x(t)$. Since $-\frac{\beta xy}{1+\eta y} \leq 0$ and $-\delta_x x \leq 0$, we obtain

$$dx \leq rx \left(1 - \frac{x}{k}\right) dt + \sigma_1 x dW_1(t).$$

Consider the auxiliary stochastic logistic equation

$$d\bar{x} = r\bar{x} \left(1 - \frac{\bar{x}}{k}\right) dt + \sigma_1 \bar{x} dW_1(t).$$

By the stochastic comparison theorem $x(t) \leq \bar{x}(t)$ for all $t \geq 0$ a.s. For the stochastic logistic equation, it is well known that, if $r > \frac{\sigma_1^2}{2}$, then

$$\lim_{t \rightarrow \infty} \sup \bar{x}(t) = k \left(1 - \frac{\sigma_1^2}{2r} \right) \text{ a.s.}$$

Therefore,

$$\lim_{t \rightarrow \infty} \sup x(t) \leq k \left(1 - \frac{\sigma_1^2}{2r} \right) \text{ a.s.}$$

To find the upper estimate for $y(t)$. Since $\frac{\beta xy}{1+\eta y} \geq 0$ and $-\gamma yz \leq 0$, we have

$$dy \leq sy \left(1 - \frac{y}{k} \right) dt + \sigma_2 y dW_2(t).$$

Consider

$$d\bar{y} = s\bar{y} \left(1 - \frac{\bar{y}}{k} \right) dt + \sigma_2 \bar{y} dW_2(t).$$

Again, by stochastic comparison $y(t) \leq \bar{y}(t)$ a.s. Hence

$$\lim_{t \rightarrow \infty} \sup y(t) \leq k \left(1 - \frac{\sigma_2^2}{2s} \right) \text{ a.s., when } s > \frac{\sigma_2^2}{2}.$$

To show the boundedness of $z(t)$. Use the third equation of the model (10) and then

applying Itô's formula to $\ln z(t)$,

$$d(\ln z) = \left(\delta y - b - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 dW_3(t).$$

Integrating from 0 to t ,

$$\ln z(t) = \ln z(0) + \int_0^t \left(\delta y(\tau) - b - \frac{1}{2} \sigma_3^2 \right) d\tau + \sigma_3 W_3(t)$$

Using the estimate for $y(t)$, $y(t) \leq k \left(1 - \frac{\sigma_2^2}{2s} \right) + \varepsilon$ for sufficiently large t . Hence

$$\delta y(t) - b \leq \delta k \left(1 - \frac{\sigma_2^2}{2s} \right) - b + \varepsilon.$$

Under the condition $\delta k \left(1 - \frac{\sigma_2^2}{2s} \right) < b$, we obtain a negative drift term. Therefore

$$\lim_{t \rightarrow \infty} \sup z(t) < \infty \text{ a.s.}$$

Thus, the solution is ultimately bounded almost surely. ■

Theorem 4.2 (Uniform Ultimate Boundedness) Assume that $\delta \leq \gamma$, and define

$$\hat{r} := \max\{r, s\}, \quad \hat{d} := \min\{\delta_x, a, b\} > 0.$$

Then every positive solution $(x(t), y(t), z(t))$ of system (10) satisfies

$$\lim_{t \rightarrow \infty} \sup(x(t) + y(t) + z(t)) \leq \frac{\hat{r}k}{\hat{d}} \text{ a.s.}$$

Consequently, for every $\varepsilon > 0$

$$B_\varepsilon = \left\{ (x, y, z) \in \mathbb{R}_+^3 : x + y + z \leq \frac{\hat{r}k}{\hat{d}} + \varepsilon \right\}$$

is an absorbing bounded set

Proof Suppose that $w(t) = x(t) + y(t) + z(t)$. Applying Itô's formula and summing the three equations of system (10), we obtain

$$\begin{aligned} dx = & \left[rx \left(1 - \frac{x+y}{k} \right) + sy \left(1 - \frac{x+y}{k} \right) - \delta_x x - ay - bz + (\delta - \gamma)yz \right] dt \\ & + \sigma_1 x \left(1 - \frac{x+y}{k} \right) dW_1(t) + \sigma_2 y \left(1 - \frac{x+y}{k} \right) dW_2(t) + \sigma_3 z dW_3(t). \end{aligned}$$

Since $\delta \leq \gamma$, we have $(\delta - \gamma)yz \leq 0$. Moreover, $rx + sy \leq \hat{r}(x + y)$, and

$$\delta_x x + ay + bz \geq \hat{d}(x + y + z) = \hat{d}w.$$

Hence

$$dw \leq \left[\hat{r}(x + y) \left(1 - \frac{x+y}{k} \right) - \hat{d}w \right] dt + dM_t,$$

where,

$$\begin{aligned} \hat{d}w = & \int_0^t \sigma_1 x(\tau) \left(1 - \frac{x(\tau)+y(\tau)}{k} \right) dW_1(\tau) + \int_0^t \sigma_2 y(\tau) \left(1 - \frac{x(\tau)+y(\tau)}{k} \right) dW_2(\tau) \\ & + \int_0^t \sigma_3 z(\tau) dW_3(\tau). \end{aligned}$$

is a continuous local martingale. Now, consider

$$f(u) = u \left(1 - \frac{u}{k} \right), \quad u \geq 0.$$

Since f is concave on $[0, \infty)$, $f(u) \leq k$, for all $u \geq 0$. Therefore,

$$(x + y) \left(1 - \frac{x+y}{k}\right) \leq k.$$

Consequently,

$$dw \leq [\hat{r}k - \hat{d}w]dt + dM_t.$$

Define $Y(t) = e^{\hat{d}t}w(t)$. Using Itô's product rule,

$$dY = e^{\hat{d}t}dw + \hat{d}e^{\hat{d}t}w dt.$$

Substituting the previous inequality gives

$$dY \leq \hat{r}ke^{\hat{d}t}dt + e^{\hat{d}t}dM_t.$$

Integrating from 0 to t

$$Y(t) \leq w_0 + \hat{r}k \int_0^t e^{\hat{d}\tau} d\tau + \int_0^t e^{\hat{d}\tau} dM_\tau.$$

Therefore

$$w(t) \leq w_0 e^{-\hat{d}t} + \frac{\hat{r}k}{\hat{d}} (1 - e^{-\hat{d}t}) + e^{-\hat{d}t} \int_0^t e^{\hat{d}\tau} dM_\tau.$$

Thus $w(t) \leq \frac{\hat{r}k}{\hat{d}} + w_0 e^{-\hat{d}t} R(t)$, where $R(t) = e^{-\hat{d}t} \int_0^t e^{\hat{d}\tau} dM_\tau$. Since M_t is a continuous local martingale and $\hat{d} > 0$, the exponential martingale estimate together with the strong law for continuous local martingales yields $R(t) \rightarrow 0$, as $t \rightarrow \infty$, a.s. Hence $\lim_{t \rightarrow \infty} \sup w(t) \leq \frac{\hat{r}k}{\hat{d}}$, a.s.

That is,

$$\lim_{t \rightarrow \infty} \sup (x(t) + y(t) + z(t)) \leq \frac{\hat{r}k}{\hat{d}}, \text{ a.s.}$$

Finally, for any $\varepsilon > 0$, there exists a random time T_ε such that

$$x(t) + y(t) + z(t) \leq \frac{\hat{r}k}{\hat{d}} + \varepsilon, \quad t \geq T_\varepsilon,$$

almost surely. Therefore, the set

$$B_\varepsilon = \left\{ (x, y, z) \in \mathbb{R}_+^3 : x + y + z \leq \frac{\hat{r}k}{\hat{d}} + \varepsilon \right\}$$

is absorbing and bounded. ■

Theorem 4.3 (Local Existence, Uniqueness and Positivity). Consider the stochastic system (10) with initial value $(x_0, y_0, z_0) \in \text{Int}(\mathbb{R}_+^3)$. Then there exists a unique positive local solution $(x(t), y(t), z(t))$ on the interval $[0, \tau_e)$ a.s, where τ_e denotes the explosion time.

Proof. To establish positivity and local solvability, introduce the logarithmic transformation

$$N(t) = \ln x(t), T(t) = \ln y(t) \text{ and } I(t) = \ln z(t).$$

Since $x = e^N$, $y = e^T$ and $z = e^I$, the positivity of (x, y, z) is equivalent to the existence of finite solutions (N, T, I) . Applying Itô's formula to the function $\phi(u) = \ln u$, for the first equation yields

$$dN = \left(\frac{f_1(x, y)}{x} - \frac{1}{2} \frac{g_1^2(x, y)}{x^2} \right) dt + \frac{g_1(x, y)}{x} dW_1(t),$$

where,

$$f_1(x, y) = rx \left(1 - \frac{x+y}{k} \right) - \frac{\beta xy}{1+\eta y} - \delta_x x \text{ and } g_1(x, y) = \sigma_1 x \left(1 - \frac{x+y}{k} \right).$$

Consequently,

$$dN = \left[r \left(1 - \frac{x+y}{k} \right) - \frac{\beta y}{1+\eta y} - \delta_x - \frac{1}{2} \sigma_1^2 \left(1 - \frac{x+y}{k} \right)^2 \right] dt + \sigma_1 \left(1 - \frac{x+y}{k} \right) dW_1(t)$$

Substituting $x = e^N$ and $y = e^T$, gives

$$dN = \left[r \left(1 - \frac{e^N + e^T}{k} \right) - \frac{\beta e^T}{1+\eta e^T} - \delta_x - \frac{1}{2} \sigma_1^2 \left(1 - \frac{e^N + e^T}{k} \right)^2 \right] dt + \sigma_1 \left(1 - \frac{e^N + e^T}{k} \right) dW_1(t)$$

Similarly, applying Itô's formula to $T = \ln y$, we obtain

$$dT = \left[s \left(1 - \frac{x+y}{k} \right) + \frac{\beta x}{1+\eta y} - a - \gamma z - \frac{1}{2} \sigma_2^2 \left(1 - \frac{x+y}{k} \right)^2 \right] dt + \sigma_2 \left(1 - \frac{x+y}{k} \right) dW_2(t),$$

or equivalently

$$dT = \left[s \left(1 - \frac{e^N + e^T}{k} \right) + \frac{\beta e^N}{1 + \eta e^T} - a - \gamma e^I - \frac{1}{2} \sigma_2^2 \left(1 - \frac{e^N + e^T}{k} \right)^2 \right] dt \\ + \sigma_2 \left(1 - \frac{e^N + e^T}{k} \right) dW_2(t).$$

Likewise, for $I = \ln z$ we have

$$dI = \left(\frac{\delta yz - bz}{z} - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 dW_3(t),$$

hence

$$dI = \left[\delta e^T - b - \frac{1}{2} \sigma_3^2 \right] dt + \sigma_3 dW_3(t).$$

Therefore, the transformed system becomes

$$dN = F_1(N, T, I)dt + G_1(N, T)dW_1(t),$$

$$dT = F_2(N, T, I)dt + G_2(N, T)dW_2(t),$$

$$dI = F_3(N, T, I)dt + G_3(N, T)dW_3(t),$$

where F_i and G_i are compositions of exponential and rational functions. Observe that

$$1 + \eta e^T > 0 \quad \text{for every } (N, T, I) \in \mathbb{R}^3.$$

Hence every coefficient F_i and G_i is continuously differentiable on $\mathbb{R}^3$. In particular, the drift and diffusion coefficients satisfy the local Lipschitz condition on every bounded subset of $\mathbb{R}^3$.

By the standard local existence and uniqueness theorem for stochastic differential equations, there exists a unique maximal local solution $(N(t), T(t), I(t))$, defined on $[0, \tau_e)$, where τ_e denotes the explosion time. Finally, define $x(t) = e^{N(t)}$, $y(t) = e^{T(t)}$, $z(t) = e^{I(t)}$. Since the exponential function is strictly positive $x(t) > 0$, $y(t) > 0$, $z(t) > 0$ for all $t < \tau_e$ almost surely. Therefore system (10) admits a unique positive local solution on $[0, \tau_e)$ for every initial value $(x_0, y_0, z_0) \in \text{Int}(\mathbb{R}_+^3)$. This completes the proof. ■

Theorem 4.4 (Global Positivity and Non-Explosion). The model (10) has a unique solution $(x(t), y(t), z(t))$ for $t \in [0, \tau_e)$ and for every $(x_0, y_0, z_0) \in \text{Int}(\mathbb{R}_+^3)$ and $\mathbb{P}\{(x(t), y(t), z(t)) \in \text{Int}(\mathbb{R}_+^3) : t \geq 0\} = 1$.

Proof It is enough to confirm that $\mathbb{P}\{\tau_\infty = \infty\} = 1$. If $\kappa_0 \in \mathbb{Z}_+$ is large enough so that (x_0, y_0, z_0) belong to the closed ball $\mathcal{B}(\kappa_0) \subset \mathbb{R}_+^3$. For $\kappa \geq \kappa_0$, define the stop-time by

$$\tau_\kappa = \inf \left\{ t \in [0, \tau_0) : x \notin \left(\frac{1}{\kappa}, \kappa\right) \text{ or } y \notin \left(\frac{1}{\kappa}, \kappa\right) \text{ or } z \notin \left(\frac{1}{\kappa}, \kappa\right) \right\}. \quad (11)$$

Now, $\inf \emptyset = \infty$. So, τ_κ is growing as $\kappa \rightarrow \infty$. Let $\tau_\infty = \lim_{\kappa \rightarrow \infty} \tau_\kappa$; then, $\tau_\infty \leq \tau_e$ a.s. If $\mathbb{P}\{\tau_\infty = \infty\} = 1$, then $\mathbb{P}\{\tau_e = \infty\} = 1$. If $\tau_\infty \neq \infty$, then $T > 0$ and $\varepsilon \in (0, 1)$ exist with

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \quad (12)$$

Hence, put $\Omega_\kappa = \{\tau_\kappa \leq T\}$, then $\kappa_1 \geq \kappa_0$ is an integer satisfies

$$\mathbb{P}\{\tau_\kappa \leq T\} \geq \varepsilon, \text{ for all } \kappa \geq \kappa_1. \quad (13)$$

Define a function $\mathcal{V}: \text{Int}(\mathbb{R}_+^3) \rightarrow \text{Int}(\mathbb{R}_+)$ as follows

$$\mathcal{V}(x, y, z) = (x - 1 - \ln y) + (y - 1 - \ln y) + (z - 1 - \ln z), \quad (14)$$

where $\mathcal{V}(x, y, z) > 0$ for all $(x, y, z) \in \text{Int}(\mathbb{R}_+^3)$. By Itô's formula

$$\mathcal{L}\mathcal{V} = V_t(X, t) + V_x(X, t)f(X, t) + \frac{1}{2} \text{trace}[g^T(X, t)V_{xx}(X, t)g(X, t)]$$

By a simple calculation we get,

$$\begin{aligned} \mathcal{L}\mathcal{V} &= (x - 1) \left[rx \left(1 - \frac{x+y}{k} \right) - \frac{\beta y}{1+\eta y} - \delta_x \right] \\ &+ (y - 1) \left[s \left(1 - \frac{x+y}{k} \right) + \frac{\beta x}{1+\eta y} - a - \gamma z \right] + (z - 1) [\delta y - b] \\ &+ \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \left(1 - \frac{x+y}{k} \right)^2 + \frac{1}{2} \sigma_3^2 \end{aligned}$$

Taking the differential of $\mathcal{V}(x, y, z)$, one gets

$$d\mathcal{V}(x, y, z) = \mathfrak{f}(x, y, z)dt + \mathfrak{g}(x, y, z)dW, \quad (15)$$

where $\mathfrak{g}(x, y, z) = \sigma_1(x - 1) + \sigma_2(y - 1) + \sigma_3(z - 1)$, and

$$\mathfrak{f}(x, y, z) = \frac{1}{2} (\sigma_1^2 + \sigma_2^2) \left(1 - \frac{x+y}{k} \right)^2 + \frac{1}{2} \sigma_3^2 - (x - 1) \left[rx \left(1 - \frac{x+y}{k} \right) - \frac{\beta y}{1+\eta y} - \delta_x \right]$$

$$-(y-1) \left[s \left(1 - \frac{x+y}{k} \right) + \frac{\beta x}{1+\eta y} - a - \gamma z \right] - (z-1)[\delta y - b]$$

Then, there is $M > 0$ so, $|\mathfrak{f}(x, y, z)| \leq M$, for all $(x, y, z) \in \mathbb{R}_+^3$. It follows (15) that

$$\int_0^{\tau_m \wedge T} d\mathcal{V}(x, y, z) \leq \int_0^{\tau_m \wedge T} M dt + \int_0^{\tau_m \wedge T} g(x, y, z) dW(t), \quad (16)$$

where $\tau_k \wedge T = \min\{\tau_k, T\}$. Considering the aforementioned inequality's expectations, yield

$$\mathbb{E}\mathcal{V}(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)) \leq \mathcal{V}(x(0), y(0), z(0)) + MT \quad (17)$$

Note that no less than one of $x(\tau_k, \omega)$, $y(\tau_k, \omega)$, and $z(\tau_k, \omega)$ lies in $\{\kappa, 1/\kappa\}$, for every $\omega \in \Omega_\kappa$; consequently,

$$\mathcal{V}(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)) \geq (\kappa - 1 - \ln \kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right) \quad (18)$$

Hence, from (13)

$$\begin{aligned} \mathbb{E}\mathcal{V}(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T)) &\geq \mathbb{E}[\mathbb{I}_{\Omega_\kappa(\omega)} \mathcal{V}(x(\tau_k \wedge T), y(\tau_k \wedge T), z(\tau_k \wedge T))] \\ &\geq \varepsilon(\kappa - 1 - \ln \kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right). \end{aligned} \quad (19)$$

Here $\mathbb{I}_{\Omega_\kappa(\omega)}$ epitomizes the inductor function of Ω_κ . By (18), yield

$$\mathcal{V}(x(0), y(0), z(0)) + MT \geq \varepsilon(\kappa - 1 - \ln \kappa) \wedge \left(\frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right). \quad (20)$$

$\kappa \rightarrow \infty$ implies to a contradiction: $\infty > \mathcal{V}(x(0), y(0), z(0)) + MT = \infty$: Therefore, $\mathbb{P}\{\tau_\infty = \infty\} = 1$. Thus, $\mathbb{P}\{(x(t), y(t), z(t)) \in \text{Int}(\mathbb{R}_+^3): t \geq 0\} = 1$.

Theorem 4.5 (Stochastically Ultimately) The solutions of system (10) are stochastically ultimately bounded, for every initial value $w_0 = (x_0, y_0, z_0) \in \mathbb{R}_+^3$. That is $\lim_{t \rightarrow \infty} \sup \mathbb{E}|w(t)|^\theta < \infty$.

Proof From Theorem 4.4, we have $\mathbb{P}\{w(t) \in \text{Int}(\mathbb{R}_+^3): t \geq 0\} = 1$. Choose appropriate Lyapunov function $\mathcal{V}(t, (x, y, z)) := (e^t x^\theta, e^t y^\theta, e^t z^\theta)$, $\theta > 0$.

Suppose that $\mathcal{V}_1(t, x) = e^t x^\theta$ for $\theta > 0$. The Itô formula implies that

$$\begin{aligned}
 \mathcal{LV}_1(t, x) &= \frac{\partial \mathcal{V}_1(t, x)}{\partial t} + \frac{\partial \mathcal{V}_1(t, x)}{\partial x} f_1(t, x, y, z) + \frac{1}{2} \frac{\partial^2 \mathcal{V}_1(t, x)}{\partial x^2} g_1^2(t, x, y, z) \\
 &= e^t x^\theta + \theta e^t x^{\theta-1} \left[r x \left(1 - \frac{x+y}{k} \right) - \frac{\beta xy}{1+\eta y} - \delta_x x \right] + \frac{1}{2} \theta (\theta - 1) e^t x^{\theta-2} \sigma_1^2 x^2 \left(1 - \frac{x+y}{k} \right)^2 \\
 &= e^t x^\theta \left\{ 1 + \theta r \left(1 - \frac{x+y}{k} \right) - \theta \frac{\beta y}{1+\eta y} - \theta \delta_x + \frac{1}{2} \theta (\theta - 1) \sigma_1^2 \left(1 - \frac{x+y}{k} \right)^2 \right\}
 \end{aligned}$$

To extract the dominant negative term, first note that $-r\theta \frac{x+y}{k}$ is important negative term, also $-\theta \frac{\beta y}{1+\eta y} < 0$. Now, write

$$\left(1 - \frac{x+y}{k} \right)^2 = 1 - \frac{2(x+y)}{k} + \frac{(x+y)^2}{k^2}.$$

Then the random term become

$$\frac{1}{2} \theta (\theta - 1) \sigma_1^2 \left(1 - \frac{2(x+y)}{k} + \frac{(x+y)^2}{k^2} \right).$$

Now we analyze the limits one by one. We have, the first term x^θ is acceptable and does not cause any problem. The second term $-x^\theta(x+y)$ and it is an excellent negative term that helps with stability. The third term (the most dangerous) $x^\theta(x+y)^2$. This is the part that needs careful handling.

Using the inequality $(x+y)^2 \leq 2(x^2 + y^2)$, then

$$x^\theta (x+y)^2 \leq 2x^{\theta+2} + 2x^\theta y^2$$

Now a new problem has arisen $x^\theta y^2$. Here we use Young's inequality.

$$ab \leq \frac{a^p}{p} + \frac{a^q}{q}, \frac{1}{p} + \frac{1}{q} = 1.$$

Take $a = x^\theta, b = y^2$. So, we get

$$x^\theta y^2 \leq C_1 x^{\theta+m} + C_2 y^{2+n}$$

That is, we replace the mixed term with a set of polynomial terms. This is useful because logistic growth in drift generates negative higher-order limits. From $rx \left(1 - \frac{x+y}{k} \right)$, we get $-\frac{r}{k} x^2$ and $-\frac{r}{k} xy$. When multiplied by $x^{\theta-1}$, yields $-\frac{r}{k} x^{\theta+1}, -\frac{r}{k} x^\theta y$. These strong negative terms can absorb the positive terms resulting from noise. After collecting the terms and using: Young inequality, Hölder inequality and absorption of small terms, we reach $\mathcal{LV}_1(t, x) \leq e^t [C_1 + C_2 x^\theta - C_3 x^{\theta+1}]$, for some constants C_1, C_2 and C_3 . This is sufficient because $-x^{\theta+1}$

asymptotically dominates x^θ . Thus $\phi(x) := C_2 x^\theta - C_3 x^{\theta+1}$ it has a finite maximum value because $\phi(x) \rightarrow -\infty$ as $x \rightarrow \infty$. So there exists $M_1(\theta) > 0$ (a maximum constant) such that $\mathcal{L}\mathcal{V}_1(t, x) \leq M_1(\theta)e^t$. Probabilistic meaning, the random part adds energy to the system, stochastic excitation. As for the logistic boundaries, they dissipate energy.

Step 6: Integration and Expectation. From Itô formula $\mathbb{E}[\mathcal{V}_1(t)] - \mathcal{V}_1(0) = \mathbb{E} \int_0^t \mathcal{L}\mathcal{V}_1(s) ds$.

Hence $e^t \mathbb{E}[x^\theta(t)] - \mathbb{E}[x_0^\theta] \leq M_1 e^t$. So, $\lim_{t \rightarrow \infty} \sup \mathbb{E}[x^\theta(t)] \leq M_1$. Similarly, take $\mathcal{V}_1 = e^t y^\theta$.

Applying Itô formula we get, $\mathcal{L}\mathcal{V}_1 \leq e^t [C_3 y^\theta - C_4 y^{\theta+1}]$. Hence $\lim_{t \rightarrow \infty} \sup \mathbb{E}[y^\theta(t)] \leq M_2$. For z , we have

$dz = (\delta yz - bz)dt + \sigma_3 z dW_3$. Take, $\mathcal{V}_3 = e^t z^\theta$, we get $\mathcal{L}\mathcal{V}_3 = e^t z^\theta \left[1 + \theta(\delta y - b) + \frac{1}{2}\theta(\theta - 1)\sigma_3^2 \right]$. Since $\mathbb{E}y^\theta$ is bounded, we conclude that $\lim_{t \rightarrow \infty} \sup \mathbb{E}[z^\theta(t)] \leq M_3$. Adding the variables to get $|w|^\theta \leq 3^{\theta/2}(x^\theta + y^\theta + z^\theta)$. Thus $\lim_{t \rightarrow \infty} \sup \mathbb{E}|w(t)|^\theta \leq M_4(\theta) < \infty$. Using Chebyshev's inequality $\mathbb{P}\{|w(t)| > R\} \leq \frac{\mathbb{E}|w(t)|^\theta}{R^\theta}$. Since $\mathbb{E}|w(t)|^\theta$ is bounded, then $\mathbb{P}\{|w(t)| > R\} \rightarrow 0$ as $R \rightarrow \infty$. Consequently, the solutions are stochastically ultimately bounded. ■

Theorem 4.6 (Uniform Continuity of Sample Paths). Let be the unique global positive solution of system (22) with initial value $(x_0, y_0, z_0) \in \mathbb{R}_+^3$. Assume that for every $p \geq 2$,

$$\sup_{t \geq 0} \mathbb{E}|x(t)|^p < \infty, \sup_{t \geq 0} \mathbb{E}|y(t)|^p < \infty \text{ and } \sup_{t \geq 0} \mathbb{E}|z(t)|^p < \infty.$$

Then every sample path of $(x(t), y(t), z(t))$ is uniformly continuous on $[0, \infty)$ almost surely.

Proof Suppose that $f_1(x, y) = rx \left(1 - \frac{x+y}{k} \right) - \frac{\beta xy}{1+\eta y} - \delta_x x$, and $g_1(x, y) = \sigma_1 x \left(1 - \frac{x+y}{k} \right)$. The first equation of system (10) admits the integral representation

$$x(t) = x_0 + \int_0^t f_1(x(u), y(u)) du + \int_0^t g_1(x(u), y(u)) dW_1(u).$$

Hence for $0 \leq s < t$,

$$x(t) - x(s) = \int_s^t f_1(x(u), y(u)) du + \int_s^t g_1(x(u), y(u)) dW_1(u)$$

Using $|a + b|^\theta \leq 2^{\theta-1}(|a|^\theta + |b|^\theta)$, $\theta > 2$, we obtain $\mathbb{E}|x(t) - x(s)|^\theta \leq 2^{\theta-1}I_1 + 2^{\theta-1}I_2$, where $I_1 = \mathbb{E} \left| \int_s^t f_1(u) du \right|^\theta$ and $I_2 = \mathbb{E} \left| \int_s^t g_1(u) dW_1(u) \right|^\theta$. To estimate the drift term, applying Hölder's inequality, $I_1 \leq (t-s)^{\theta-1} \int_s^t \mathbb{E}|f_1(u)|^\theta du$. Now $|f_1| \leq r|x| + \frac{r}{k}|x|(|x| + |y|) + \beta|x||y| + \delta_x|x|$. Therefore, there exists a constant C_1 such that $|f_1|^\theta \leq C_1(|x|^{2\theta} + |y|^{2\theta} + 1)$. Taking expectations and using the bounded-moment assumption

$\mathbb{E}|f_1(u)|^\theta \leq C_2$, where C_2 is independent of u . Consequently, $I_1 \leq C_2(t-s)^\theta$. To estimate the diffusion term, applying the Burkholder–Davis–Gundy inequality $I_2 \leq C_\theta \mathbb{E} \left(\int_s^t |g_1(u)|^2 du \right)^{\theta/2}$. Using Hölder's inequality $I_2 \leq C_\theta(t-s)^{\frac{\theta}{2}-1} \int_s^t \mathbb{E}|g_1(u)|^\theta du$. Since $g_1(x, y) = \sigma_1 x \left(1 - \frac{x+y}{k} \right)$, there exists C_3 such that $|g_1|^\theta \leq C_3(1 + |x|^{2\theta} + |y|^{2\theta})$. Hence $\mathbb{E}|g_1(u)|^\theta \leq C_4$, uniformly in u . Therefore, $I_2 \leq C_5(t-s)^{\frac{\theta}{2}}$. Combining the above estimates yields

$\mathbb{E}|x(t) - x(s)|^\theta \leq C_6 \left[(t-s)^\theta + (t-s)^{\frac{\theta}{2}} \right]$. Since $t-s \leq 1$, then $(t-s)^\theta \leq (t-s)^{\frac{\theta}{2}}$, and thus $\mathbb{E}|x(t) - x(s)|^\theta \leq C_7 (t-s)^{\frac{\theta}{2}}$. Exactly the same argument applied to the second and third equations gives $\mathbb{E}|y(t) - y(s)|^\theta \leq C_8 (t-s)^{\frac{\theta}{2}}$, and $\mathbb{E}|z(t) - z(s)|^\theta \leq C_9 (t-s)^{\frac{\theta}{2}}$. Hence $\mathbb{E}\|X(t) - X(s)\|^\theta \leq C|t-s|^{\frac{\theta}{2}}$, where $X(t) = (x(t), y(t), z(t))$.

Since $\theta > 2$, then $\frac{\theta}{2} = 1 + \frac{\theta-2}{2} > 1$. Therefore the assumptions of the Kolmogorov–Chentsov continuity theorem are satisfied. It follows that $X(t)$ possesses a modification whose sample paths are Hölder continuous of every order $\nu < \frac{\theta-2}{2\theta}$. In particular, every sample path is continuous. Because every sample path is locally Hölder continuous on $[0, \infty)$, and the moment bounds are uniform in time, the Hölder estimate is independent of the location of the interval. Hence every sample path is uniformly continuous on $[0, \infty)$ almost surely. Thus $(x(t), y(t), z(t))$ has uniformly continuous sample paths on $[0, \infty)$ with probability one. ■

5. Mathematical Description of The Development of Cancerous Tumors

In this section, we investigate the long-term behavior of the system. To facilitate the subsequent stability analysis, we introduce two hypotheses that will be used throughout the study.

$$(H1): \frac{L}{\eta} \max\{r, a\} + \frac{1}{2} \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\} < \min\{r-s, r-\delta\}$$

$$(H2): r - a - \frac{\sigma_1^2}{2} < 0, s - b - \frac{\sigma_2^2}{2} < 0, c - \delta - \frac{\sigma_3^2}{2} < 0 \quad (21)$$

The condition (H1) indicates that:

Immune pressure + viral therapy > tumor expansion + stochastic disturbances

The conditions mentioned in (H2) mean that the destructive effect of noise $\sigma_i^2/2$ has become greater than the reproductive capacity of the cells.

Theorem 5.1 (Stochastic Permanent) If the hypothesis (H_1) justifies, then the system (10) is stochastically permanent.

Proof To establish stochastic permanence, it is sufficient to prove that the solution remains ultimately bounded away from both 0 and $+\infty$ with high probability. Let $X(t) = (x(t), y(t), z(t))$ be the unique positive global solution corresponding to any initial value $(x_0, y_0, z_0) \in \mathbb{R}_+^3$. From the positivity theorem proved previously, we already know that $x(t) > 0, y(t) > 0, z(t) > 0$ almost surely for all $t \geq 0$. First, we need to Construct a Lyapunov function. Define $V(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Clearly, $V(x, y, z) > 0, V(x, y, z) \rightarrow \infty$ whenever any component tends to zero.

Applying Itô's formula to $1/x$, we obtain $d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx + \frac{1}{x^3}(dx)^2$. Since $(dx)^2 = \sigma_1^2 x^2 \left(1 - \frac{x+y}{k}\right)^2 dt$,

we have

$$d\left(\frac{1}{x}\right) = -\frac{1}{x} \left[r \left(1 - \frac{x+y}{k}\right) - \frac{\beta y}{1+\eta y} - \delta_x \right] dt - \frac{\sigma_1}{x} \left(1 - \frac{x+y}{k}\right) dW_1(t) + \frac{\sigma_1^2}{x} \left(1 - \frac{x+y}{k}\right)^2 dt.$$

Hence $\mathcal{L}V_1 = -\frac{1}{x} \left[r \left(1 - \frac{x+y}{k}\right) - \frac{\beta y}{1+\eta y} - \delta_x \right] + \frac{\sigma_1^2}{x} \left(1 - \frac{x+y}{k}\right)^2$. Using $\frac{\beta y}{1+\eta y} \leq \frac{\beta}{\eta}$, and $\left(1 - \frac{x+y}{k}\right)^2 \leq L$, for some positive constant L , we obtain $\mathcal{L}V_1 \leq -\frac{1}{x} \left(r - \delta_x - \frac{\beta}{\eta} \right) + \frac{L\sigma_1^2}{x}$. Therefore, $\mathcal{L}V_1 \leq -\frac{C_1}{x}$, where $C_1 := r - \delta_x - \frac{\beta}{\eta} - L\sigma_1^2 > 0$ under hypothesis (H1). Similarly, for $V_2 = \frac{1}{y}$, Itô's formula gives

$$\mathcal{L}V_2 = -\frac{1}{y} \left[s \left(1 - \frac{x+y}{k} \right) + \frac{\beta x}{1+\eta y} - a - \gamma z \right] + \frac{\sigma_2^2}{y} \left(1 - \frac{x+y}{k} \right)^2.$$

Using positivity of all variables, $\frac{\beta x}{1+\eta y} \geq 0$ and $\gamma z \geq 0$, we deduce $\mathcal{L}V_2 \leq -\frac{s-a}{y} + \frac{L\sigma_2^2}{y}$. Thus, $\mathcal{L}V_2 \leq -\frac{C_2}{y}$, where $C_2 := s - a - L\sigma_2^2 > 0$. For $V_3 = \frac{1}{z}$, we have $d\left(\frac{1}{z}\right) = -\frac{1}{z^2}dz + \frac{1}{z^3}(dz)^2$. Hence $\mathcal{L}V_3 = -\frac{\delta y - b}{z} + \frac{\sigma_3^2}{z}$. Since $y > 0$, $\mathcal{L}V_3 \leq -\frac{\delta - b - \sigma_3^2}{z}$. Therefore, $\mathcal{L}V_3 \leq -\frac{C_3}{z}$, where $C_3 := \delta - b - \sigma_3^2 > 0$. To do drift estimate for the total Lyapunov function. Let $V = V_1 + V_2 + V_3$. Then $\mathcal{L}V = \mathcal{L}V_1 + \mathcal{L}V_2 + \mathcal{L}V_3$. Consequently, $\mathcal{L}V \leq -C_1 \frac{1}{x} - C_2 \frac{1}{y} - C_3 \frac{1}{z}$.

Let $C := \min\{C_1, C_2, C_3\} > 0$. Then $\mathcal{L}V \leq -CV$. Thus, $dV(X(t)) \leq -CV(X(t))dt + dM_t$, where $M(t)$ is a local martingale. Taking expectations, $\frac{d}{dt} \mathbb{E}[V(X(t))] \leq -C\mathbb{E}[V(X(t))]$. By Gronwall's inequality, $\mathbb{E}[V(X(t))] \leq V(X_0)e^{-Ct}$.

Hence, $\sup_{t \geq 0} \mathbb{E}[V(X(t))] < \infty$. Therefore, $\sup_{t \geq 0} \mathbb{E}\left[\frac{1}{x(t)} + \frac{1}{y(t)} + \frac{1}{z(t)}\right] < \infty$. To show that the lower bounded away from zero. Using Chebyshev's inequality, for any $\varepsilon > 0$, $\mathbb{P}\{x(t) < \varepsilon\} = \mathbb{P}\left\{\frac{1}{x(t)} > \frac{1}{\varepsilon}\right\} \leq \varepsilon \mathbb{E}\left[\frac{1}{x(t)}\right]$. Since $\sup_{t \geq 0} \mathbb{E}\left[\frac{1}{x(t)}\right] < \infty$, there exists a constant $K_1 > 0$ such that $\mathbb{P}\{x(t) < \varepsilon\} \leq K_1 \varepsilon$. Similarly, $\mathbb{P}\{y(t) < \varepsilon\} \leq K_2 \varepsilon$ and $\mathbb{P}\{z(t) < \varepsilon\} \leq K_3 \varepsilon$.

Thus, for any $\xi \in (0,1)$, one can choose $\varepsilon > 0$ sufficiently small such that $\lim_{t \rightarrow \infty} \inf \mathbb{P}\{x(t) \geq \varepsilon\} \geq 1 - \xi$, $\lim_{t \rightarrow \infty} \inf \mathbb{P}\{y(t) \geq \varepsilon\} \geq 1 - \xi$, and $\lim_{t \rightarrow \infty} \inf \mathbb{P}\{z(t) \geq \varepsilon\} \geq 1 - \xi$. To show the ultimate Boundedness of the solution. From the logistic structure of the first two equations, $x(t) + y(t)$ is ultimately bounded in expectation. Moreover, from standard moment estimates for stochastic differential equations, there exists $H > 0$ such that $\lim_{t \rightarrow \infty} \sup \mathbb{P}\{|X(t)| \leq H\} \geq 1 - \xi$. Hence the solution is bounded from above with arbitrarily high probability. Combining the lower persistence estimate and the upper boundedness estimate, we conclude that for every $\xi \in (0,1)$, there exist positive constants $m = M(\xi)$, $M = M(\xi)$, such that $\lim_{t \rightarrow \infty} \inf \mathbb{P}\{m \leq x(t), y(t), z(t) \leq M\} \geq 1 - \xi$. Therefore, the stochastic system is stochastically permanent. ■

Theorem 5.2 (Stochastic Extinction) The solution $w(t) = (x(t), y(t), z(t))$ of system (10) will be extinct with probability one for every given initial value $w(0) = (x(0), y(0), z(0)) \in \mathbb{R}_+^3$ when (H2) is fulfilled. namely

$$\lim_{t \rightarrow \infty} x(t) = 0, \lim_{t \rightarrow \infty} y(t) = 0, \lim_{t \rightarrow \infty} z(t) = 0 \text{ almost surely.}$$

Proof To establish extinction, we analyze the asymptotic behavior of the logarithmic processes $\ln x(t)$, $\ln y(t)$, $\ln z(t)$. The proof is divided into several steps. First, to show the extinction of $x(t)$: Apply Itô's formula to the function $V_1(x) = \ln x$. Since $d(\ln x) = \frac{1}{x}dx + \frac{1}{2x^2}(dx)^2$ and $(dx)^2 = \sigma_1^2 x^2 \left(1 - \frac{x+y}{k}\right)^2 dt$,

we obtain

$$\begin{aligned} d(\ln x) &= \left[r \left(1 - \frac{x+y}{k} \right) - \frac{\beta y}{1+\eta y} - \delta_x - \frac{\sigma_1^2}{2} \left(1 - \frac{x+y}{k} \right)^2 \right] dt \\ &\quad + \sigma_1 \left(1 - \frac{x+y}{k} \right) dW_1(t). \end{aligned}$$

Because $0 \leq 1 - \frac{x+y}{k} \leq 1$, and $\frac{\beta y}{1+\eta y} \geq 0$, we have $d(\ln x) \leq \left(r - \delta_x - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 \left(1 - \frac{x+y}{k} \right) dW_1(t)$.

Integrating from 0 to t , $\ln x(t) \leq \ln x(0) + \left(r - \delta_x - \frac{\sigma_1^2}{2} \right) t + M_1(t)$, where $M_1(t) := \int_0^t \sigma_1 \left(1 - \frac{x(\tau)+y(\tau)}{k} \right) dW_1(\tau)$

is a continuous local martingale. Using the strong law of large numbers for martingales, $\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0$ a.s.

Hence, $\lim_{t \rightarrow \infty} \sup \frac{\ln x(t)}{t} \leq r - \delta_x - \frac{\sigma_1^2}{2}$. By (H2), $r - \delta_x - \frac{\sigma_1^2}{2} < 0$. Therefore, $\lim_{t \rightarrow \infty} \sup \frac{\ln x(t)}{t} < 0$ a.s. Consequently,

$\lim_{t \rightarrow \infty} x(t) = 0$ a.s. To show the extinction $y(t)$. Apply Itô's formula to $V_2(y) = \ln y$, we have $d(\ln y) = \frac{1}{y} dy + \frac{1}{2y^2} (dy)^2$. Since $(dy)^2 = \sigma_2^2 y^2 \left(1 - \frac{x+y}{k}\right)^2 dt$, It follows that

$$d \ln y(t) = \left[s \left(1 - \frac{x+y}{k}\right) - \frac{\beta x}{1+\eta y} - a - \gamma z - \frac{\sigma_2^2}{2} \left(1 - \frac{x+y}{k}\right)^2 \right] dt + \sigma_2 \left(1 - \frac{x+y}{k}\right) dW_2(t).$$

Using $1 - \frac{x+y}{k} \leq 1$, $-\gamma z \leq 0$, and the boundedness $\frac{\beta x}{1+\eta y} \leq \beta x$, we obtain

$$d \ln y(t) \leq \left(s - a - \frac{\sigma_2^2}{2} \right) dt + \beta x(t) dt + \sigma_2 \left(1 - \frac{x(t)+y(t)}{k}\right) dW_2(t).$$

Integrating from 0 to t , $\ln y(t) \leq \ln y(0) + \left(s - a - \frac{\sigma_2^2}{2} \right) t + \beta \int_0^t x(\tau) d\tau + M_2(t)$, where, $M_2(t) := \int_0^t \sigma_2 \left(1 - \frac{x(\tau)+y(\tau)}{k}\right) dW_2(\tau)$. So, $x(t) \rightarrow 0$ a.s. Hence $\frac{1}{t} \int_0^t x(\tau) d\tau \rightarrow 0$ a.s. Dividing by t ,

$$\frac{\ln y(t)}{t} \leq \frac{\ln y(0)}{t} + s - a - \frac{\sigma_2^2}{2} + \beta \frac{1}{t} \int_0^t x(\tau) d\tau + \frac{M_2(t)}{t}.$$

Again $\frac{M_2(t)}{t} \rightarrow 0$ a.s. Thus, $\lim_{t \rightarrow \infty} \sup \frac{\ln y(t)}{t} \leq s - a - \frac{\sigma_2^2}{2}$. By (H2), $s - a - \frac{\sigma_2^2}{2} < 0$. Therefore, $\lim_{t \rightarrow \infty} y(t) = 0$ a.s.

To show the Extinction of $z(t)$. Applying Itô's formula to $V_3(z) = \ln z$, gives $d \ln z = \left(\delta y - b - \frac{\sigma_3^2}{2} \right) dt + \sigma_3 dW_3(t)$.

Integrating from 0 to t , $\ln z(t) = \ln z(0) + \delta \int_0^t y(\tau) d\tau - \left(b + \frac{\sigma_3^2}{2} \right) t + M_3(t)$, where $M_3(t) = \sigma_3 W_3(t)$. Divided by t ,

$$\frac{\ln z(t)}{t} = \frac{\ln z(0)}{t} + \frac{\delta}{t} \int_0^t y(\tau) d\tau - b - \frac{\sigma_3^2}{2} + \frac{M_3(t)}{t}. \text{ Since } y(t) \rightarrow 0 \text{ a.s., we have } \frac{1}{t} \int_0^t y(\tau) d\tau \rightarrow 0. \text{ Also } \frac{M_3(t)}{t} \rightarrow 0 \text{ a.s.}$$

Hence $\lim_{t \rightarrow \infty} \sup \frac{\ln z(t)}{t} \leq -b - \frac{\sigma_3^2}{2} < 0$. Therefore, $\lim_{t \rightarrow \infty} z(t) = 0$ a.s. Consequently, we have shown that $x(t) \rightarrow 0, y(t) \rightarrow 0, z(t) \rightarrow 0$ almost surely as $t \rightarrow \infty$. Hence every population in the stochastic tumor-virus-immune model becomes extinct with probability one. Therefore, the system is extinct almost surely. ■

6. Numerical Analysis of Results

The simulation results indicate that the three variables x, y , and z remain within stable limits over the long term, reflecting the stability effect resulting from the structural conditions of the system. When the intensity of noise increases or the values of γ and δ change, it can lead to different behaviors such as explosion or oscillation. The comparable estimation formulas are as follows:

$$\begin{aligned} x_{k+1} &= x_k + \left[r x_k \left(1 - \frac{x_k+y_k}{k}\right) - \frac{\beta x_k y_k}{1+\eta y_k} - \delta x_k \right] \Delta t + x_k \left(1 - \frac{x_k+y_k}{k}\right) \left[\sigma_1 \sqrt{\Delta t} \xi_{k,1} + \frac{\sigma_1^2}{2} (\xi_{k,1}^2 - 1) \Delta t \right] \\ y_{k+1} &= y_k + \left[s y_k \left(1 - \frac{x_k+y_k}{k}\right) + \frac{\beta x_k y_k}{1+\eta y_k} - a y_k - \gamma y_k z_k \right] \Delta t + y_k \left(1 - \frac{x_k+y_k}{k}\right) \left[\sigma_2 \sqrt{\Delta t} \xi_{k,2} + \frac{\sigma_2^2}{2} (\xi_{k,2}^2 - 1) \Delta t \right] \\ z_{k+1} &= z_k + (\delta z_k - b) z_k \Delta t + z_k \left[\sigma_3 \sqrt{\Delta t} \xi_{k,3} + \frac{\sigma_3^2}{2} (\xi_{k,3}^2 - 1) \Delta t \right]. \end{aligned} \quad (22)$$

Case 0: If $\gamma = 0.05$ and $\delta = 0.03$, then these values are moderate and may lead to a relative balance between the predator and the prey. See Fig. (0).

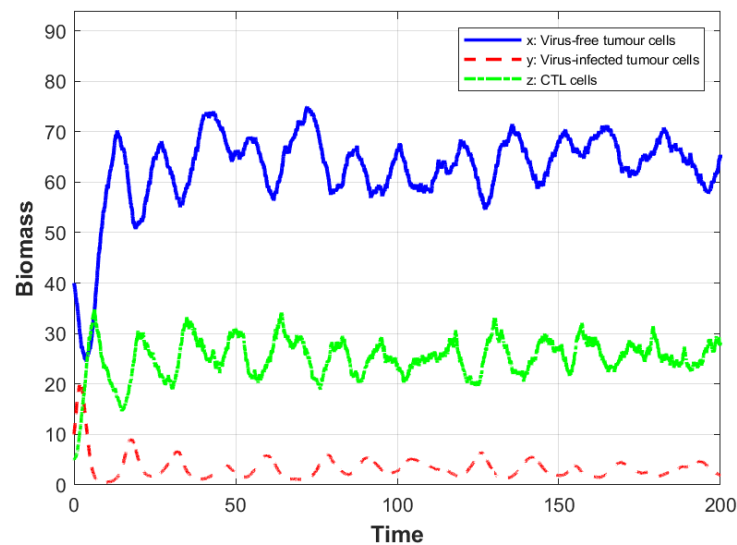


Figure 0 $\gamma = 0.05$ and $\delta = 0.03$

Case 1: Increased predation rate $\Rightarrow$ May lead to the extinction of the prey: $\gamma = 0.2$ and $\delta = 0.01$. An increase in γ means that the predator eats the prey more aggressively, while a low δ means that a large part of the prey does not turn into new predators. This could lead to the extinction of y in the long term. See Fig. (1)

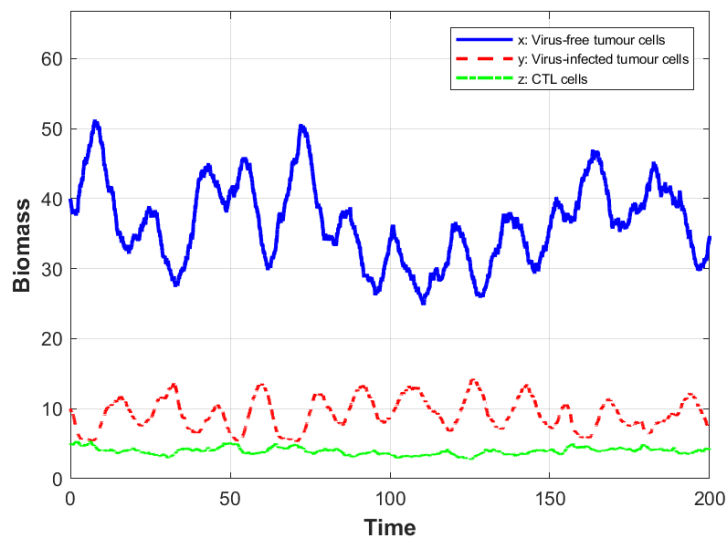


Figure 1 $\gamma = 0.2$ and $\delta = 0.01$

Analytical forecasts: Due to the high predation rate ($\gamma = 0.2$) and its weak benefit in forming predators ($\delta = 0.01$), the prey y is at risk of extinction, and the number of predators may also decline later due to food scarcity.

Case 2: Increased conversion rate $\Rightarrow$ May lead to an explosion in the number of predators: $\gamma = 0.03$ and $\delta = 0.15$. The predator does not consume much y (low), but each consumption results in many new predators δ (high), which can lead to rapid growth in z . See Fig. (2).

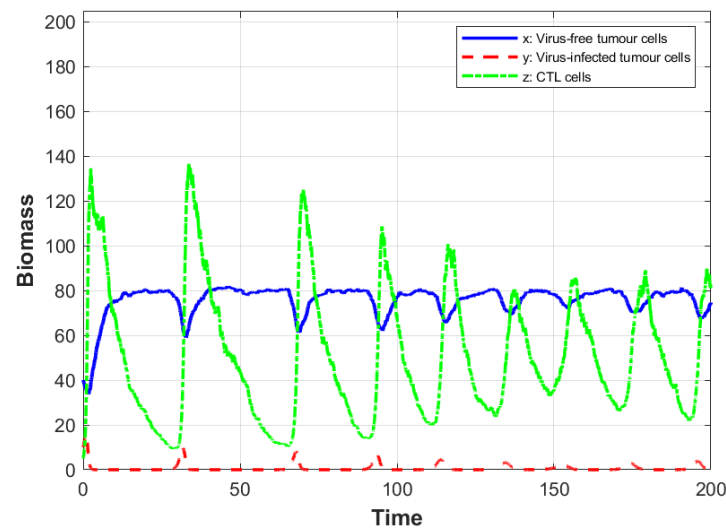


Figure 2 $\gamma = 0.03$ and $\delta = 0.15$

Case 3: Low values for both $\Rightarrow$ Inactive system and relative stability: $\gamma = 0.01$ and $\delta = 0.01$. This represents a weak interaction system between y and z , which may lead to slow stabilization or even stagnation in z . See Fig. (3).

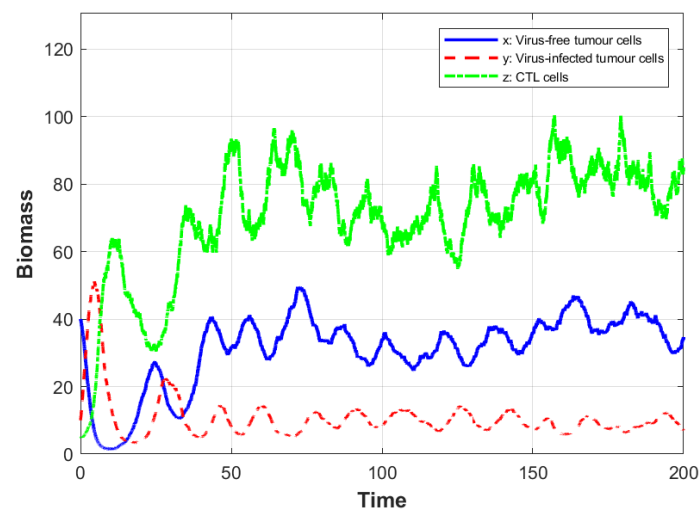


Figure 3 $\gamma = 0.01$ and $\delta = 0.01$

Notes: In this case, y is not much affected by z . Also, z may remain stable or decrease very slowly. Therefore, the system as a whole may stabilize after a while.

Case 4: High values for both $\Rightarrow$ Highly volatile dynamic system $\gamma = 0.2$ and $\delta = 0.2$.

This could cause sharp oscillations and possibly even chaotic behavior due to the strong interaction between y and z . See Fig. (4)

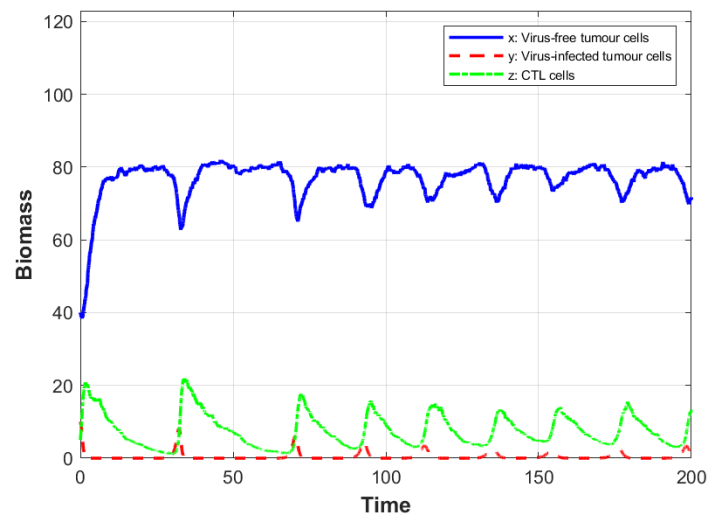


Figure 4 $\gamma = 0.2$ and $\delta = 0.2$

Notes: In this case, the predator z consumes significantly and increases rapidly and the prey y is subjected to strong pressure. Sharp fluctuations or cycles may appear, possibly irregular (chaotic-like behavior).

Case 5: Predator extinction due to weak transformation and predation $\gamma = 0.01$ and $\delta = 0.001$. The predator does not eat much and does not change from the prey; this may lead to the extinction of z over time. See Fig. (5).

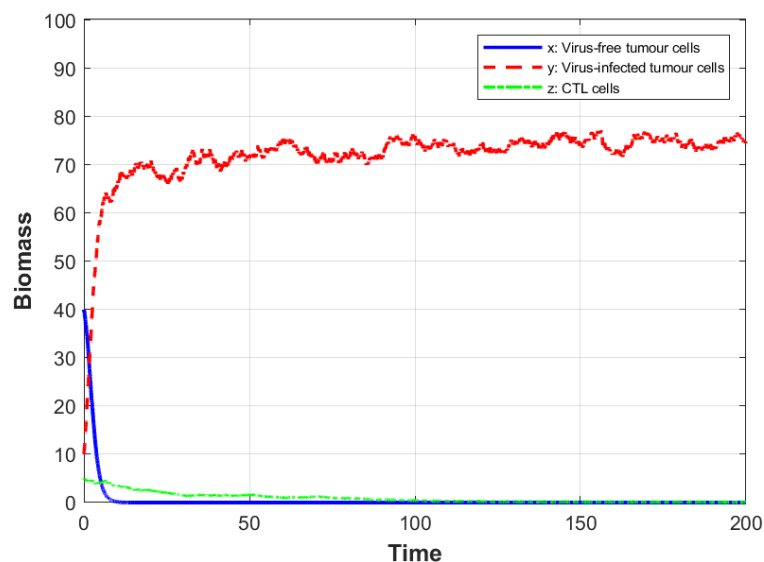


Figure 5 $\gamma = 0.01$ and $\delta = 0.001$

Over time, you will see that the number of predators z gradually decreases until it approaches zero. The reason is that: The predator does not consume much (γ is very small). Even if it is consumed, it does not result in new predators ($\delta \approx$ zero). The other factor contributing to the decrease in z is the death rate $b = 0.1$.

7. Discussion

This study presents a stochastic model of tumor–virus–immune interactions that incorporates environmental and physiological variability through white-noise perturbations. Key theorems demonstrate biological principles such as tumor carrying capacity and the feasibility of solutions, ensuring that cell populations evolve within physiological

limits despite stochastic effects. The model indicates that cancer cells do not grow indefinitely, viral therapy avoids numerical explosions, and environmental noise does not disrupt model dynamics. In oncolytic virotherapy, the model reliably predicts changes in tumor size, viral load, and immune activity, enhancing therapeutic dose estimation and response prediction. Additionally, the coexistence of uninfected and infected tumor cells with Cytotoxic T lymphocytes supports a state of "Chronic Tumor Control," aiming for stable tumor management rather than eradication, thus achieving a resilient biological homeostasis.

8. Conclusion

A stochastic model of tumor–virus–immune interactions integrates logistic tumor growth, viral dynamics, immune clearance, and environmental noise. The model ensures mathematical soundness with unique solutions under nonnegative initial conditions, bounded trajectories, and uniform continuity, supporting long-term dynamics. Stability analysis reveals that viral infection and immune activation affect tumor progression. The findings highlight the synergistic effects of oncolytic viruses and immune cells in tumor suppression while recognizing environmental fluctuations. This model provides a framework for exploring virotherapy strategies and future cancer treatments involving stochastic processes.

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