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SS-Injective Modules and Rings

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Abstract

We introduce and investigate SS-injectivity as a generalization of both soc-injectivity and small injectivity. A right module M over a ring R is said to be SS-N-injective (where N is a right R-module) if every R-homomorphism from a semisimple small submodule of N into M extends to N. A module M is said to be SS-injective (resp. strongly SS-injective), if M is SS-R-injective (resp. SS-N-injective for every right R-module N). Some characterizations and properties of (strongly) SS-injective modules and rings are given. Some results on soc-injectivity are extended to SS-injectivity.

Key words and phrases: Small Injective rings (modules); Soc-Injective rings (modules); SS-Injective rings (modules); Perfect rings; quasi-Frobenius rings.

Mathematics subject classification: Primary: 16D50, 16D60, 16D80; Secondary: 16P20, 16P40, 16L60.

1. Introduction

Throughout this paper, R is an associative ring with identity, and all modules are unitary right R-modules. For a right R-module M, we write $\mathrm{soc}(M)$, $J(M)$, $Z(M)$, $Z_2(M)$, $E(M)$ and $\mathrm{End}(M)$ for the socle, the Jacobson radical, the singular submodule, the second singular submodule, the injective hull and the endomorphism ring of M, respectively. Also, we use S_r , S_ℓ , Z_r , Z_ℓ , Z_2^r and J to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, the right second singular ideal, and the Jacobson radical of R, respectively. For a submodule N of M, we write $N \subseteq^{ess} M$, $N \ll M$, $N \subseteq^{\oplus} M$, and $N \subseteq^{max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand, and a maximal submodule of M, respectively. If X is a subset of a right R-module M. The right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp. $l_R(X)$). If $M = R$, we write $r_R(X) = r(X)$ and $l_R(X) = l(X)$.

Let M and N be right R-modules, M is called soc-N-injective if every R-homomorphism from the $\mathrm{soc}(N)$ into M extends to N. A right R-module M is called soc-injective, if M is soc-R-injective. A right R-module M is called strongly soc-injective, if M is soc-N-injective for all right R-module N [1].

Recall that a right R-module M is called mininjective [2] (resp. small injective [3], principally small injective [4]) if every R-homomorphism from any simple (resp. small, principally small) right ideal to M extends to R. A ring R is called right mininjective (resp. small injective, principally small injective) ring, if it is right mininjective (resp. small injective, principally small injective) as right R-module. A ring R is called right Kasch if every simple right R-module embeds in R (see for example [5]). Recall that a ring R is called semilocal if R/J is a semisimple [6]. Also, a ring R is said to be right perfect if every right R-module has projective cover. Recall that a ring R is said to be quasi-Frobenius (or QF) ring if it is right (or left) artinian and right (or left) self-injective; or equivalently, every injective right R-module is projective.

In this paper, we introduce and investigate the notions of SS-injective and strongly SS-injective modules and rings. Examples are given to show that the (strong) SS-injectivity is distinct from that of mininjectivity, principally small injectivity, small injectivity, simple J-injectivity, and (strong) soc-injectivity. Some characterizations and properties of (strongly) SS-injective modules and rings are given.

In Section 2, we give some basic properties of SS-injective modules. For examples, we prove that a ring R is right universally mininjective if and only if every simple right ideal is SS-injective. We also prove that if M is projective right R-module, then every quotient of an SS-M-injective right R-module is SS-M-injective if and only if $\mathrm{soc}(M) \cap J(M)$ is projective. We show that if every simple singular right R-module is SS-injective, then S_r is projective and $r(a) \subseteq^{\oplus} R_R$ for all $a \in S_r \cap J$.

In Section 3, we show that a right R-module M is strongly SS-injective if and only if every small submodule A of a right R-module N, every R-homomorphism $\alpha: A \longrightarrow M$ with $\alpha(A)$ semisimple extends to N. In particular, R is semiprimitive if every simple right R-module is strongly SS-injective, but not conversely. We also prove that if R is a right perfect ring, then a right R-module M is strongly soc-injective if and only if M is strongly SS-injective. A results ([1, Theorem 3.6 and Proposition 3.7]) are extended. We prove that a ring R is right artinian if and only if every direct sum of strongly SS-injective right R-modules is injective, and R is QF ring if and only if every strongly SS-injective right R-module is projective.

In Section 4, we extend the results ([1, Proposition 4.6 and Theorem 4.12]) from a soc-injective ring to an SS-injective ring (see Proposition 4.9 and Corollary 4.10).

In Section 5, we show that a ring R is QF if and only if it is strongly SS-injective and right noetherian with essential right socle if and only if it is strongly SS-injective, $l(J^2)$ is countable generated left ideal, $S_r \subseteq^{ess} R_R$, and the chain $r(x_1) \subseteq r(x_2 x_1) \subseteq \cdots \subseteq r(x_n x_{n-1} \ldots x_1) \subseteq \cdots$ terminates for every infinite sequence $x_1, x_2, \ldots$ in R (see Theorem 5.9 and Theorem 5.11). Finally, we prove that a ring R is QF if and only if R is strongly left and right SS-injective, left Kasch, and J is left t-nilpotent (see Theorem 5.14), extending a result of I. Amin, M. Yousif and N. Zeyada [1, Propostion 5.8] on strongly soc-injective rings.

General background material can be found in [7], [8] and [9].

2. SS-Injective Modules

Definition 2.1. Let N be a right R-module. A right R-module M is said to be SS-N-injective, if for any semisimple small submodule K of N, any right R-homomorphism $f: K \longrightarrow M$ extends to N. A module M is said to be SS-quasi-injective if M is SS-M-injective. M is said to be SS-injective if M is SS-R-injective. A ring R is said to be right SS-injective if the right R-module R_R is SS-injective.

Definition 2.2. A right R-module M is said to be strongly SS-injective if M is SS-N-injective, for all right R-module N. A ring R is said to be strongly right SS-injective if the right R-module R_R is strongly SS-injective.

Example 2.3.

(1) Every soc-injective module is SS-injective, but not conversely (see Example 5.7).

(2) Every small injective module is SS-injective, but not conversely (see Example 5.5).

(3) Every $\mathbb{Z}$-module is SS-injective. In fact, if M is a $\mathbb{Z}$-module, then M is small injective (by [3, Theorem 2.8) and hence it is SS-injective.

(4) The two classes of principally small injective rings and SS-injective rings are different (see [5, Example 5.2], Example 4.4 and Example 5.5).

(5) Every strongly soc-injective module is strongly SS-injective, but not conversely (see Example 5.7).

(6) Every strongly SS-injective module is SS-injective, but not conversely (see Example 5.6).

Theorem 2.4. The following statements hold:

(1) Let N be a right R-module and let $\{M_i : i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i \in I} M_i$ is SS-N-injective if and only if each M_i is SS-N-injective, $i \in I$.

(2) Let M, N and K be right R-modules with $K \subseteq N$. If M is SS-N-injective, then M is SS-K-injective.

(3) Let M, N and K be right R-modules with $M \cong N$. If M is SS-K-injective, then N is SS-K-injective.

(4) Let M, N and K be right R-modules with $K \cong N$ and M is SS-K-injective. Then M is SS-N-injective.

(5) Let M, N and K be right R-modules with N is a direct summand of M. If M is SS-K-injective, then N is SS-K-injective.

Proof. Clear. ■

Corollary 2.5.

(1) If N is a right R-module, then a finite direct sum of SS-N-injective modules is again SS-N-injective. Moreover, a finite direct sum of SS-injective (resp. strongly SS-injective) modules is again SS-injective (resp. strongly SS-injective).

(2) A direct summand of an SS-quasi-injective (resp., SS-injective, strongly SS-injective) module is again SS-quasi-injective (resp., SS-injective, strongly SS-injective).

Proof. (1) Take the index I to be a finite set and apply Theorem 2.4 (1).

(2) This follows from Theorem 2.4 (5). ■

Proposition 2.6. Every SS-injective right R-module is a right mininjective.

Proof. Let I be a simple right ideal of R. By [10, Lemma 3.8, p. 29] we have that either I is nilpotent or a direct summand of R. If I is a nilpotent, then $I \subseteq J$ by [11, Corollary 6.2.8, p. 181] and hence I is a simple small right ideal of R. Thus every SS-injective right R-module is right mininjective. ■

It easy to prove the following proposition.

Proposition 2.7. Let N be a right R-module. If $J(N)$ is a small submodule of N, then a right R-module M is SS-N-injective if and only if any R-homomorphism $f: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ extends to N.

Proposition 2.8. Let N be a right R-module and $\{A_i : i = 1, 2, \dots, n\}$ be a family of finitely generated right R-modules. Then N is SS-$\bigoplus_{i=1}^{n} A_i$-injective if and only if N is SS-A_i-injective, for all $i = 1, 2, \dots, n$.

Proof. ($\Rightarrow$) This follows from Theorem 2.4 ((2), (4)).
($\Leftarrow$) By [12, Proposition (I.4.1) and Proposition (I.1.2), p. 28 and 16] we have $\mathrm{soc}(\bigoplus_{i=1}^{n} A_i) \cap J(\bigoplus_{i=1}^{n} A_i) = (\mathrm{soc} \cap J)(\bigoplus_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} (\mathrm{soc} \cap J)(A_i) = \bigoplus_{i=1}^{n} \left(\mathrm{soc}(A_i) \cap J(A_i)\right)$. For $j = 1, 2, \dots, n$ consider the following diagram:

$$\begin{array}{ccc} K_j = \mathrm{soc}(A_j) \cap J(A_j) & \xrightarrow{\;i_2\;} & A_j \\ \downarrow{\scriptstyle i_{K_j}} & & \downarrow{\scriptstyle i_{A_j}} \\ \bigoplus_{i=1}^{n} \left(\mathrm{soc}(A_i) \cap J(A_i)\right) & \xrightarrow{\;i_1\;} & \bigoplus_{i=1}^{n} A_i \\ \downarrow{\scriptstyle f} & & \\ N & & \end{array}$$

where i_1, i_2 are inclusion maps and i_{K_j}, i_{A_j} are injection maps. By hypothesis, there exists an R-homomorphism $h_j : A_j \longrightarrow N$ such that $h_j i_2 = f i_{K_j}$, also there exists exactly one R-homomorphism $h: \bigoplus_{i=1}^{n} A_i \longrightarrow N$ satisfying $h_j = h i_{A_j}$ by [8, Theorem 4.1.6 (2), p. 83]. Thus $f i_{K_j} = h_j i_2 = h i_{A_j} i_2 = h i_1 i_{K_j}$ for all $j = 1, 2, \dots, n$. Let $(a_1, a_2, \dots, a_n) \in \bigoplus_{i=1}^{n} \left(\mathrm{soc}(A_i) \cap J(A_i)\right)$, thus $a_j \in \mathrm{soc}(A_j) \cap J(A_j)$, for all $j = 1, 2, \dots, n$, and
$f((a_1, a_2, \dots, a_n)) = f\left(i_{K_1}(a_1)\right) + f\left(i_{K_2}(a_2)\right) + \cdots + f\left(i_{K_n}(a_n)\right) = (h i_1)((a_1, a_2, \dots, a_n))$. Thus $f = h i_1$ and the proof is complete. ■

Corollary 2.9.

(1) Let $1 = e_1 + e_2 + \cdots + e_n$ in R, where the e_i are orthogonal idempotents. Then M is SS-injective if and only if M is SS-e_iR-injective for every $i = 1, 2, \dots, n$.

(2) For idempotents e and f of R. If $eR \cong fR$ and M is SS-eR-injective, then M is SS-fR-injective.

Proof. (1) From [7, Corollary 7.3, p. 96], we have $R = \bigoplus_{i=1}^{n} e_i R$, thus it follows from Proposition 2.8 that M is SS-injective if and only if M is SS-e_iR-injective for all $1 \le i \le n$.
(2) This follows from Theorem 2.4 (4). ■

Corollary 2.10. A right R-module M is SS-injective if and only if M is SS-P-injective, for every finitely generated projective right R-module P.

Proof. By Proposition 2.8 and Theorem 2.4 ((2), (4)). ■

Proposition 2.11. The following statements are equivalent for a right R-module M:

(1) Every right R-module is SS-M-injective.

(2) Every simple submodule of M is SS-M-injective.

(3) $\mathrm{soc}(M) \cap J(M) = 0$.

Proof. (1)⇒(2) and (3)⇒(1) are obvious.
(2)⇒(3) Assume that $\mathrm{soc}(M) \cap J(M) \neq 0$, thus $\mathrm{soc}(M) \cap J(M) = \bigoplus_{i \in I} x_i R$ where $x_i R$ is a simple small submodule of M, for each $i \in I$. Therefore $x_i R$ is SS-M-injective for each $i \in I$ by hypothesis. For any $i \in I$, the inclusion map from $x_i R$ to M is split, so we have that $x_i R \subseteq^{\oplus} M$. Since $x_i R$ is small submodule of M, it follows that $x_i R = 0$ and hence $x_i = 0$ for all $i \in I$ and this a contradiction. ■

A ring R is called right universally mininjective ring if it is satisfies the condition $S_r \cap J = 0$ (see for example [2, Lemma 5.1]).

Corollary 2.12. The following statements are equivalent for a ring R:

(1) R is right universally mininjective.

(2) Every right R-module is SS-injective.

(3) Every simple right ideal is SS-injective.

Proof. By Proposition 2.11. ■

Theorem 2.13. (SS-Baer's condition) The following statement are equivalent for a ring R:

(1) M is an SS-injective right R-module.

(2) If $S_r \cap J = A \oplus B$, and $\alpha: A \longrightarrow M$ is an R-homomorphism, then there exists $m \in M$ such that $\alpha(a) = ma$ for all $a \in A$ and $mB = 0$.

Proof. Clear. ■

Theorem 2.14. If M is a projective right R-module, then the following statements are equivalent:

(1) Every quotient of an SS-M-injective right R-module is SS-M-injective.

(2) Every quotient of a soc-M-injective right R-module is SS-M-injective.

(3)

(4) Every quotient of an injective right R-module is SS-M-injective.

(5) Every sum of two SS-M-injective submodules of a right R-module is SS-M-injective.

(6) Every sum of two soc-M-injective submodules of a right R-module is SS-M-injective.

(7) Every sum of two injective submodules of a right R-module is SS-M-injective.

(8) Every semisimple small submodule of M is projective.

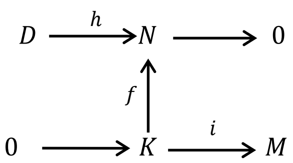
(9) Every simple small submodule of M is projective.

(10) $\mathrm{soc}(M) \cap J(M)$ is projective.

Proof. (1)⇒(2)⇒(3), (4)⇒(5)⇒(6) and (9)⇒(7)⇒(8) are obvious.

(8)⇒(9) Since $\mathrm{soc}(M) \cap J(M)$ is a direct sum of simple submodules of M and since every simple in $J(M)$ is small in M, thus $\mathrm{soc}(M) \cap J(M)$ is projective.

(3)⇒(7) Let D and N be right R-modules and consider the diagram:



where K is a semisimple small submodule of M, h is a right R-epimorphism, f is a right R-homomorphism, and i is the inclusion map. We can take D to be injective R-module (by [13, Proposition 5.2.10, p. 148]). Since N is SS-M-injective, then we can extend f to an R-homomorphism $\alpha: M \longrightarrow N$. By projectivity of M, thus α can be lifted to an R-homomorphism $\tilde{\alpha}: M \longrightarrow D$ such that $h\tilde{\alpha} = \alpha$. Let $\tilde{f}: K \longrightarrow D$ be the restriction of $\tilde{\alpha}$ over K. Obviously, $h\tilde{f} = f$ and this implies that K is projective.

(7)⇒(1) Let $h: N \longrightarrow L$ be an R-epimorphism, where N and L are right R-modules, and N is SS-M-injective. Let K be any semisimple small submodule of M, $f: K \longrightarrow L$ be any R-homomorphism, and i is the inclusion map. By hypothesis, K is projective, thus f can be lifted to R-homomorphism $g: K \longrightarrow N$ such that $hg = f$. Since N is SS-M-injective, then there exists R-homomorphism $\tilde{g}: M \longrightarrow N$ such that $\tilde{g}i = g$. Put $\beta = h\tilde{g}: M \longrightarrow L$. Thus $\beta i = h\tilde{g}i = hg = f$. Hence L is an SS-M-injective right R-module.

(1)⇒(4) Let N_1 and N_2 be two SS-M-injective submodules of a right R-module N. Then $N_1 + N_2$ is a homomorphic image of the direct sum $N_1 \oplus N_2$. Since $N_1 \oplus N_2$ is SS-M-injective, thus $N_1 + N_2$ is SS-M-injective by hypothesis.

(6)⇒(3) Let E be an injective right R-module and $N \hookrightarrow E$. Let $Q = E \oplus E, K = \{(n,n) \mid n \in N\}, \bar{Q} = Q/K, H_1 = \{y + K \in \bar{Q} \mid y \in E \oplus 0\}$ and $H_2 = \{y + K \in \bar{Q} \mid y \in 0 \oplus E\}$. Then $\bar{Q} = H_1 + H_2$. Since $(E \oplus 0) \cap K = 0$ and $(0 \oplus E) \cap K = 0$, thus $E \cong H_i$, $i = 1, 2$. Since $H_1 \cap H_2 = \{y + K \in \bar{Q} \mid y \in N \oplus 0\} = \{y + K \in \bar{Q} \mid y \in 0 \oplus N\}$, thus $H_1 \cap H_2 \cong N$ under $y \mapsto y + K$ for all $y \in N \oplus 0$. By hypothesis, $\bar{Q}$ is SS-M-injective. Since H_1 is injective, thus $\bar{Q} = H_1 \oplus A$ for some $A \hookrightarrow \bar{Q}$, so $A \cong (H_1 + H_2)/H_1 \cong H_2/(H_1 \cap H_2) \cong E/N$. By Theorem 2.4 (5), E/N is SS-M-injective. ■

Corollary 2.15. The following statements are equivalent for a ring R:

(1) Every quotient of an SS-injective right R-module is SS-injective.

(2) Every quotient of a soc-injective right R-module is SS-injective.

(3) Every quotient of a small injective right R-module is SS-injective.

(4) Every quotient of an injective right R-module is SS-injective.

(5) Every sum of two SS-injective submodules of any right R-module is SS-injective.

(6) Every sum of two soc-injective submodules of any right R-module is SS-injective.

(7) Every sum of two small injective submodules of any right R-module is SS-injective.

(8) Every sum of two injective submodules of any right R-module is SS-injective.

(9) Every semisimple small submodule of any projective right R-module is projective.

(10) Every semisimple small submodule of any finitely generated projective right R-module is projective.

(11) Every semisimple small submodule of R_R is projective.

(12) Every simple small submodule of R_R is projective.

(13) $S_r \cap J$ is projective.

(14) S_r is projective (R is a right PS-ring).

Proof. The equivalence between (1), (2), (4), (5), (6), (8), (11), (12) and (13) is from Theorem 2.14.

(1)⇒(3)⇒(4), (5)⇒(7)⇒(8) and (9)⇒(10)⇒(13) are clear.

(14)⇒(9) By [1, Corollary 2.9].

(13)⇒(14) Let $S_r = (S_r \cap J) \oplus A$, where $A = \oplus_{i \in I} S_i$ and S_i is a right simple and direct summand of R_R, for all $i \in I$. Thus A is projective, but $S_r \cap J$ is projective, so it follows that S_r is projective. ■

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Theorem 2.16. If every simple singular right R-module is SS-injective, then $r(a) \subseteq^{\oplus} R_R$ for every $a \in S_r \cap J$ and S_r is projective.

Proof. Let $a \in S_r \cap J$ and let $A = RaR + r(a)$. Thus there exists $B \hookrightarrow R_R$ such that $A \oplus B \subseteq^{ess} R_R$. Assert that $A \oplus B \neq R_R$, then we find $I \subseteq^{max} R_R$ such that $A \oplus B \subseteq I$, and so $I \subseteq^{ess} R_R$. By hypothesis, R/I is SS-injective. Consider the map $\alpha: aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ which is well define R-homomorphism. Thus, there exists $c \in R$ with $1 + I = ca + I$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$ which leads to $1 \in I$, a contradiction. Thus $A \oplus B = R_R$ and hence $RaR + (r(a) \oplus B) = R$. Since $RaR \ll R_R$, then $r(a) \subseteq^{\oplus} R_R$. Put $r(a) = (1-e)R$, for some $e^2 = e \in R$, so it follows that $ax = aex$ (because $(1-e)x \in r(a)$, and so $a(1-e)x = 0$) for all $x \in R$ and this leads to $aR = aeR$. Let $\gamma: eR \longrightarrow aeR$ be defined by $\gamma(er) = aer$ for all $r \in R$. Then γ is a well defined R-epimorphism. Clearly, $\ker(\gamma) = \{er: aer = 0\} = \{er: er \in r(a)\} = eR \cap r(a) = 0$. Hence γ is an isomorphism and so aR is projective. Since $S_r \cap J$ is a direct sum of simple small right ideals, thus $S_r \cap J$ is projective and it follows from Corollary 2.15 that S_r is projective. ■

Corollary 2.17. A ring R is right mininjective and every singular simple right R-module is SS-injective if and only if R is a right universally mininjective.

Proof. By Theorem 2.16 and [2, Lemma 5.1]. ■

Recall that a ring R is called zero insertive if $aRb = 0$ for all $a, b \in R$ with $ab = 0$ (see [3]). Note that if R is zero insertive ring, then $RaR + r(a) \subseteq^{ess} R_R$ for every $a \in R$ (see [3, Lemma 2.11]).

Proposition 2.18. Let R be a zero insertive ring. If every simple singular right R-module is SS-injective, then R is right universally mininjective.

Proof. Let $a \in S_r \cap J$. We claim that $RaR + r(a) = R$, thus $r(a) = R$ (since $RaR \ll R$), so $a = 0$ and this means that $S_r \cap J = 0$. Otherwise, if $RaR + r(a) \subsetneq R$, then there exists a maximal right ideal I of R such that $RaR + r(a) \subseteq I$. Since $I \subseteq^{ess} R_R$ by Lemma 2.1.22, then R/I is SS-injective by hypothesis. Consider $\alpha: aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ for all $r \in R$ which is well defined R-homomorphism. Thus $1 + I = ca + I$ for some $c \in R$. Since $ca \in RaR \subseteq I$, then $1 \in I$ and this contradicts the maximality of I, so we must have $RaR + r(a) = R$ and this ends the proof. ■

Theorem 2.19. If M is a finitely generated right R-module, then the following statements are equivalent:

(1) $\mathrm{soc}(M) \cap J(M)$ is a noetherian R-module.

(2) $\mathrm{soc}(M) \cap J(M)$ is finitely generated.

(3) Any direct sum of SS-M-injective right R-modules is SS-M-injective.

(4) Any direct sum of soc-M-injective right R-modules is SS-M-injective.

(5) Any direct sum of injective right R-modules is SS-M-injective.

(6) $K^{(S)}$ is SS-M-injective for every injective right R-module K and for any index set S.

(7) $K^{(\mathbb{N})}$ is SS-M-injective for every injective right R-module K.

Proof. (1)⇒(2) and (3)⇒(4)⇒(5)⇒(6)⇒(7) Clear.

(2)⇒(3) Let $E = \bigoplus_{i \in I} M_i$ be a direct sum of SS-M-injective right R-modules and $f: N \longrightarrow E$ be a right R-homomorphism where N is a semisimple small submodule of M. Since $\mathrm{soc}(M) \cap J(M)$ is finitely generated, thus N is finitely generated and hence $f(N) \subseteq \bigoplus_{i \in I_1} M_i$, for a finite subset I_1 of I. Since a finite direct sums of SS-M-injective right R-modules is SS-M-injective, thus $\bigoplus_{i \in I_1} M_i$ is SS-M-injective and hence f can be extended to an R-homomorphism $g: M \longrightarrow E$. Thus E is SS-M-injective.

(7)⇒(1) Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of submodules of $\mathrm{soc}(M) \cap J(M)$. For each $i \geq 1$, let $E_i = E(M/N_i)$ and $E = \bigoplus_{i=1}^{\infty} E_i$. For every $i \geq 1$, we put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1 \\ j \neq i}}^{\infty} E_j \right)$, then M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus \left(\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} E_j \right)$ is SS-M-injective, so it follows from Theorem 2.4 (5) that E is SS-M-injective. Define $f: U = \bigcup_{i=1}^{\infty} N_i \longrightarrow E$ by $f(m) = (m + N_i)_i$. It is clear that f is a well defined R-homomorphism. Since M is finitely generated, thus $\mathrm{soc}(M) \cap J(M)$ is a semisimple small submodule of M and hence $\bigcup_{i=1}^{\infty} N_i$ is a semisimple small submodule of M, so f can be extended to a right R-homomorphism $g: M \longrightarrow E$. Since M is finitely generated, then we have $g(M) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$ for some n and hence $f(U) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$. Since $\pi_i f(x) = \pi_i \left((x + N_j)_{j \geq 1} \right) = x + N_i$ for all $x \in U$ and $i \geq 1$, where $\pi_i: \bigoplus_{j \geq 1} E(M/N_i) \longrightarrow E(M/N_i)$ be the projection map. Thus $\pi_i f(U) = U/N_i$ for all $i \geq 1$. Since $f(U) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$. Thus $U/N_i = \pi_i f(U) = 0$, for all $i \geq n+1$, so $U = N_i$ for all $i \geq n+1$ and hence the chain $N_1 \subseteq N_2 \subseteq \cdots$ terminates at N_{n+1}. Thus $\mathrm{soc}(M) \cap J(M)$ is a noetherian R-module. ■

Corollary 2.20. If N is a finitely generated right R-module, then the following statements are equivalent:

(1) $\mathrm{soc}(N) \cap J(N)$ is finitely generated.

(2) $M^{(S)}$ is SS-N-injective for every soc-N-injective right R-module M and for any index set S.

(3) $M^{(S)}$ is SS-N-injective for every SS-N-injective right R-module M and for any index set S.

(4) $M^{(\mathbb{N})}$ is SS-N-injective for every soc-N-injective right R-module M.

(5) $M^{(\mathbb{N})}$ is SS-N-injective for every SS-N-injective right R-module M.

Proof. By Theorem 2.19. ■

Corollary 2.21. The following statements are equivalent :

(1) $S_r \cap J$ is finitely generated.

(2) Any direct sum of SS-injective right R-modules is SS-injective.

(3) Any direct sum of soc-injective right R-modules is SS-injective.

(4) Any direct sum of small injective right R-modules is SS-injective.

(5) Any direct sum of injective right R-modules is ss-injective.

(6) $M^{(S)}$ is SS-injective for every injective right R-module M and for any index set S.

(7) $M^{(S)}$ is SS-injective for every soc-injective right R-module M and for any index set S.

(8) $M^{(S)}$ is SS-injective for every small injective right R-module M and for any index set S.

(9) $M^{(S)}$ is SS-injective for every SS-injective right R-module M and for any index set S.

(10) $M^{(\mathbb{N})}$ is SS-injective for every injective right R-module M.

(11) $M^{(\mathbb{N})}$ is SS-injective for every soc-injective right R-module M.

(12) $M^{(\mathbb{N})}$ is SS-injective for every small injective right R-module M.

(13) $M^{(\mathbb{N})}$ is SS-injective for every SS-injective right R-module M.

Proof. By applying Theorem 2.19 and Corollary 2.20. ■

3. Strongly SS-Injective Modules

Proposition 3.1. A right R-module M is a strongly SS-injective if and only if every R-homomorphism $\alpha: A \longrightarrow M$ extends to N, for all right R-module N, where $A \ll N$ and $\alpha(A)$ is a semisimple submodule in M.

Proof. (⇐) Clear.

(⇒) Let A be a small submodule of N, and $\alpha: A \longrightarrow M$ be an R-homomorphism with $\alpha(A)$ is a semisimple submodule of M. If $B = \ker(\alpha)$, then α induces an R-homomorphism $\tilde{\alpha}: A/B \longrightarrow M$ defined by $\tilde{\alpha}(a+B) = \alpha(a)$, for all $a \in A$. Clearly, $\tilde{\alpha}$ is well define because if $a_1 + B = a_2 + B$ we have $a_1 - a_2 \in B$, so $\alpha(a_1) = \alpha(a_2)$, that is $\tilde{\alpha}(a_1 + B) = \tilde{\alpha}(a_2 + B)$. Since M is strongly SS-injective and A/B is semisimple and small in N/B, thus $\tilde{\alpha}$ extends to an R-homomorphism $\gamma: N/B \longrightarrow M$. If $\pi: N \longrightarrow N/B$ is the canonical map, then the R-homomorphism $\beta = \gamma\pi: N \longrightarrow M$ is an extension of α such that if $a \in A$, then $\beta(a) = (\gamma\pi)(a) = \gamma(a+B) = \tilde{\alpha}(a+B) = \alpha(a)$as desired. ■

Corollary 3.2.

(1) Let M be a semisimple right R-module. If M is a strongly SS-injective, then M is a small injective.

(2) If every simple right R-module is strongly SS-injective, then R is a semiprimitive ring.

Proof. (1) By Proposition 3.1.
(2) By (1) and applying [3, Theorem 2.8]. ■

Remark 3.3. The converse of Corollary 3.2 is not true (see Example 3.8).

Theorem 3.4. If M is a strongly SS-injective (or just SS-$E(M)$-injective) right R-module, then for every semisimple small submodule A of M, there is an injective R-module E_A such that $M = E_A \oplus T_A$ where $T_A \hookrightarrow M$ with $T_A \cap A = 0$. Moreover, if $A \neq 0$, then E_A can be taken $A \subseteq^{ess} E_A$.

Proof. Let A be a semisimple small submodule of M. If $A = 0$, we end the proof by taking $E_A = 0$ and $T_A = M$. Suppose that $A \neq 0$ and let i_1, i_2 and i_3 be inclusion maps and $D_A = E(A)$ is the injective hull of A in $E(M)$. Since M is strongly SS-injective, thus M is SS-$E(M)$-injective. Since A is a semisimple small submodule of M, so it follows from [8, Lemma 5.1.3 (a)] that A is a semisimple small submodule in $E(M)$ and hence there exists an R-homomorphism $\alpha: E(M) \longrightarrow M$ such that $\alpha i_2 i_1 = i_3$. Put $\beta = \alpha i_2: D_A \longrightarrow M$, thus β is an extension of i_3. Since $A \subseteq^{ess} D_A$, β is an R-monomorphism. Put $E_A = \beta(D_A)$. Since E_A is an injective submodule of M, thus $M = E_A \oplus T_A$ for some $T_A \hookrightarrow M$. Since $\beta(A) = A$, $A \subseteq \beta(D_A) = E_A$ and this means that $T_A \cap A = 0$. Moreover, define $\tilde{\beta} = \beta: D_A \longrightarrow E_A$, thus $\tilde{\beta}$ is an isomorphism. Since $A \subseteq^{ess} D_A$, thus $\tilde{\beta}(A) \subseteq^{ess} E_A$. But $\tilde{\beta}(A) = \beta(A) = A$, so $A \subseteq^{ess} E_A$. ■

Corollary 3.5. If M is a right R-module has a semisimple small submodule A such that $A \subseteq^{ess} M$, then the following statements are equivalent:

(1) M is injective.

(2) M is strongly SS-injective.

(3) M is SS-$E(M)$-injective.

Proof. (1)⇒(2) and (2)⇒(3) are obvious.
(3)⇒(1) By Theorem 3.4, we can write $M = E_A \oplus T_A$ where E_A injective and $T_A \cap A = 0$. Since $A \subseteq^{ess} M$, thus $T_A = 0$ and hence $M = E_A$. Therefore M is an injective R-module. ■

Example 3.6. $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly SS-injective. In particular, $\mathbb{Z}_4$ is not SS-$\mathbb{Z}_{2^\infty}$-injective.

Proof. Assume that $\mathbb{Z}_4$ is strongly SS-injective $\mathbb{Z}$-module. Let $A = <\bar{2}> = \{\bar{0}, \bar{2}\}$. It is clear that A is a semisimple small and essential submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module. By Corollary 3.5, $\mathbb{Z}_4$ is injective $\mathbb{Z}$-module and this a contradiction. Thus $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly SS-injective. Moreover, Since $E(\mathbb{Z}_{2^2}) = \mathbb{Z}_{2^\infty}$ as $\mathbb{Z}$-module, thus $\mathbb{Z}_4$ is not SS-$\mathbb{Z}_{2^\infty}$-injective, by Corollary 3.5. ■

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Corollary 3.7. Let M be a right R-module such that $\mathrm{soc}(M) \cap J(M) \ll M$ (in particular, if M is finitely generated). If M is strongly SS-injective, then $M = E \oplus T$, where E is injective and $T \cap \mathrm{soc}(M) \cap J(M) = 0$. Moreover, if $\mathrm{soc}(M) \cap J(M) \neq 0$, then we can take $\mathrm{soc}(M) \cap J(M) \subseteq^{ess} E$.

Proof. By taking $A = \mathrm{soc}(M) \cap J(M)$ and applying Theorem 3.4. ■

The following example shows that the converse of Theorem 3.4 and Corollary 3.7 is not true.

Example 3.8. Let $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module. Since $J(M) = 0$ and $\mathrm{soc}(M) = M$, thus $\mathrm{soc}(M) \cap J(M) = 0$. So, we can write $M = 0 \oplus M$ with $M \cap \big(\mathrm{soc}(M) \cap J(M)\big) = 0$. Let $N = \mathbb{Z}_8$ as $\mathbb{Z}$-module. Since $J(N) =< \bar{2} >$ and $\mathrm{soc}(N) =< \bar{4} >$. Define $\gamma: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ by $\gamma(\bar{4}) = \bar{3}$, thus γ is a $\mathbb{Z}$-homomorphism. Assume that M is strongly SS-injective, thus M is SS-N-injective, so there exists $\mathbb{Z}$-homomorphism $\beta: N \longrightarrow M$ such that $\beta \circ i = \gamma$, where i is the inclusion map from $\mathrm{soc}(N) \cap J(N)$ to N. Since $\beta\big(J(N)\big) \subseteq J(M)$, thus $\bar{3} = \gamma(\bar{4}) = \beta(\bar{4}) \in \beta\big(J(N)\big) \subseteq J(M) = 0$ and this contradiction, so M is not strongly SS-injective $\mathbb{Z}$-module.

Corollary 3.9. The following statements are equivalent:

(1) $\mathrm{soc}(M) \cap J(M) = 0$, for all right R-module M.

(2) Every right R-module is strongly SS-injective.

(3) Every simple right R-module is strongly SS-injective.

Proof. By Proposition 2.11. ■

Lemma 3.10. Let M and C be right R-modules and $N \hookrightarrow M$ with M/N is a semisimple. Then every R-homomorphism from a submodule (resp. semisimple submodule) A of M to C can be extended to an R-homomorphism from M to C if and only if every R-homomorphism from a submodule (resp. semisimple submodule) B of N to C can be extended to an R-homomorphism from M to C.

Proof. (⇒) is obtained directly.

(⇐) let f be an R-homomorphism from a submodule A of M to C. Since M/N is a semisimple, there exists $L \hookrightarrow M$ such that $A + L = M$ and $A \cap L \subseteq N$ (see [6, Proposition 2.1]). Thus there exists an R-homomorphism $g: M \longrightarrow C$ such that $g(x) = f(x)$ for all $x \in A \cap L$. Define $h: M \longrightarrow C$ such that for any $x = a + \ell$, $a \in A$, $\ell \in L$, $h(x) = f(a) + g(\ell)$. Thus h is a well define R-homomorphism, because if $a_1 + \ell_1 = a_2 + \ell_2$, $a_i \in A$, $\ell_i \in L$, $i = 1,2$, then $a_1 - a_2 = \ell_2 - \ell_1 \in A \cap L$, that is $f(a_1 - a_2) = g(\ell_2 - \ell_1)$ which leads to $h(a_1 + \ell_1) = h(a_2 + \ell_2)$. Therefore h is a well define R-homomorphism and extension of f. ■

Corollary 3.11. For right R-modules M and N, the following hold:

(1) If M is finitely generated and $M/J(M)$ is semisimple right R-module, then N is soc-M-injective if and only if N is SS-M-injective.

(2) If $M/\mathrm{soc}(M)$ is semisimple right R-module, then N is soc-M-injective if and only if N is M-injective.

(3) If R/S_r is semisimple as right R-module, then N is soc-injective if and only if N is injective.

(4) If R/S_r is semisimple as right R-module, then N is SS-injective if and only if N is small injective.

Proof. (1) (⇒) Clear.

(⇐) Since N is a right SS-M-injective, thus every R-homomorphism from a semisimple small submodule of M to N extends to M. Since M is finitely generated, thus $J(M) \ll M$ and hence every R-homomorphism from any semisimple submodule of $J(M)$ to N extends to M. Since $M/J(M)$ is semisimple, thus every R-homomorphism from any semisimple submodule of M to N extends to M by Lemma 3.10. Therefore, N is soc-M-injective right R-module.

(2) (⇒) Since N is soc-M-injective. Thus every R-homomorphism from any submodule of $\mathrm{soc}(M)$ to N extends to M. Since $M/\mathrm{soc}(M)$ is semisimple, thus Lemma 3.10 implies that every R-homomorphism from any submodule of M to N extends to M. Hence N is M-injective.

(⇐) Clear.

(3) By (2).

(4) Since R/S_r is semisimple as right R-module, thus $J(R/S_r) = 0$. By [8, Theorem 9.1.4(b)], we have $J \subseteq S_r$ and hence $J = J \cap S_r$. Thus N is SS-injective if and only is N is small injective. ■

Corollary 3.12. Let R be a semilocal ring, then $S_r \cap J$ is finitely generated if and only if S_r is finitely generated.

Proof. Suppose that $S_r \cap J$ is finitely generated. By Corollary 2.21, every direct sum of soc-injective right R-modules is SS-injective. Thus it follows from Corollary 3.11 (1) and [1, Corollary 2.11] that S_r is finitely generated. ■

Theorem 3.13. If R is a right perfect ring, then M is a strongly soc-injective right R-module if and only if M is a strongly SS-injective.

Proof. (⇒) Clear.

(⇐) Let R be a right perfect ring and M be a strongly SS-injective right R-module. Since R is a semilocal ring, thus it follows from [14, Theorem 3.5] that every right R-module N is semilocal and hence $N/J(N)$ is semisimple right R-module. Since R is a right perfect ring, the Jacobson radical of every right R-module is small by [13, Theorem 4.3 and 4.4, p. 69]. Thus $N/J(N)$ is semisimple and $J(N) \ll N$, for any $N \in$ Mod-R. Since M is strongly SS-injective it follows Lemma 3.10 implies that M is strongly soc-injective. ■

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Corollary 3.14. A ring R is QF if and only if every strongly SS-injective right R-module is projective.

Proof. (⇒) If R is QF ring, then R is a right perfect ring, so by Theorem 3.13 and [1, Proposition 3.7] we have that every strongly SS-injective right R-module is projective.

(⇐) By hypothesis we have that every injective right R-module is projective and hence R is QF ring (see for instance [11, Proposition 12.5.13]). ■

Theorem 3.15. The following statements are equivalent for a ring R:

(1) Every direct sum of strongly SS-injective right R-modules is injective.

(2) Every direct sum of strongly soc-injective right R-modules is injective.

(3) R is right artinian.

Proof. (1)⇒(2) Clear.

(2)⇒(3) Since every direct sum of strongly soc-injective right R-modules is injective. Thus R is right noetherian and right semiartinian by [1, Theorem 3.3 and Theorem 3.6], so it follows from [15, Proposition VIII.5.2, p. 189] that R is right artinian.

(3)⇒(1) By hypothesis, R is right perfect and right noetherian. It follows from Theorem 3.13 and [1, Theorem 3.3] that every direct sum of strongly SS-injective right R-modules is strongly soc-injective. Since R is right semiartinian, so [1, Theorem 3.6] implies that every direct sum of strongly SS-injective right R-modules is injective. ■

Recall that a submodule K of a right R-module M is called t-essential in M (written $K \subseteq^{tes} M$) if for every submodule L of M, $K \cap L \subseteq Z_2(M)$ implies that $L \subseteq Z_2(M)$ (see [16]). A right R-module M is said to be t-semisimple if every submodule A of M there exists a direct summand B of M such that $B \subseteq^{tes} A$ (see [16]). A ring R is said to be right V-ring (GV-ring, SI-ring, respectively) if every simple (simple singular, singular, respectively) right R-module is injective. A right R-module is called strongly s-injective if every R-homomorphism from K to M extends to N for every right R-module N, where $K \subseteq Z(N)$ (see [17]). In the next results, we will give the connection between injectivity and strongly s-injectivity and we characterize V-rings, GV-rings, SI-rings and semisimple rings by this connection.

Theorem 3.16. If R is a right t-semisimple, then a right R-module M is injective if and only if M is strongly s-injective.

Proof. (⇒) Obvious.

(⇐) Let M be a strongly s-injective, $Z_2(M)$ is injective by [17, Proposition 3, p. 27]. Thus every R-homomorphism $f: K \longrightarrow M$, where $K \subseteq Z_2^r$ extends to R by [17, Lemma 1, p. 26]. Since R is a right t-semisimple, thus R/Z_2^r is a right semisimple by [16, Theorem 2.3]. So by applying Lemma 3.10, we conclude that M is injective. ■

Corollary 3.17. A ring R is right SI and right t-semisimple if and only if it is semisimple.

Proof. (⇒) Since R is a right SI-ring, thus every right R-module is strongly s-injective by [17, Theorem 1, p. 29]. By Theorem 3.16, we have that every right R-module is injective and hence R is semisimple ring.

(⇐) Clear. ■

Corollary 3.18. If R is a right t-semisimple ring. Then R is right V-ring if and only if R is right GV-ring.

Proof. By [17, Proposition 5, p. 28] and Theorem 3.16. ■

Corollary 3.19. If R is a right t-semisimple ring, then R/S_r is noetherian right R-module if and only if R is right noetherian.

Proof. If R/S_r is noetherian right R-module, then every direct sum of injective right R-modules is strongly s-injective by [17, Proposition 6]. Since R is right t-semisimple, so it follows from Theorem 3.16 that every direct sum of injective right R-modules is injective and hence R is right noetherian. The converse is clear. ■

4. SS-Injective Rings

We recall that the dual of a right R-module M is $M^d = Hom_R(M, R_R)$ and clearly that M^d is a left R-module.

Proposition 4.1. The following statements are equivalent for a ring R:

(1) R is a right SS-injective ring.

(2) If K is a semisimple right R-module, P and Q are finitely generated projective right R-modules, $\beta: K \longrightarrow P$ is an R-monomorphism with $\beta(K) \ll P$ and $f: K \longrightarrow Q$ is an R-homomorphism, then f can be extended to an R-homomorphism $h: P \longrightarrow Q$.

(3) If M be a right semisimple R-module and f is a nonzero R-monomorphism from M to R_R with $f(M) \ll R_R$, then $M^d = Rf$.

Proof. (2)⇒(1) Clear.

(1)⇒(2) Since Q finitely generated, there is an R-epimorphism $\alpha_1: R^n \longrightarrow Q$ for some $n \in \mathbb{Z}^+$. Since Q is a projective, there is an R-homomorphism $\alpha_2: Q \longrightarrow R^n$ such that $\alpha_1\alpha_2 = I_Q$. Define $\tilde{\beta}: K \longrightarrow \beta(K)$ by $\tilde{\beta}(a) = \beta(a)$ for all $a \in K$. Since R is a right SS-injective ring by hypothesis, it follows from Proposition 2.8 and Corollary 2.5 (1) that R^n is a right SS-P-injective R-module. So there exists an R-homomorphism $h: P \longrightarrow R^n$ such that $hi = \alpha_2 f \tilde{\beta}^{-1}$. Put $g = \alpha_1 h: P \longrightarrow Q$. Thus $gi = (\alpha_1 h)i = \alpha_1(\alpha_2 f \tilde{\beta}^{-1}) = f\tilde{\beta}^{-1}$ and hence $(g\beta)(a) = g\left(i(\beta(a))\right) = (f\tilde{\beta}^{-1})(\beta(a)) = f(a)$ for all $a \in K$. Therefore, there is an R-homomorphism $g: P \longrightarrow Q$ such that $g\beta = f$.

(1)⇒(3) Let $g \in M^d$, we have $gf^{-1}: f(M) \longrightarrow R_R$, since $f(M)$ is a semisimple small right ideal of R and R is a right SS-injective ring (by hypothesis), $gf^{-1} = a$. for some $a \in R$. Therefore, $g = af$ and hence $M^d = Rf$.
(3)⇒(1) Let $f: K \longrightarrow R$ be a right R-homomorphism, where K is a semisimple small right ideal of R and $i: K \longrightarrow R$ be the inclusion map, thus by (3) we have $K^d = Ri$ and hence $f = ci$ in K^d for some $c \in R$. Thus there is $c \in R$ such that $f(a) = ca$ for all $a \in K$ and this implies that R is a right SS-injective ring. ■

Example 4.2.

(1) Every universally mininjective ring is SS-injective, but not conversely (see Example 5.6).

(2) The two classes of universally mininjective rings and soc-injective rings are different (see Example 5.6 and Example 5.7).

Lemma 4.3. Let R be a right SS-injective ring. Then:

(1) R is a right mininjective ring.

(2) $lr(a) = Ra$ for all $a \in S_r \cap J$.

(3) $r(a) \subseteq r(b)$, $a \in S_r \cap J$, $b \in R$ implies $Rb \subseteq Ra$.

(4) $l\big(bR \cap r(a)\big) = l(b) + Ra$, for all $a \in S_r \cap J$, $b \in R$.

(5) $l(K_1 \cap K_2) = l(K_1) + l(K_2)$, for all semisimple small right ideals K_1 and K_2 of R.

Proof. Clear. ■

The following is an example of a right mininjective ring which is not right SS-injective.

Example 4.4. (The Björk Example [5, Example 2.5, p. 38]). Let F be a field and let $a \longmapsto \bar{a}$ be an isomorphism $F \longrightarrow \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. By [5, Example 2.5 and 5.2, p. 38 and 97] we have R is a right principally injective and local ring. It is mentioned in [1, Example 4.15], that R is not right soc-injective. Since R is local, thus by Corollary 3.11 (1), R is not right SS-injective ring.

Proposition 4.5. Let R be a right SS-injective ring. Then :

(1) If Ra is a simple left ideal of R, then $\text{soc}(aR) \cap J(aR)$ is zero or simple.

(2) $rl(S_r \cap J) = S_r \cap J$ if and only if $rl(N) = N$ for all semisimple small right ideals N of R.

Proof. (1) Suppose that $\text{soc}(aR) \cap J(aR)$ is a nonzero. Let x_1R and x_2R be any simple small right ideals of R with $x_i \in aR$, $i = 1, 2$. If $x_1R \cap x_2R = 0$, then by Lemma 4.3 (5), $l(x_1) + l(x_2) = R$. Since $x_i \in aR$, thus $x_i = ar_i$ for some $r_i \in R$, $i = 1, 2$, that is $l(a) \subseteq l(ar_i) = l(x_i)$, $i = 1, 2$. Since Ra is a simple, then $l(a) \subseteq^{max} R$, that is $l(x_1) = l(x_2) = l(a)$. Therefore, $l(a) = R$ and hence $a = 0$ and this contradicts the minimality of Ra. Thus $\text{soc}(aR) \cap J(aR)$ is simple.
(2) Suppose that $rl(S_r \cap J) = S_r \cap J$ and let N be a semisimple small right ideal of R, trivially we have $N \subseteq rl(N)$. If $N \cap xR = 0$ for some $x \in rl(N)$, then by Lemma 4.3 (5), $l(N \cap xR) = l(N) + l(xR) = R$, since $x \in rl(N) \subseteq rl(S_r \cap J) = S_r \cap J$. If $y \in l(N)$, then $yx = 0$, that is $y(xr) = 0$ for all $r \in R$ and hence $l(N) \subseteq l(xR)$. Thus $l(xR) = R$, so $x = 0$ and this means that $N \subseteq^{ess} rl(N)$. Since $N \subseteq^{ess} rl(N) \subseteq rl(S_r \cap J) = S_r \cap J$, it follows that $N = rl(N)$. The converse is trivial. ■

Recall that a right ideal I of R is said to be lie over summand of R_R, if there exists a direct decomposition $R_R = A_R \oplus B_R$ with $A \subseteq I$ and $B \cap I \ll R_R$ (see [18]) which leads to $I = A \oplus (B \cap I)$.

Lemma 4.6. Let K be an m-generated semisimple right ideal lies over summand of R_R. If R is a right SS-injective ring, then every R-homomorphism from K to R_R can be extended to an endomorphism of R_R.

Proof. Let $\alpha: K \longrightarrow R$ be a right R-homomorphism. By hypothesis, $K = eR \oplus B$, for some $e^2 = e \in R$, where B is an m-generated semisimple small right ideal of R. Now, we need to prove that $K = eR \oplus (1-e)B$. Clearly, $eR + (1-e)B$ is a direct sum. Let $x \in K$, then $x = a + b$, for some $a \in eR, b \in B$, so we can write $x = a + eb + (1-e)b$ and this implies that $x \in eR \oplus (1-e)B$. Conversely, let $x \in eR \oplus (1-e)B$. Thus $x = a + (1-e)b$, for some $a \in eR, b \in B$. We obtain $x = a + (1-e)b = (a - eb) + b \in eR \oplus B$. It is obvious that $(1-e)B$ is an m-generated semisimple small right ideal. Since R is a right SS-injective, then there exists $\gamma \in \text{End}(R_R)$ such that $\gamma_{|(1-e)B} = \alpha_{|(1-e)B}$. Define $\beta: R_R \longrightarrow R_R$ by $\beta(x) = \alpha(ex) + \gamma\big((1-e)x\big)$, for all $x \in R$ which is well defined R-homomorphism. If $x \in K$, then $x = a + b$ where $a \in eR$ and $b \in (1-e)B$, so $\beta(x) = \alpha(ex) + \gamma\big((1-e)x\big) = \alpha(a) + \gamma(b) = \alpha(a) + \alpha(b) = \alpha(x)$ which yields β is an extension of α. ■

Corollary 4.7. Let S_r be a finitely generated and lies over summand of R_R. Then R is a right SS-injective ring if and only if R is a right soc-injective .

Proof. By Lemma 4.6. ■

Recall that a ring R is called right minannihilator if $rl(K) = K$ for every simple right ideal K of R (see [2]) (equivalently, for every simple small right ideal K of R).

Corollary 4.8. For a right SS-injective ring R, the following hold:

(1) If $rl(S_r \cap J) = S_r \cap J$, then R is right minannihilator.

(2) If $S_\ell \subseteq S_r$, then:

(a) $S_\ell = S_r$.

(b) R is a left minannihilator ring.

Proof. (1) By Proposition 4.5 (2).
(2) (a) By [2, Proposition 1.14 (4)].
(b) By Lemma 4.3 (2). ■

Proposition 4.9. The following statements are equivalent for a right SS-injective ring R:

(1) $S_\ell \subseteq S_r$.
(2) $S_\ell = S_r$.
(3) R is a left mininjective ring.

Proof. (1)⇒(2) By Corollary 4.8 (2) (a).
(2)⇒(3) By Corollary 4.8 (2) and [5, Corollary 2.34, p. 53], we must show that R is right minannihilator ring. Let aR be a simple small right ideal, then Ra is a simple small left ideal by [2, Theorem 1.14]. Let $0 \neq x \in rl(aR)$, then $l(a) \subseteq l(x)$. Since $l(a) \subseteq^{max} R$, thus $l(a) = l(x)$ and hence Rx is simple left ideal, that is $x \in S_r$. Now, if $Rx = Re$ for some $e^2 = e \in R$, then $e = rx$ for some $0 \neq r \in R$. Since $(e-1)e = 0$, then $(e-1)rx = 0$, that is $(e-1)ra = 0$ and this implies that $ra \in eR$. Thus $raR \subseteq eR$, but eR is semisimple right ideal, so $raR \subseteq^{\oplus} R$ and hence $ra = 0$. Therefore, $rx = 0$, that is $e = 0$, a contradiction. Thus $x \in J$ and hence $x \in S_r \cap J$. Therefore, $aR \subseteq rl(aR) \subseteq S_r \cap J$. Now, let $aR \cap yR = 0$ for some $y \in rl(aR)$, thus $l(aR) + l(yR) = l(aR \cap yR) = R$. Since $y \in rl(aR)$, thus $l(aR) \subseteq l(yR)$ and hence $l(yR) = R$, that is $y = 0$. Therefore, $aR \subseteq^{ess} rl(aR)$, so $aR = rl(aR)$ as desired.
(3)⇒(1) Follows from [5, Corollary 2.34, p. 53]. ■

Recall that a ring R is said to be right minfull if it is semiperfect, right mininjective and $\text{soc}(eR) \neq 0$ for each local idempotent $e \in R$ (see [5]). A ring R is called right min-PF, if it is a semiperfect, right mininjective, $S_r \subseteq^{ess} R_R$, $lr(K) = K$ for every simple left ideal $K \subseteq eR$ for some local idempotent $e \in R$ (see [5]).

Corollary 4.10. Let R be a right SS-injective ring, semiperfect with $S_r \subseteq^{ess} R_R$. Then R is a right minfull ring and the following statements hold:

(1) Every simple right ideal of R is essential in a summand.
(2) $\text{soc}(eR)$ is simple and essential in eR for every local idempotent $e \in R$. Moreover, R is right finitely cogenerated.
(3) For every semisimple right ideal I of R, there exists $e^2 = e \in R$ such that $I \subseteq^{ess} rl(I) \subseteq^{ess} eR$.
(4) $S_r \subseteq S_\ell \subseteq rl(S_r)$.
(5) If I is a semisimple right ideal of R and aR is a simple right ideal of R with $I \cap aR = 0$, then $rl(I \oplus aR) = rl(I) \oplus rl(aR)$.
(6) $rl(\oplus_{i=1}^n a_iR) = \oplus_{i=1}^n rl(a_iR)$, where $\oplus_{i=1}^n a_iR$ is a direct sum of simple right ideals.
(7) The following statements are equivalent:
 (a) $S_r = rl(S_r)$.
 (b) $K = rl(K)$, for every semisimple right ideals K of R.
 (c) $kR = rl(kR)$, for every simple right ideals kR of R.
 (d) $S_r = S_\ell$.
 (e) $\text{soc}(Re)$ is a simple for all local idempotent $e \in R$.
 (f) $\text{soc}(Re) = S_r e$, for all local idempotent $e \in R$.
 (g) R is a left mininjective.
 (h) $L = lr(L)$, for every semisimple left ideals L of R.
 (i) R is a left minfull ring.
 (j) $S_r \cap J = rl(S_r \cap J)$.
 (k) $K = rl(K)$, for every semisimple small right ideals K of R.
 (l) $L = lr(L)$, for every semisimple small left ideals L of R.
(8) If R satisfies any condition of (7), then $r(S_\ell \cap J) \subseteq^{ess} R_R$.

Proof. (1), (2), (3), (4), (5) and (6) are obtained by Corollary 2.1.32 (1) and [1, Theorem 4.12].
(7) The equivalence of (a), (b), (c), (d), (e), (f), (g), (h) and (i) follows from Corollary 3.11 and [1, Theorem 4.12].
(b)⇒(j) Clear.
(j)⇔(k) By Proposition 4.5 (2).
(k)⇒(c) By Corollary 4.8 (1).
(h)⇒(l) Clear.
(l)⇒(d) Let Ra be a simple left ideal of R. By hypothesis, $lr(A) = A$ for any simple small left ideal A of R. Since $lr(A) = A$, for any simple left ideal A of R, $lr(Ra) = Ra$. Thus R is a right min-PF ring and it follows from [2, Theorem 3.14] that $S_r = S_\ell$.
(8) Let K be a right ideal of R such that $r(S_\ell \cap J) \cap K = 0$. Then $K\, r(S_\ell \cap J) = 0$ and we have $K \subseteq lr(S_\ell \cap J) = S_\ell \cap J = S_r \cap J$. Now, $r\big((S_\ell \cap J) + l(K)\big) = r(S_\ell \cap J) \cap K = 0$. Since R is left Kasch, then $(S_\ell \cap J) + l(K) = R$ by [9, Corollary 8.28 (5), p. 281]. Thus $l(K) = R$ and hence $K = 0$, so $r(S_\ell \cap J) \subseteq^{ess} R_R$. ■

N. Zeyada, S. Hussein and A. Amin [19] introduced the notion almost-injective, a right R-module M is called almost-injective if $M = E \oplus K$, where E is injective and K has zero radical. They proved that, every almost-injective right R-module is an injective if and only if every almost-injective is a quasi-continuous if and only if R is a semilocal ring (see [19, Theorem 2.12]). After reflect of [19, Theorem 2.12] we found it is not true always and the reason is due to the R-homomorphism $h: (L+J)/J \longrightarrow K$ in the proof of the part of the Theorem 2.12 in [19] is not well define, so most of the other results in [19] are not necessary to be correct, because they are based on [19, Theorem2.12]. The following examples show that the contradiction in [19, Theorem 2.12] is exist.

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Example 4.11. In particular from the proof of the part (3)⇒(1) in [19, Theorem 2.12], we consider $R = \mathbb{Z}_8$ and $M = K =< \bar{4} >$. Thus $M = E \oplus K$, where $E = 0$ is a trivial injective R-module and $J(K) = 0$. Let $f: L \longrightarrow K$ is the identity map, where $L = K$. So, the map homomorphism $h: (L + J)/J \longrightarrow K$ which is given by $h(\ell + J) = f(\ell)$ is not well define, because $J = \bar{4} + J$ but $h(J) = f(\bar{0}) = \bar{0} \neq \bar{4} = f(\bar{4}) = h(\bar{4} + J)$.

Example 4.12.

(1) Let R be an artinian ring. Assume that R is not semisimple ring, then R is not right V-ring. Thus there is simple right R-module is not injective. Therefore, there is almost-injective right R-module is not injective. So it follows from [19, Theorem 2.12] that R is not semilocal. Hence, R is not right artinian and this a contradiction. Thus every right artinian ring is semisimple, but this is not true in general (see below example).

(2) The ring $\mathbb{Z}_8$ is semilocal. Since $< \bar{4} >= \{\bar{0}, \bar{4}\}$ is almost-injective as $\mathbb{Z}_8$-module, then $< \bar{4} >$ is injective $\mathbb{Z}_8$-module by [19, Theorem 2.12]. Thus $< \bar{4} >\subseteq^{\oplus} \mathbb{Z}_8$ and this a contradiction.

Theorem 4.13. The following statements are equivalent for a ring R:

(1) R is a semiprimitive and every almost-injective right R-module is quasi-continuous.

(2) R is a right SS-injective and right minannihilator ring, J is a right artinian, and every almost-injective right R-module is quasi-continuous.

(3) R is a semisimple ring.

Proof. (1)⇒(2) and (3)⇒(1) are clear.

(2)⇒(3) Let M be a right R-module with zero Jacobson radical and let K be a nonzero submodule of M. Thus $K \oplus M$ is a quasi-continuous. By [20, Corollary 2.14, p. 23], K is an M-injective. Thus $K \subseteq^{\oplus} M$ and hence M is semisimple. In particular, R/J is a semisimple R-module and hence R/J is artinian by [8, Theorem 9.2.2 (b), p. 219], so R is semilocal ring. Since J is a right artinian, then R is a right artinian. So, it follows from Corollary 4.10 (7) that R is right and left mininjective. Thus [2, Corollary 4.8] implies that R is QF ring. By hypothesis $R \oplus (R/J)$ is quasi-continuous (since R is self-injective), so again by [20, Corollary 2.14, p. 23] we have that R/J is an injective. Since R is QF ring, then R/J is a projective (see [8, Theorem 13.6.1]). Thus the canonical map $\pi: R \longrightarrow R/J$ is a splits and hence $J \subseteq^{\oplus} R$, that is $J = 0$. Therefore, R is semisimple. ■

5. Strongly SS-Injective Rings

A ring R is called a right Ikeda-Nakayama ring if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R (see [5, p. 148]). In the next proposition, the strongly SS-injectivity gives a new version of Ikeda-Nakayama rings.

Proposition 5.1. Let R be a strongly right SS-injective ring, then $l(N \cap K) = l(N) + l(K)$ for all semisimple small right ideals N and all right ideals K of R.

Proof. Suppose that $x \in l(N \cap K)$ and define $\alpha: N + K \longrightarrow R_R$ by $\alpha(a + b) = xa$ for all $a \in N$ and $b \in K$. Clearly, α is well define, because if $a_1 + b_1 = a_2 + b_2$, then $a_1 - a_2 = b_2 - b_1$, that is $x(a_1 - a_2) = 0$, so $\alpha(a_1 + b_1) = \alpha(a_2 + b_2)$. Define the R-homomorphism $\tilde{\alpha}: (N + K)/K \longrightarrow R_R$ by $\tilde{\alpha}(a + K) = xa$ for all $a \in N$ which induced by α. Since $(N + K)/K \subseteq \operatorname{soc}(R/K) \cap J(R/K)$ and R is a strongly right SS-injective, $\tilde{\alpha}$ can be extended to an R-homomorphism $\gamma: R/K \longrightarrow R_R$. If $\gamma(1 + K) = y$, for some $y \in R$, then $y(a + b) = xa$, for all $a \in N$ and $b \in K$. In particular, $ya = xa$ for all $a \in N$ and $yb = 0$ for all $b \in K$. Hence $x = (x - y) + y \in l(N) + l(K)$. Therefore, $l(N \cap K) \subseteq l(N) + l(K)$. Since the converse is always holds, thus the proof is complete. ■

Recall that a ring R is said to be right simple J-injective if for any small right ideal I and any R-homomorphism $\alpha: I \longrightarrow R_R$ with simple image, $\alpha = c.$ for some $c \in R$ (see [14]).

Corollary 5.2. Every strongly right SS-injective ring is a right simple J-injective.

Proof. By Proposition 3.1. ■

Remark 5.3. The converse of Corollary 5.2 is not true (see Example 5.6).

Proposition 5.4. Let R be a right Kasch and strongly right SS-injective. Then:

(1) $rl(K) = K$, for every small right ideal K of R. Moreover, R is right minannihilator.

(2) If R is left Kasch, then $r(J) \subseteq^{ess} R_R$.

Proof.(1) By Corollary 5.2 and [14, Lemma 2.4].

(2) Let K be a right ideal of R and $r(J) \cap K = 0$. Then $Kr(J) = 0$ and we obtain $K \subseteq lr(J) = J$, because R is left Kasch. By (1), we have $r\big(J + l(K)\big) = r(J) \cap K = 0$ and this means that $J + l(K) = R$ (since R is left Kasch). Thus $K = 0$ and hence $r(J) \subseteq^{ess} R_R$. ■

The following examples show that the three classes of rings: strongly SS-injective rings, soc-injective rings and small injective rings are different.

Example 5.5. Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}:\ p$ does not divide $n\}$, the localization ring of $\mathbb{Z}$ at the prime p. Then R is a commutative local ring and it has zero socle but not principally small injective (see [4, Example 4]). Since $S_r = 0$, thus R is strongly soc-injective ring and hence R is strongly SS-injective ring.

Example 5.6. Let $R = \left\{\begin{pmatrix} n & x \\ 0 & n \end{pmatrix}:\ n \in \mathbb{Z}, x \in \mathbb{Z}_2\right\}$. Thus R is a commutative ring, $J = S_r = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}:\ x \in \mathbb{Z}_2\right\}$ and R is small injective (see [3, Example (i)]). Let $A = J$ and $B = \left\{\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}:\ n \in \mathbb{Z}\right\}$, then $l(A) = \left\{\begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix}:\ n \in \mathbb{Z}, y \in \mathbb{Z}_2\right\}$ and $l(B) = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}:\ y \in \mathbb{Z}_2\right\}$. Thus $l(A) + l(B) = \left\{\begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix}:\ n \in \mathbb{Z}, y \in \mathbb{Z}_2\right\}$. Since $A \cap B = 0$, then $l(A \cap B) = R$ and this implies that $l(A) + l(B) \neq l(A \cap B)$. Therefore R is not strongly SS-injective and not strongly soc-injective by Proposition 5.1.

Example 5.7. Let $F = \mathbb{Z}_2$ be the field of two elements, $F_i = F$ for $i = 1, 2, \ldots$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \bigoplus_{i=1}^{\infty} F_i$. If R is the subring of Q generated by 1 and S, then R is a von Neumann regular ring (see [17, Example (1), p. 28]). Since R is commutative, thus every simple R- module is injective by [9, Corollary 3.73. Thus R is V-ring and hence and hence $J(N) = 0$ for every right R-module N. It follows from Corollary 3.9 that every R-module is a strongly SS-injective. In particular, R is a strongly SS-injective ring. But R is not soc-injective (see [17, Example (1)]).

Example 5.8. Let $R = \mathbb{Z}_2[x_1, x_2, \ldots]$ where $\mathbb{Z}_2$ is the field of two elements, $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2 \neq 0$ for all i and j. If $m = x_i^2$, then R is a commutative, local, soc-injective ring with $J = \text{span}\{m, x_1, x_2, \ldots\}$, and R has simple essential socle $J^2 = \mathbb{Z}_2 m$ (see [1, Example 5.7]). It follows from [1, Example 5.7] that the R-homomorphism $\gamma: J \longrightarrow R$ which is given by $\gamma(a) = a^2$ for all $a \in J$ with simple image can not extend to R, then R is not simple J-injective and not small injective, so it follows from Corollary 5.2 that R is not strongly SS-injective.

Recall that a ring R is called right minsymmetric if aR is simple, $a \in R$, implies that Ra is simple left ideal (see [2]). Every right mininjective ring is right minsymmetric by [2, Theorem 1.14].

Theorem 5.9. A ring R is QF if and only if R is a strongly right SS-injective and right noetherian ring with $S_r \subseteq^{ess} R_R$.

Proof. (⇒) This is clear.
(⇐) By Lemma 4.3 (1), R is a right minsymmetric. It follows from [3, Lemma 2.2] that R is right perfect. Thus, R is strongly right soc-injective, by Theorem 3.13. Since $S_r \subseteq^{ess} R_R$, so it follows from [1, Corollary 3.2] that R is a self-injective and hence R is QF. ■

Corollary 5.10. For a ring R, the following statements are true:

(1) R is a semisimple if and only if $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective.

(2) R is QF if and only if R is a strongly right SS-injective, semiperfect with essential right socle and R/S_r is noetherian as right R-module.

Proof. (1) Suppose that $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective, then R is a right noetherian right V-ring by [1, Proposition 3.12], so it follows from Corollary 3.9 that R is a strongly right SS-injective. Thus R is QF by Theorem 5.9. But $J = 0$, so R is a semisimple. The converse is clear.
(2) By [2, Theorem 2.9], $J = Z_r$. Since R/Z_2^r is a homomorphic image of R/Z_r and R is a semilocal ring, thus R is a right t-semisimple. By Corollary 3.19, R is right noetherian, so it follows from Theorem 5.9 that R is QF. The converse is clear. ■

Theorem 5.11. A ring R is QF if and only if R is strongly right SS-injective, $l(J^2)$ is a countable generated left ideal, $S_r \subseteq^{ess} R_R$ and the chain $r(x_1) \subseteq r(x_2 x_1) \subseteq \cdots \subseteq r(x_n x_{n-1} \ldots x_2 x_1) \subseteq \cdots$ terminates for every infinite sequence $x_1, x_2, \ldots$ in R.

Proof. (⇒) Clear.
(⇐) By [3, Lemma 2.2], R is right perfect. Since $S_r \subseteq^{ess} R_R$, thus R is right Kasch by [2, Theorem 3.7]. Since R is a strongly right SS-injective, R is a right simple J-injective, by Corollary 5.2. Now, by Proposition 5.4 (1) we have $rl(S_r \cap J) = S_r \cap J$, so Corollary 4.10 (7) leads to $S_r = S_\ell$. By [5, Lemma 3.36, p. 73], $S_2^r = l(J^2)$. The result now follows from [14, Theorem 2.18]. ■

Remark 5.12. The condition $S_r \subseteq^{ess} R_R$ in Theorem 5.9 and Theorem 5.11 can be not be deleted, for example, $\mathbb{Z}$ is a strongly SS-injective noetherian ring but not QF.

The following two results are extension of Proposition 5.8 in [1].

Corollary 5.13. A ring R is QF ring if and only if it is left perfect, strongly left and right SS-injective ring.

Proof. By Corollary 5.2 and [14, Corollary 2.12]. ■

Theorem 5.14. For a ring R, the following statements are equivalent:

(1) R is a QF ring.

(2) R is a strongly left and right SS-injective, right Kasch and J is left t-nilpotent.

(3) R is a strongly left and right SS-injective, left Kasch and J is left t-nilpotent.

Proof. (1)⇒(2) and (1)⇒(3) are clear.
(3)⇒(1) Suppose that xR is simple right ideal. Thus either $rl(x) = xR \subseteq^{\oplus} R_R$ or $x \in J$. If $x \in J$, then $rl(x) = xR$ (since R is right minannihilator), so Theorem 3.4 implies that $rl(x) \subseteq^{ess} E \subseteq^{\oplus} R_R$. Therefore, $rl(x)$ is an essential in a direct summand of R_R for every simple right ideal xR. Let K be a left maximal ideal of R. Since R is a left Kasch, thus $r(K) \neq 0$ by [9, Corollary 8.28, p. 281]. Choose $0 \neq y \in r(K)$, so $K \subseteq l(y)$ and we conclude that $K = l(y)$. Since $Ry \cong R/l(y)$, thus Ry is simple left ideal. But R is a left mininjective ring, so yR is a simple right ideal by [2, Theorem 1.14] and this implies that $r(K) \subseteq^{ess} eR$ for some $e^2 = e \in R$ (since $r(K) = rl(y)$). Thus R is semiperfect by [5, Lemma 4.1, p. 79] and hence R is a left perfect (since J is left t-nilpotent), so it follows from Corollary 5.13 that R is QF.
(2)⇒(1) is similar to proof of (3)⇒(1). ■

Theorem 5.15. The ring R is QF if and only if R is a strongly left and right SS-injective, left and right Kasch, and the chain $l(a_1) \subseteq l(a_1a_2) \subseteq \cdots \subseteq l(a_1a_2 \dots a_n) \subseteq \cdots$ terminates for every $a_1, a_2, \dots \in Z_\ell$.

Proof. (⇒) Clear.
(⇐) By Proposition 5.4, $l(J)$ is essential in ${}_RR$. Thus $J \subseteq Z_\ell$. Let $a_1, a_2, \dots \in J$, we have $l(a_1) \subseteq l(a_1a_2) \subseteq \cdots \subseteq l(a_1a_2 \dots a_n) \subseteq \cdots$. Thus there exists $k \in \mathbb{N}$ such that $l(a_1 \dots a_k) = l(a_1 \dots a_k a_{k+1})$ (by hypothesis). Suppose that $a_1 \dots a_k \neq 0$, so $R(a_1 \dots a_k) \cap l(a_{k+1}) \neq 0$ (since $l(a_{k+1})$ is essential in ${}_RR$). Thus $ra_1 \dots a_k \neq 0$ and $ra_1 \dots a_k a_{k+1} = 0$ for some $r \in R$, a contradiction. So, $a_1 \dots a_k = 0$ and hence J is left t-nilpotent, so it follows from Theorem 5.14 that R is QF. ■

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الموديولات والحلقات الاغمارية من النمط-SS

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المستخلص :

قدمنا وناقشنا الاغمارية من النمط-SS كتعميم الى كلاً من الاغمارية من النمط-soc والاغمارية الصغيرة. الموديول الايمن M على الحلقة R يقال انه اغماري من النمط-N-SS (حيث N هو موديول ايمن على الحلقة R) اذا كان كل تماثل موديولي على الحلقة R من موديول جزئي صغير شبه بسيط من N الى M يمكن توسيعه الى N. الموديول M نسميه موديول اغماري من النمط-SS (اغماري قوي من النمط-SS) اذا كان M هو موديول اغماري من النمط-R-SS (موديول اغماري من النمط-N-SS لكل موديول ايمن N على الحلقة R). بعض تشخيصات وخصائص الموديولات والحلقات الاغمارية (القوية) من النمط-SS قد اعطيت. بعض النتائج على الاغمارية من النمط-soc قد تم توسيعها الى الاغمارية من النمط-SS.