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JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)



# Error Estimation for Polynomial Approximations in $L_{p,w^*}$ Space

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## ARTICLE INFO

### Article history:

Received: 08/06/2026

Revised form: 20 /06/2026

Accepted : 21 /06/2026

Available online: 30 /09/2026

### Keywords:

Approximation theory  
Weighted normed space  
Jackson inequality  
Whitney inequality  
Markov inequality  
Error of approximation,  
Modulus of smoothness

## ABSTRACT

In this research ,we conduct an analytical study to estimate the approximate error of functions using algebraic polynomials in the normed linear space  $L_{p,w^*}(\mathbb{I})$  ,  $\mathbb{I} = [d, b]$ ,  $1 \leq p \leq \infty$ . We first address the limitations of the weighted integral operator and then establish a mathematical relationship between the norm of the weight functions  $L_{p,w^*}(\mathbb{I})$  and the norm of continuous functions  $C(\mathbb{I})$  on the uniform partition of  $\mathbb{I}$  the interval  $\mathbb{I}$ . We found a relationship between higher-order smoothness and the approximation error ,and we also determined approximation constants that depend on the properties of the normed linear space  $L_{p,w^*}$ . Furthermore, we established the interpolation of functions  $f$  on the interval  $[d, h]$  , where  $h \leq b$ , to minimize the difference between the function and its derivatives and the nearest algebraic polynomials in representation. We also demonstrated that the uniform  $C -$  normed space on the interval  $\mathbb{I}$  can be controlled by the normed linear space  $L_{p,w^*}$ .

MSC..

<https://doi.org/10.29304/jqcm.2026.18.32968>

## 1.Introduction

The theory of important theorems which defines the cornerstone of analysis , is of great importance in functional analysis , numerical analysis and differential equation , it relies on representing complex function using simpler binary representations , among these functions are algebraic polynomials starting in space  $\prod_n$  ,as well as geometric polynomials .In this research , we focused on the approximate motion of a weighted function in external space  $L_{p,w^*}(\mathbb{I})$  on the interval  $\mathbb{I} = [d, b]$  for values  $1 \leq p \leq \infty$  , wherever " $w^*(t)$ " is a weighted function [1], [2], [3] and which determines the nature of the norm used in the weighted integral along the interval:

$$(\|f\|_{L_{p,w^*}(\mathbb{I})}) = \left( \int_{\mathbb{I}} |f(t)w^*(t)|^p dt \right)^{\frac{1}{p}}, f \in L_{p,w^*}(\mathbb{I}), t \in \mathbb{I} \text{ [4], [5]}.$$

The interval  $\mathbb{I}$  was divided into subintervals  $\mathcal{L}_i = [t_i, t_{i+1}]$  ,  $1 \leq i \leq n$  with length  $h_i = |\mathcal{L}_i|$  , and this implies that find the close relationship between  $L_{p,w^*}$  and  $C(\mathbb{I})$  [6], this deepens the local analysis .In this research , we also

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estimated the magnitude of the error  $E_{n-1}(f)_{w^*,p}$  [7], which represents the smallest distance between the function  $f \in L_{p,w^*}(\mathbb{I})$  and its best approximation using  $n - 1$  polynomials in  $L_{p,w^*}$ . To achieve these ideas, several analytical tools were integrated among them, some important inequalities in approximation theory were employed including Markov's inequality [8], [9] in determining the modulus of derivatives of polynomials and finding a relationship with the overall modulus on the interval  $\mathcal{L}_i$ . Whitney's inequality [8], [9] was also used and applied to the weighted space  $L_{p,w^*}$ , due to the equivalence of moduli between  $L_p$  and  $L_{p,w^*}$ , since the weighted function  $w^*(t)$  is a decreasing and bounded function, this allows for relating the smoothness modulus  $\omega_{m-1}^\varphi(f, \delta)_{w^*,p}$  to the approximation error of the derivative [1], [4]. Jackson's inequality [8], [9] is another important inequality that finds the upper limits of the approximation error in founded on the stability of the function, this ensures that algebraic polynomials provide the optimal approximation of the function on the interval  $L_{p,w^*}$  and that the constants do not depend on the segment length  $h$ . We can also link the analytical results of this research in  $L_{p,w^*}$  space with the topological characteristics of open sets of the (Delta-Semi-Open) type and (fan sets) discussed in the literature [12], [13], [14].

## 2. Auxiliary Results

In this section, we will present some important Lemmas for the relationship between the normed linear space  $L_{p,w^*}$  and  $\mathbb{C}$ -normed space, which are necessary to complete the main ideas of the research.

**Lemma 2.1:** Let  $\mathbb{I} = [d, b]$ ,  $1 \leq p \leq \infty$  and  $w^*$  weighted function on  $\mathbb{I}$ , there is  $0 < m = \inf w^*(t) \leq w^*(t) \leq \sup w^*(t) = M < \infty$ , if  $\mathbb{P}_n^* \in \Pi_n$  is a polynomial of degree  $\leq n$ , and  $r = \min \left\{ \frac{h_i}{2}, \frac{1}{2\mathcal{A}_0 n^2} \right\}$ ,  $\mathcal{L}_i = [t_i, t_{i+1}] \subset \mathbb{I}$ ,  $h = |\mathcal{L}_i|$ , then

$$\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} |\mathbb{J}|^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}$$

**Proof:** We define  $M = \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} = \max_{t \in \mathcal{L}_i} |\mathbb{P}_n^*(t)|$ .

Since  $\mathbb{P}_n^*$  is continuous (closed, bounded), it achieves its maximum, hence there exists  $t_0 \in \mathcal{L}_i$ , such that  $|\mathbb{P}_n^*(t)| = M$ . By apply Markov's inequality on  $\mathbb{I}$  then  $\|\mathbb{P}_n^{*'}\|_{C(\mathbb{I})} \leq \mathcal{A}_0 n^2 \|\mathbb{P}_n^*\|_{C(\mathbb{I})}$ ,  $\mathcal{A}_0 > 0$ .

Since  $\mathcal{L}_i \subset \mathbb{I}$ ,  $\|\mathbb{P}_n^{*'}\|_{C(\mathcal{L}_i)} \leq \|\mathbb{P}_n^{*'}\|_{C(\mathbb{I})}$  then  $\|\mathbb{P}_n^{*'}\|_{C(\mathcal{L}_i)} \leq \mathcal{A}_0 n^2 \|\mathbb{P}_n^*\|_{C(\mathbb{I})}$ , by the major amount Theorem [15], for any  $t \in \mathcal{L}_i$ ,  $\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A}_0 n^2 M$ .

$$\begin{aligned} |\mathbb{P}_n^*(t) - \mathbb{P}_n^*(t_0)| &\leq \|\mathbb{P}_n^{*'}\|_{C(\mathcal{L}_i)} |t - t_0| \leq \|\mathbb{P}_n^{*'}\|_{C(\mathbb{I})} |t - t_0| \\ &\leq \mathcal{A}_0 n^2 \|\mathbb{P}_n^*\|_{C(\mathbb{I})} |t - t_0|. \end{aligned}$$

We choose  $r > 0$ ,  $r = \min \left\{ \frac{h_i}{2}, \frac{1}{2\mathcal{A}_0 n^2} \right\}$ . Define  $\mathbb{J} = \mathcal{L}_i \cap [t_0 - r, t_0 + r]$ ,  $\mathbb{J} \subset \mathcal{L}_i$ ,  $r \leq |\mathbb{J}|$ , for all  $t \in \mathbb{J}$ , that is  $t \in \mathbb{J}$ , " $t \in [t_0 - r, t_0 + r]$ ".

Hence

$$(t_0 - r \leq t \leq t_0 + r \Rightarrow -r \leq t - t_0 \leq r \Rightarrow |t - t_0| \leq r = \frac{1}{2\mathcal{A}_0 n^2})$$

$$|\mathbb{P}_n^*(t) - \mathbb{P}_n^*(t_0)| \leq \mathcal{A}_0 n^2 \|\mathbb{P}_n^*\|_{C(\mathbb{I})} r \leq \frac{1}{2} \|\mathbb{P}_n^*\|_{C(\mathbb{I})}$$

$t \in \mathbb{J}$ , since  $|\mathbb{P}_n^*(t_0)| = \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)}$  then  $|\mathbb{P}_n^*(t)| \geq |\mathbb{P}_n^*(t_0)| - |\mathbb{P}_n^*(t) - \mathbb{P}_n^*(t_0)| \geq \frac{1}{2} \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)}$

Since  $w^*(t) \geq m_{\min} > 0$  and  $\mathbb{J} \subset \mathcal{L}_i$

$$\|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}^p = \int_{\mathcal{L}_i} |\mathbb{P}_n^*(\mathfrak{t}) w^*(\mathfrak{t})|^p d\mathfrak{t} \geq \int_{\mathbb{J}} |\mathbb{P}_n^*(\mathfrak{t}) w^*(\mathfrak{t})|^p d\mathfrak{t} \geq m_{\min}^p \left( \frac{1}{2} \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \right)^p |\mathbb{J}|$$

$$\text{Hence } \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)} \geq m_{\min} \frac{\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)}}{2} |\mathbb{J}|^{\frac{1}{p}}$$

$$\text{Therefore } \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq 2m_{\min} |\mathbb{J}|^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)} = \mathcal{A} |\mathbb{J}|^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}$$

$$\mathcal{A} = \mathcal{A}(m_{\min}), \text{ hence } \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} |\mathbb{J}|^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}.$$

**Lemma 2.2:** Let  $\mathcal{L}_i$  be a partition of the interval  $\mathbb{I}$  such that  $h = \mathfrak{t}_{i+1} - \mathfrak{t}_i$ , suppose that the partition is regular, for some fixed " $\mathcal{A}_1, \mathcal{A}_2$ " and  $(\frac{\mathcal{A}_1}{n} \leq h_i \leq \frac{\mathcal{A}_2}{n})$ ,  $i = 0, \dots, m-1$ .

If  $\mathbb{P}_n^* \in \prod_n$  then there is a fixed  $\mathcal{A} > 0$  such that

$$\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} h^{\frac{2}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}.$$

**Proof:** By Lemma (2.1), for every point  $\mathcal{L}_i$

$$\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} |\mathbb{J}|^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}$$

$$\text{since } |\mathbb{J}|^{\frac{-1}{p}} \leq r^{\frac{-1}{p}}, \text{ therefore } \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} r^{\frac{-1}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}.$$

By assumption  $\frac{\mathcal{A}_1}{n} \leq h_i$  hence  $\frac{\mathcal{A}_1}{2n} \leq \frac{h_i}{2}$ . Substituting into lemma (2.1) then

$$\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} n^{\frac{2}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}.$$

### 3.Prime Effects

We will demonstrate the prime ideas of the research, where we will find the point estimation of the approximation.

**Theorem 3.1:** If  $\mathfrak{f} \in L_{p,w^*}[d, h]$ ,  $h < b$ , then there is a polynomial  $\mathbb{P}_{n-1}^* \in \prod_{r-1}$  interpolating fat every points inside  $[d, h]$ , such that

$$\|\mathfrak{f} - \mathbb{P}_{n-1}^*\|_{L_{p,w^*}} \leq \mathcal{A} h \omega_{m-1}^{\mathcal{O}}(\mathfrak{f}', h)_{w^*,p}, m \leq n$$

**Proof:** i. If is  $\mathfrak{f}$  a function of degree  $< n$  where is then the polynomial  $\mathbb{P}_{n-1}^*$  is identical to the function  $\mathfrak{f}$ , and therefore there is no difference in approximating  $\mathfrak{f}$  using  $\mathbb{P}_{n-1}^*$ , i.e. the difference is zero.

ii. If is  $\mathfrak{f}$  function of degree not  $< n$ , then we need the polynomial  $\mathbb{P}_{n-2}^*$ .

This polynomial provides the best approximation of  $\mathfrak{f}'$  on  $[d, h]$ ,  $h < b$  because  $\mathfrak{f}'$  is of degree  $n-1$  and  $\mathbb{P}_{n-2}^*$  is the "best approximation" of  $\mathfrak{f}'$  more accurate in representing the derivative of  $\mathfrak{f}$ . Since  $\mathbb{P}_{n-2}^*$  represents the optimal polynomial for approximating  $\mathfrak{f}'$ , therefore, the approximation error can be represented as follows:

$$E_{n-2}(\mathfrak{f}') = E_{n-2} = \|\mathfrak{f}' - \mathbb{P}_{n-2}^*\|_{L_{p,w^*}[d,h]}$$

Let  $Q_{\mathfrak{t}}[g]$  denote an integral operator on the interval  $[a, \mathfrak{t}]$ ,  $\mathfrak{t} \leq h$  and represents the cumulative value of the function  $g(\mathfrak{z})$  from  $a$  to  $\mathfrak{t}$ , suppose that  $Q_{\mathfrak{t}}$  bounded, positive, and linear operator. Let

$$\mathbb{P}_{n-1}^*(\mathfrak{t}) = \mathfrak{f}(d) + \frac{\mathfrak{f}(h) - \mathfrak{f}(d)}{Q_h[g]} Q_{\mathfrak{t}}[g], \text{ in the theoretical analysis, we can take } Q_{\mathfrak{t}}[g] = \int_a^{\mathfrak{t}} g(\mathfrak{z}) d\mathfrak{z} \quad \ni g(\mathfrak{z}) =$$

$$\mathbb{P}_{n-2}^*(\mathfrak{z}) + E_{n-2}$$

$$"Q_h[g] = \int_d^h g(\mathfrak{z}) d\mathfrak{z}"$$

Which includes the approximation set of the previous errors ,and this formula also ensure  $\mathbb{P}_{n-2}^*(h) = \mathfrak{f}(h)$  ,  
 $\mathbb{P}_{n-1}^*(d) = \mathfrak{f}(d)$  hence  $g(z) \geq 0$ .

Let  $\mathbb{Q}^*(\mathfrak{t}) = \mathfrak{f}'(\mathfrak{t}) - \mathbb{P}_{n-2}^*(\mathfrak{t})$  , on  $[d, h]$  and  $\mathfrak{t} = th$  ,  $d \leq t \leq 1$ .

Suppose  $G(t) = \mathbb{Q}^*(th) = (\mathfrak{f}'(th) - \mathbb{P}_{n-2}^*(th))\omega^*(th)$

where  $G(t)$  a function defined on  $[d, 1]$  which is same as  $\mathbb{Q}^*$  but compressed horizontally on  $[d, 1]$

$$\|\mathbb{Q}^*\|_{L_{p,\omega^*}[d,h]}^p = \int_d^h |\mathbb{Q}^*(\mathfrak{t})\omega^*(\mathfrak{t})|^p d\mathfrak{t}$$

At  $\mathfrak{t} = a$  then  $t = \frac{d}{h}$  , if  $\mathfrak{t} = h$  then  $t = 1$ ,hence

$$\|\mathbb{Q}^*\|_{L_{p,\omega^*}}^p = \int_d^h |\mathbb{Q}^*(\mathfrak{t})\omega^*(\mathfrak{t})|^p d\mathfrak{t} = h \int_{\frac{d}{h}}^1 |\mathbb{Q}^*(ht)\omega^*(ht)|^p dt$$

$$\|\mathbb{Q}^*\|_{L_{p,\omega^*}[d,h]} = \|\mathfrak{f}' - \mathbb{P}_{n-2}^*\|_{L_{p,\omega^*}[d,h]} = h^{\frac{1}{p}} \|G\|_{L_{p,\omega^*}[\frac{d}{h},1]}$$

$$\begin{aligned} E_{n-2}(\mathfrak{f}') &= \|\mathfrak{f}' - \mathbb{P}_{n-2}^*\|_{L_{p,\omega^*}[d,h]} = h^{\frac{1}{p}} \|\mathfrak{f}'(th) - \mathbb{P}_{n-2}^*(th)\|_{L_{p,\omega^*}[\frac{d}{h},1]} \\ &\geq ch^{\frac{1}{p}} \|\mathfrak{f}' - \mathbb{P}_{n-2}^*\|_{L_{p,\omega^*}[d,h]} \end{aligned}$$

We have for  $\mathfrak{t} \in [d, h]$

$$\|\mathfrak{f} - \mathbb{P}_{n-1}^*\|_{L_{p,\omega^*}[d,h]} = \left\| \mathfrak{f}(\mathfrak{t}) - \mathfrak{f}(d) - \frac{\mathfrak{f}(h) - \mathfrak{f}(d)}{Q_h[g]} Q_{\mathfrak{t}}[g] \right\|_{L_{p,\omega^*}[d,h]}$$

Since  $(\int_d^{\mathfrak{t}} \mathfrak{f}'(z) dz = \mathfrak{f}(\mathfrak{t}) - \mathfrak{f}(d))$  and  $(\int_d^h \mathfrak{f}'(z) dz = \mathfrak{f}(h) - \mathfrak{f}(d))$

$$\begin{aligned} &= \left\| \int_d^{\mathfrak{t}} \mathfrak{f}'(z) dz - \frac{\int_d^h \mathfrak{f}'(z) dz}{Q_h[g]} Q_{\mathfrak{t}}[g] \right\|_{L_{p,\omega^*}[d,h]} \\ &= \left\| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz - \frac{\int_d^h \mathfrak{f}'(z) dz}{Q_h[g]} Q_{\mathfrak{t}}[g] + \int_d^{\mathfrak{t}} \mathbb{P}_{n-2}^*(z) dz \right\|_{L_{p,\omega^*}[d,h]}, \\ &\|\mathfrak{f} - \mathbb{P}_{n-1}^*\|_{L_{p,\omega^*}[d,h]} = \\ &\left\| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz + \int_d^{\mathfrak{t}} \mathbb{P}_{n-2}^*(z) dz - \mathcal{M} \int_d^{\mathfrak{t}} (\mathbb{P}_{n-2}^*(z) + E_{n-2}) dz \right\|_{L_{p,\omega^*}[d,h]}, \mathcal{M} = \frac{\int_d^h \mathfrak{f}'(z) dz}{Q_h[g]} \\ &\leq \left\| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz \right\|_{L_{p,\omega^*}[d,h]} + \left\| \int_d^{\mathfrak{t}} (\mathbb{P}_{n-2}^*(z) - \mathcal{M}(\mathbb{P}_{n-2}^*(z) + E_{n-2})) dz \right\|_{L_{p,\omega^*}[d,h]} \\ &= \|I_1\|_{L_{p,\omega^*}[d,h]} + \|I_2\|_{L_{p,\omega^*}[d,h]}, \\ &I_1 = \left\| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz \right\|_{L_{p,\omega^*}[d,h]} \end{aligned}$$

Let  $\mathbb{Q}^*(z) = \mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)$

$$\left| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz \right| \leq \int_d^{\mathfrak{t}} |\mathbb{Q}^*(z)\omega^*(z)| dz$$

by applying Holder's inequality to  $[d, \mathfrak{t}]$

$$\begin{aligned}
 \int_d^{\mathfrak{t}} \mathbb{Q}^*(z) w^*(z) dz &\leq \left( \int_d^{\mathfrak{t}} dz \right)^{\frac{1}{q}} \left( \int_d^{\mathfrak{t}} |\mathbb{Q}^*(z) w^*(z)|^p dz \right)^{\frac{1}{p}} \\
 \left( \int_d^{\mathfrak{t}} |\mathbb{Q}^*(z) w^*(z)|^p dz \right)^{\frac{1}{p}} &\leq \left( \int_d^h |\mathbb{Q}^*(z) w^*(z)|^p dz \right)^{\frac{1}{p}}, \quad \mathfrak{t} \leq h \\
 &= \|\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)\|_{L_{p,w^*}[d,h]} \\
 &= E_{n-2}(\mathfrak{f}')_{p,w^*} \\
 \left| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz \right| &\leq (\mathfrak{t} - d)^{1-\frac{1}{p}} E_{n-2}(\mathfrak{f}')_{w^*,p} \\
 &\leq h^{1-\frac{1}{p}} E_{n-2}(\mathfrak{f}')_{w^*,p} \\
 \left\| \int_d^{\mathfrak{t}} (\mathfrak{f}'(z) - \mathbb{P}_{n-2}^*(z)) dz \right\|_{L_{p,w^*}[d,h]} &\leq E_{n-2}(\mathfrak{f}')_{w^*,p} \left\| h^{1-\frac{1}{p}} \right\|_{L_{p,w^*}[d,h]} \\
 \|I_1\|_{L_{p,w^*}[d,h]} &\leq \frac{h}{p^{\frac{1}{p}}} E_{n-2}(\mathfrak{f}')_{w^*,p} \leq h E_{n-2}(\mathfrak{f}')_{w^*,p} \\
 \|I_2\|_{L_{p,w^*}[d,h]} &= \left\| \int_d^{\mathfrak{t}} \left( \mathbb{P}_{n-2}^*(z) - \mathcal{M}(\mathbb{P}_{n-2}^*(z) + E_{n-2}(\mathfrak{f}')_{w^*,p}) \right) dz \right\|_{L_{p,w^*}[d,h]}
 \end{aligned}$$

We have

$$\begin{aligned}
 &\mathbb{P}_{n-2}^*(z) + E_{n-2}(\mathfrak{f}')_{w^*,p} \geq 0, \text{ then} \\
 &|\mathbb{P}_{n-2}^*(z) w^*(z)|^p \leq E_{n-2}^p(\mathfrak{f}')_{w^*,p} \text{ and} \\
 &\int_d^h |\mathbb{P}_{n-2}^*(z) w^*(z)|^p dz \leq \int_d^h E_{n-2}^p(\mathfrak{f}')_{w^*,p} dz \\
 &\left| (1 - \mathcal{M}) \int_d^{\mathfrak{t}} \mathbb{P}_{n-2}^*(z) w^*(z) dz - \mathcal{M} \int_d^{\mathfrak{t}} E_{n-2}(\mathfrak{f}')_{w^*,p} dz \right| \\
 &\leq (1 - \mathcal{M}) \int_d^{\mathfrak{t}} |\mathbb{P}_{n-2}^*(z) w^*(z)| dz + \mathcal{M} \int_d^{\mathfrak{t}} E_{n-2}(\mathfrak{f}')_{w^*,p} dz \\
 &\leq (1 - \mathcal{M}) h E_{n-2}(\mathfrak{f}')_{w^*,p} + \mathcal{M} h E_{n-2}(\mathfrak{f}')_{w^*,p}, \quad 1 - \mathcal{M} > 0, \quad 0 < \mathcal{M} \leq 1 \\
 &= h E_{n-2}(\mathfrak{f}')_{w^*,p}
 \end{aligned}$$

$$\|I_2\|_{L_{p,w^*}[d,h]} \leq h E_{n-2}(\mathfrak{f}')_{w^*,p}$$

$$\|\mathfrak{f} - \mathbb{P}_{n-1}^*\|_{L_{p,w^*}[d,h]} \leq \|I_1\|_{L_{p,w^*}[d,h]} + \|I_2\|_{L_{p,w^*}[d,h]} \leq 2h E_{n-2}(\mathfrak{f}')_{w^*,p}.$$

Using the reference that contains Whitney's inequality, and since the weight function is positive and bounded on  $[d, h]$ , the space  $L_{p,w^*}$  is norm-equivalent to  $L_p$ . Consequently, Whitney's inequality extends from  $L_p$  to  $L_{p,w^*}$ , hence  $\|\mathfrak{f} - \mathbb{P}_{n-1}^*\|_{L_{p,w^*}[d,h]} \leq \mathcal{A} h \omega_{m-1}^{\varphi}(\mathfrak{f}', \delta)_{w^*,p}$ ,  $\mathcal{A} = \mathcal{A}(n, p)$ .

**Theorem 3.2:** Let  $1 \leq p \leq \infty$  and  $\{\mathfrak{t}_i\}_{i=0}^m$  be a fraction of the interval  $\mathbb{I}$  such that  $h_i = \mathfrak{t}_{i+1} - \mathfrak{t}_i \approx \frac{1}{n}$ , if  $\mathbb{P}_n^* \in \Pi_n$  then for some fixed  $K(z) > 0$  such that

$$\|\mathbb{P}_n^*\|_{C(\mathbb{I})} \leq K(z) \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathbb{I})}$$

**Proof:** Since  $\cup_{i=0}^{m-1} \mathcal{L}_i = \mathbb{I} = [d, b]$  such that  $\mathcal{L}_i = [\mathfrak{t}_i, \mathfrak{t}_{i+1}]$

$$\|\mathbb{P}_n^*\|_{C(\mathbb{I})} = \max_{0 \leq i \leq m-1} \|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)}. \text{ By Lemma (2.2)}$$

$$\|\mathbb{P}_n^*\|_{C(\mathcal{L}_i)} \leq \mathcal{A} n^{\frac{2}{p}} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}$$

$$\text{since } \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathbb{I})}^p = \sum_{i=0}^{m-1} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathcal{L}_i)}^p$$

$Max_{0 \leq i \leq m-1} \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathbb{L}_i)} \leq \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathbb{I})}$ , therefore

$$\|\mathbb{P}_n^*\|_{C(\mathbb{I})} \leq K(\mathbb{Z}) \|\mathbb{P}_n^*\|_{L_{p,w^*}(\mathbb{I})}, \quad K(\mathbb{Z}) = \mathcal{A} n^{\frac{2}{p}}.$$

**Theorem 3.3:** Let  $f \in L_{p,w^*}$ , and  $\mathbb{P}_{n-1}^* \in \Pi_{n-1}$ , then there is occur a fixed  $\mathcal{A} > 0$  so that

$$|f(t) - \mathbb{P}_{n-1}^*(t)| \leq A h \omega_{m-1}^\varphi(f', \delta)_{w^*,p}, \text{ where } \mathcal{A} = \mathcal{A}(1 + K(n-1)), \forall t \in \mathbb{I}.$$

**Proof:** Let  $\mathbb{Q}_{n-1}^* \in \Pi_{n-1}$  is a polynomial who is the "best approximation" to  $f$  then  $\forall t \in \mathbb{I}$

$$\begin{aligned} f(t) - \mathbb{P}_{n-1}^*(t) &= (f(t) - \mathbb{Q}_{n-1}^*(t)) + (\mathbb{Q}_{n-1}^*(t) - \mathbb{P}_{n-1}^*(t)) \\ |f(t) - \mathbb{P}_{n-1}^*(t)| &\leq |f(t) - \mathbb{Q}_{n-1}^*(t)| + |\mathbb{Q}_{n-1}^*(t) - \mathbb{P}_{n-1}^*(t)| \end{aligned} \quad \dots (1)$$

Sine  $\mathbb{Q}_{n-1}^*(t) - \mathbb{P}_{n-1}^*(t) \in \Pi_{n-1}$  then

By theorem (3.2)

$$|(\mathbb{Q}_{n-1}^*(t) - \mathbb{P}_{n-1}^*(t))w^*(t)| \leq K(n-1) \|\mathbb{Q}_{n-1}^*(t) - \mathbb{P}_{n-1}^*(t)\|_{L_{w^*,p}(\mathbb{I})} \quad \dots (2)$$

$$\|\mathbb{Q}_{n-1}^* - \mathbb{P}_{n-1}^*\|_{L_{w^*,p}(\mathbb{I})} \leq \|f - \mathbb{Q}_{n-1}^*\|_{L_{w^*,p}(\mathbb{I})} + \|f - \mathbb{P}_{n-1}^*\|_{L_{w^*,p}(\mathbb{I})} \quad \dots (3)$$

Hence, from (1), (2), (3), we obtain

$$\begin{aligned} |(f(t) - \mathbb{P}_{n-1}^*(t))w^*(t)| &\leq K(n-1)|f - \mathbb{Q}_{n-1}^*| + K(n-1)(\|f - \mathbb{Q}_{n-1}^*\|_{L_{w^*,p}(\mathbb{I})} \\ &+ \|f - \mathbb{P}_{n-1}^*\|_{L_{w^*,p}(\mathbb{I})}) \end{aligned} \quad \dots (4)$$

$$|(f - \mathbb{Q}_{n-1}^*)w^*(t)| \leq \|f - \mathbb{Q}_{n-1}^*\|_{C(\mathbb{I})} \quad \dots (5)$$

Using Jackson's inequality, thus

$$|(f - \mathbb{Q}_{n-1}^*)w^*(t)| \leq \|f - \mathbb{Q}_{n-1}^*\|_{C(\mathbb{I})} \leq \mathcal{A} h \omega_{m-1}^\varphi(f', \delta)_{w^*,p}$$

To this end, using (4), (5) and Theorem (3.1) we obtain

$$|(f(t) - \mathbb{P}_{n-1}^*(t))w^*(t)| \leq \mathcal{A} h \omega_{m-1}^\varphi(f', \delta)_{w^*,p} + K_1(n-1) h \omega_{m-1}^\varphi(f', \delta)_{w^*,p}, \quad K_1 = K(n-1)\mathcal{A}$$

$$|(f(t) - \mathbb{P}_{n-1}^*(t))w^*(t)| \leq A(n-1) h \omega_{m-1}^\varphi(f', \delta)_{w^*,p}$$

$t \in I$ ,  $A(n-1) = c + K_1(n-1)$ , since  $m = \min_{t \in I} |w^*(t)|$  then

$$|f(t) - \mathbb{P}_{n-1}^*(t)| \leq \frac{A(n-1) h}{m} \omega_{m-1}^\varphi(f', \delta)_{w^*,p}$$

## References

- [1] N. Z. A. Al-Sada, "Copositive approximation by Hermite extension," *J. Interdiscip. Math.*, vol. 25, no. 4, 2022.
- [2] N. Zuhair, "Pointwise Estimates for Finding the Error of Best Approximation by Spline Positive Algebraic Polynomials and Copositive," *Iraqi J. Sci.*, vol. 63, no. 11, 2022.
- [3] S. Turki Atiyah and N. Zuhair Abd AL-Sada, "Hy,wp(I) Space anew Contribution to Weighted Space Theory and Smoothness Analysis," *Bol. Soc. Parana. Matemática*, vol. 44, no. 5, 2026, doi: 10.5269/bspm.81559.
- [4] N. Z. A. Al-Sada, "Copositive spline approximation in weighted space," *J. Interdiscip. Math.*, vol. 25, no. 4, pp. 1171–1186, May 2022, doi: 10.1080/09720502.2022.2060611.
- [5] Z. A. Mohameed and N. Z. A. Al-Sada, "The  $Z^p_{[a,w]}$  Space and Weighted Smoothness Analysis," in *Proceedings of ICMMCS 2026*, Iraq (or leave empty if not sure), 2026.
- [6] E. Kreyszig, *Introductory Functional Analysis with Applications*, 1st ed. New York: Wiley, 1978. [Online]. Available: [https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE15AF53\\_kov/Kreyszig%20-%20Introductory%20Functional%20Analysis%20with%20Applications%20%281%29.pdf](https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE15AF53_kov/Kreyszig%20-%20Introductory%20Functional%20Analysis%20with%20Applications%20%281%29.pdf)
- [7] J. G. Mthawer and N. Z. Abd Al-Sada, "On the Error of Two-Sided Comonotonic Approximation," in *Nonlinear Functional Analysis and Applications*, 2024.
- [8] L. N. Trefethen, *Approximation Theory and Approximation Practice*. Philadelphia: SIAM, 2013.
- [9] G. Garrigós and E. Hernández, "Sharp Jackson and Bernstein Inequalities for N-term Approximation in Sequence Spaces with Applications," *J. Approx. Theory*, vol. 171, pp. 24–1, 2013.
- [10] N. Paskin, "Toward unique identifiers," *Proc. IEEE*, vol. 87, no. 7, pp. 1208–1227, Jul. 1999, doi: 10.1109/5.771073.
- [11] D. Dung and T. Ullrich, "Whitney type inequalities for local anisotropic polynomial approximation," *J. Approx. Theory*, 2011.

- [12] F. J. Obaid and A. R. Sadek, "On  $S\delta$ -derived and  $S\delta$ -isolated sets in supra topological spaces," presented at the THE INTERNATIONAL SCIENTIFIC CONFERENCE OF ENGINEERING SCIENCES AND ADVANCED TECHNOLOGIES, Karbala, Iraq, 2024, p. 080038. doi: 10.1063/5.0236582.
  - [13] F. J. Obaid and A. R. Sadek, "Some types of fan sets in supra topology," *J. Interdiscip. Math.*, vol. 28, no. 3-A, pp. 983–989, 2025, doi: 10.47974/JIM-2078.
  - [14] F. J. Obaid and A. R. Sadek, "Supra Topological Spaces Via Delta-Semi-Open Sets," *Iraqi J. Sci.*, pp. 353–364, Jan. 2026, doi: 10.24996/ij.s.2026.67.1.29.
  - [15] W. Rudin, *Principles of Mathematical Analysis*. New York: McGraw-Hill, 1976.
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