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(Quasi-)Injective Extending Gamma Modules

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Abstract

In this paper we introduce and study the concept of injective (Quasi-injective) extending gamma modules as a generalization of injective (Quasi-injective) gamma modules. An R_Γ −module E is called injective (Quasi-injective) extending gamma modules if each proper R_Γ −submodule in E is essential in injective (Quasi-injective) R_Γ −submodule of E. The concept of injective extending gamma modules lie between injective gamma modules and quasi-injective gamma modules.

Keywords: Gamma module, injective gamma module, quasi-injective gamma module, extending gamma module, essential gamma submodule, closed gamma submodule.

Mathematics subject classification:16D10,16D40,16D50,16D60,16D90.

1. Preliminaries:

Let R and Γ be two additive abelian groups, R is called a Γ −ring (in the sense of Barnes), if there exists a mapping $\cdot : R \times \Gamma \times R \longrightarrow R$, written $\cdot\, (r,\gamma,s) \longmapsto r\gamma s$ such that $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)c = a\alpha c + a\beta c$, $a\alpha(b+c) = a\alpha b + b\alpha c$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in R$ and $\alpha,\beta \in \Gamma$ [4]. A subset A of Γ −ring R is said to be a right (left) ideal of R if A is an additive subgroup of R and $A\Gamma R \subseteq A$($R\Gamma A \subseteq A$), where $A\Gamma R = \{a\alpha r : a \in A,\ \alpha \in \Gamma,\ r \in R\}$. If A is both right and left ideal, we say that A is an ideal of . An element 1 in Γ −ring R is unity if there exists element $\gamma_\circ \in \Gamma$ such that $r = 1\gamma_\circ r = r\gamma_\circ 1$ for every $r \in R$, in this paper we denote $\gamma_\circ \in \Gamma$ to the element such that $1\gamma_\circ$ is the unity, unities in Γ −rings differ from unities in rings, it is possible for a Γ −ring have more than one unity [9]. A Γ −ring R is called commutative, if $r\gamma s = s\gamma r$ for any $r,s \in R$ and $\gamma \in \Gamma$.

Let R be a Γ −ring and M be an additive abelian group. Then M together with a mapping $\cdot\!: R \times \Gamma \times M \rightarrow M$, written $\cdot\, (r,\gamma,m) \longmapsto r\gamma m$ such that $r\alpha(m_1+m_2) = r\alpha m_1 + r\alpha m_2$, $(r_1+r_2)\alpha m = r_1\alpha m + r_2\alpha m$, $r(\alpha+\beta)m = r\alpha m + r\beta m$, $(r_1\alpha r_2)\beta m = r_1\alpha(r_2\beta m)$ for each $r,r_1,r_2 \in R$, $\alpha,\beta \in \Gamma$ and $m,m_1,m_2 \in M$, is called a left R_Γ −module, similarly one can defined right R_Γ −module [4]. A left R_Γ −module M is unitary if there exist elements, say 1 in R and $\gamma_\circ \in \Gamma$ such that $1\gamma_\circ m = m$ for every $m \in M$.

Let M be an R_Γ −module. A nonempty subset N of M is said to be an R_Γ −submodule of M (denoted by $N \leq M$) if N is a subgroup of M and $R\Gamma N \subseteq N$, where $R\Gamma N = \{r\alpha n : r \in R, \alpha \in \Gamma, n \in N\}$ [4]. An R_Γ −module M is called simple if $R\Gamma M \neq 0$ and the only R_Γ −submodules of M are M and 0 [6], A Γ −ring R is called simple if $R\Gamma R \neq 0$ and the only ideals of R are R and 0. If X is a nonempty subset of M, then the R_Γ −submodule of M generated by X denoted by $\langle X \rangle$ and $\langle X \rangle = \cap \{N \leq M : X \subseteq N\}$, X is called the generator of $\langle X \rangle$ and $\langle X \rangle$ is finitely generated if $|X| < \infty$. In particular , if $X = \{x\}$, then$\langle X \rangle$ is called the cyclic R_Γ −submodule of M generated by x. $\langle X \rangle = \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j \gamma_j x_j : k, m \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$. If M is unitary, then $\langle X \rangle = \{\sum_{i=1}^{n} r_i \gamma_i x_i : n \in \mathbb{N}, \gamma_i \in \Gamma, r_i \in R, x_i \in X\}$ [4]. An R_Γ −submodule N of R_Γ −module M is called essential (denote by $N \leq_e M$) if every nonzero R_Γ −submodule of M has nonzero intersection with N, in this case we say that M is an essential extension of N, equivalent to, for each nonzero element m in M there is $r_1, r_2, \ldots, r_n \in R$ and $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$ such that $\sum_{i=1}^{n} r_i \gamma_i m (\neq 0) \in N$ [1]. An R_Γ −submodule N of R_Γ −module M is called direct summand of M if there exists an R_Γ −submodule K of M such that $M = N + K$ and $N \cap K = 0$, in this case M is written as $M = N \oplus K$ [2]. An R_Γ −module M is called semisimple if every R_Γ −submodule is a direct summand of M [3]. An R_Γ −submodule N of R_Γ −module M is called closed in M if it has no proper essential extension in M, equivalent to saying that the only solution of the relation $N \leq_e K \leq M$ is $N = K$ [2].

Let M and N be two R_Γ −modules. A mapping $f : M \to N$ is called homomorphism of R_Γ −modules (simply R_Γ −homomorphism) if $f(x + y) = f(x) + f(y)$ and $f(r\gamma x) = r\gamma f(x)$ for each $x, y \in M, r \in R$ and $\gamma \in \Gamma$. An R_Γ −homomorphism is R_Γ −monomorphism if it is one-to-one and R_Γ −epimorphism if it is onto, the set of all R_Γ −homomorphisms from M into N denote by $Hom_{R_\Gamma}(M, N)$ in particular if $M = N$, $Hom_{R_\Gamma}(M, N)$ denote by $End_{R_\Gamma}(M)$. If M is R_Γ −module, then $End_{R_\Gamma}(M)$ is a Γ −ring with the mapping $\cdot : End_{R_\Gamma}(M) \times \Gamma \times End_{R_\Gamma}(M) \to End_{R_\Gamma}(M)$ denoted by $\cdot$ $(f, \gamma, g) \mapsto f\gamma g$ where $f\gamma g(x) = g(f(1\gamma x))$, for $f, g \in End_{R_\Gamma}(M)$,$\gamma \in \Gamma$ and $x \in M$. All modules in this paper are unitary left R_Γ −modules, in this case M is a right $End_{R_\Gamma}(M)$ −module with the mapping $\cdot : M \times \Gamma \times End_{R_\Gamma}(M) \to End_{R_\Gamma}(M)$ by $\cdot (x, \gamma, f) \mapsto x\gamma f$ where $x\gamma f = f(1\gamma x)$, for $f \in End_{R_\Gamma}(M)$, $\gamma \in \Gamma$ and $x \in M$ [4].

The notions of injective gamma modules and quasi-injective gamma modules have been introduced by M. S. Abbas, S. A. Al-saadi and E. A. Shallal in [1] and [2]. If M and N are two R_Γ −modules, then M is called N −injective R_Γ −module if for any R_Γ −submodule A of N and for any R_Γ −homomorphism $f : A \to M$ there exists an R_Γ −homomorphism $g : N \to M$ such that $gi = f$ where i is the inclusion mapping [1]. An R_Γ −module M is injective if it is N −injective for any R_Γ −module N. It is proved in [1], that every gamma module can be embedded in injective gamma module called injective hull and denote by $E(M)$ which is unique up to isomorphism.

2. Injective Extending Gamma Modules

We extended the concept of injective extending gamma modules from category of modules [8] to the category of gamma modules which is lie between injective gamma modules [1] and quasi-injective gamma modules [2].

An R −module M is called extending if every submodule of M is essential in a direct summand of M [8].

Definition 2.1. An R_Γ −module M is called Γ −Extending if every R_Γ −submodule of M is essential in a direct summand of M.

Proposition 2.2. An R_Γ −module M is Γ −Extending if and only if each closed R_Γ −submodule of M is a direct summand of M.

Proof: Let N be closed R_Γ −submodule of Γ −Extending R_Γ −module M, then there is an R_Γ −submodule K of M such that $N \leq_e K \leq_\oplus M$, so $N = K$. Conversely , Let N be a R_Γ −submodule M, then by using Zorn's lemma N has a maximal essential extension K in M which is closed, so by hypothesis K is direct summand of M, thus M is Γ −Extending.

The following proposition follows from Corollary(3.11) in [2].

Proposition 2.3. Every quasi-injective R_Γ −module is Γ −Extending.

The converse of Proposition(2.3) is not true in general as in Example(2.4)(2).

Examples 2.4.

1- Every semisimple (simple) R_Γ −module is Γ −Extending. Since every R_Γ −submodule of M is a direct summand, then M is Γ −Extending. In particular, Z_2 as Z_Z −module is Γ −Extending.

2- Let $R = \{(n\ \ n), n \in Z\}$ and $\Gamma = \{\binom{\gamma}{0}, \gamma \in Z\}$. Then R is Γ −ring by $\cdot: R \times \Gamma \times R \to R$ with $(n\ \ n)\binom{\gamma}{0}(m\ \ m) = (n\gamma m\ \ n\gamma m)$. Let $I \neq 0$ be an ideal of R, any another ideal P of R with $P \cap I = 0$, take $0 \neq (n\ \ n) \in I$, $0 \neq \binom{\gamma}{0} \in \Gamma$ and $(m\ \ m)$ any element in P, then $(n\gamma m\ \ n\gamma m) = (n\ \ n)\binom{\gamma}{0}(m\ \ m) \in R\Gamma P \subseteq P$, also $(n\gamma m\ \ n\gamma m) = (m\ \ m)\binom{\gamma}{0}(n\ \ n) \in R\Gamma I \subseteq I$, so $n\gamma m = 0$, hence $m = 0$, thus $P = 0$, so every ideal in R is essential, therefore R is Γ −Extending. Note that R is not quasi-injective, take the ideal $I = \{(2n\ \ 2n): n \in Z\}$ and R_Γ −homomorphism $\lambda: I \to R$ by $\lambda(2n\ \ 2n) = (n\ \ n)$ for each $(2n\ \ 2n) \in I$, if R quasi-injective, then there is $g: R \to R$ which extends λ , so $(1\ \ 1) = \lambda(2\ \ 2) = g(2\ \ 2) = g\left((2\ \ 2)\binom{1}{0}(1\ \ 1)\right) = (2\ \ 2)\binom{1}{0} g(1\ \ 1)$, hence $g(1\ \ 1) = (\frac{1}{2}\ \ \frac{1}{2})$ contradiction.

Definition 2.5. An R_Γ −module M is called injective extending ($I\Gamma$ −Extending) if each proper R_Γ −submodule of M is essential in injective R_Γ −submodule of M, that is, for each proper R_Γ −submodule $N \leq M$, there exists an injective R_Γ −submodule K of M such that $N \leq_e K$.

Proposition 2.6. If M is $I\Gamma$ −Extending R_Γ −module, then each proper closed R_Γ −submodule of M is injective.

Proof: Assume $N \leq M$ is proper closed R_{Γ} −submodule, then there is an injective R_{Γ} −submodule K of with $N \leq_e K \leq M$, so $N = K$, thus N is injective.

The converse of Proposition(2.3) is not true in general, for example Z as Z_Z −module , the only proper closed Z_Z − submodule of Z which is injective is 0 but Z is not $I\Gamma$ −Extending.

The following proposition gives the converse of Proposition(2.6) under certain conditions. First we note that a semisimple R_{Γ} −module different from $I\Gamma$ −Extending. The Z_Z −module $Z_2 \oplus Z_2$ is semisimple but not $I\Gamma$ −Extending, if not, then Z_2 is injective by Proposition(2.6) contradiction. The Z_Z −module Q is injective, so $I\Gamma$ −Extending but not semisimple.

Proposition 2.7. Let M semisimple R_{Γ} −module. Then M is a $I\Gamma$ −Extending if and only if each proper closed R_{Γ} −submodule in M is injective.

Proof: For each proper R_{Γ} −submodule N of M, N has a maximal essential extension K by Zorn's lemma. It is clear that K is closed and proper, so by hypothesis, K is injective, thus M is $I\Gamma$ −Extending.

Proposition 2.8. Let M be R_{Γ} −module, if M has a proper nonessential R_{Γ} −submodule, than M is $I\Gamma$ − Extending if and only if each proper closed R_{Γ} −submodule is injective.

Proof: Assume N is a proper nonessential R_{Γ} −submodule of an R_{Γ} −module M, then by Zorn's lemma there is maximal R_{Γ} −submodule K of M such that $N \leq_e K$, clear K is closed and proper, so by hypothesis K is injective. By Proposition(1.9) in [3], there is nonzero R_{Γ} −submodule L of M such that $M = K \oplus L$, again by hypothesis L is injective, hence M is injective[1]. The obverse by Proposition(2.6).

Proposition 2.9. If M is $I\Gamma$ −Extending R_{Γ} −module, then every proper closed R_{Γ} −submodule of M is a direct summand of M. In particular, every $I\Gamma$ −Extending R_{Γ} −module is Γ −Extending.

Proof: Let N is a proper closed R_{Γ} −submodule of $I\Gamma$ −Extending R_{Γ} −module M, then N is injective R_{Γ} −submodule of M by Proposition(2.6), by Proposition(1.9) in [1] N is a direct summand of M.

Proposition 2.10. Every injective R_{Γ} −module is $I\Gamma$ −Extending.

Proof: Let N be a proper R_{Γ} −submodule of an injective R_{Γ} −module M. Then by Zorn's lemma N has a maximal essential extension K in M, clearly that K is closed, by Corollary(3.11) in [2] K is a direct summand in M, so K is injective by Examples and Remarks(1.10) (3) in [1], thus M is $I\Gamma$ −Extending.

Proposition(2.10) shows that there are a lot of $I\Gamma$ −Extending R_{Γ} −modules , for any R_{Γ} −module M it's injective hull is $I\Gamma$ −Extending . In fact every R_{Γ} −module can be embedded in $I\Gamma$ −Extending R_{Γ} −module see [1].

Proposition 2.11. Let M be $I\Gamma$ −Extending R_{Γ} −module. If M has a nontrivial closed R_{Γ} −submodule, then M is injective.

Proof: Let N be a nontrivial closed R_{Γ} −submodule of $I\Gamma$ −Extending R_{Γ} −module M. Then by Proposition(2.6) N is injective R_{Γ} −submodule and by Proposition(1.9) in [1] N is a direct summand of M, so $M = N \oplus K$ for some R_{Γ} −submodule K of M, since N is a nontrivial, then K is proper closed and again by

Proposition(2.6) K is injective and so M is injective [1].

Examples 2.12.

1- Every simple R_Γ −module is $I\Gamma$ −Extending, since the only proper closed R_Γ −submodule is 0 .In particular, the Z_Z −module $M = Z_2$ is $I\Gamma$ −Extending. Note that M is not injective [1], so the converse of Proposition(2.10) is not true in general.

2- Let $R = Z$, $\Gamma = Z$ and $M = Z$, then M is not $I\Gamma$ −Extending, since the R_Γ −submodule $\langle 2\rangle$ is not essential in any injective R_Γ −submodule. Note that M is a Γ −Extending since the only closed R_Γ −submodule of M is 0 which is direct summand, hence the converse of Proposition(2.9) is not true in general.

3- Let $M = Z_6$ as Z_Z − module , the only ideals of Z_6 are 0 , Z_6 , $\langle 2\rangle$ and $\langle 3\rangle$, then Z_6 is semisimple Z_Z −module but not $I\Gamma$ −Extending.

4- If R is semisimple R_Γ −module, then every R_Γ −module is injective [3], so $I\Gamma$ −Extending.

Direct sum of two $I\Gamma$ −Extending R_Γ −modules may not be $I\Gamma$ −Extending, for example The Z_Z −module Z_2 is $I\Gamma$ −Extending but not injective Examples(2.12)(1), the Z_Z −module $Z_2 \oplus Z_2$ is not $I\Gamma$ −Extending, if not, then Z_2 is injective by Proposition(2.6) which is a contradiction .

Proposition 2.13. If direct sum of every two $I\Gamma$ −Extending R_Γ −modules is $I\Gamma$ −Extending, then M is injective if and only if M is $I\Gamma$ −Extending.

Proof: Let M be $I\Gamma$ −Extending, since $E = E(M)$ is injective, so E is $I\Gamma$ −Extending by Proposition(2.10), hence by hypothesis $M \oplus E$ is $I\Gamma$ − Extending, by Proposition(2.6) M is injective.

The following proposition gives the converse of Proposition(2.6) under another condition.

Proposition 2.14. Let M be R_Γ −module contains a nontrivial nonessential R_Γ −submodule. Then the following statements are equivalent:

1- M is injective.
2- M is $I\Gamma$ −Extending.
3- Every proper closed R_Γ −submodule of M is injective.

Proof: $(1) \Rightarrow (2)$ By Proposition(2.10). $(2) \Rightarrow (3)$ By Proposition(2.6). $(3) \Rightarrow (1)$ Assume N is a nontrivial R_Γ −submodule which is not essential in M, then by Zorn's lemma N has a maximal essential extension R_Γ −submodule K in M which is closed in M, if $K = M$ then N is essential in M contradiction, so K is a proper by hypothesis K is injective and a direct summand of M by Proposition(1.9) in [1], so $M = K \oplus L$ for some R_Γ −submodule L of M, if $L = 0$ a contradiction, hence L is a proper again by hypothesis L is injective, therefore M is injective [1].

Proposition 2.15. An R_Γ −module M is $I\Gamma$ −Extending if and only if M contains injective hulls of each of its proper R_Γ −submodule.

Proof: For each proper R_Γ −submodule N of $I\Gamma$ −Extending R_Γ −module M, there exists injective R_Γ −submodule K of M such that $N \leq_e K \leq M$ but $N \leq E(N)$ which is minimal injective extension of N, so $N \leq E(N) \leq K \leq M$, hence $E(N) \leq M$.

Corollary 2.16. If M is $I\Gamma$ −Extending, then M is injective hull of each proper essential R_Γ −submodule of M.

Proof: For each proper essential R_Γ −submodule N of M, by Proposition(2.15) $E(N) \leq M$ but $E(N)$ is injective, therefore by Proposition(1.9) in [1] $E(N)$ is a direct summand of M, so $M = E(N) \oplus L$ for some R_Γ −submodule L of M. But $N \leq_e M$, then $L = 0$, thus $M = E(N)$.

In the following proposition we a characterization of $I\Gamma$ −Extending R_Γ −modules in which every proper R_Γ −submodule lies under injective direct summand.

Proposition 2.17. An R_Γ −module M is $I\Gamma$ −Extending if and only if for every proper R_Γ −submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that M_1 is injective, $N \leq_e M_1$ and $N \oplus M_2 \leq_e M$.

Proof: Assume that N is a proper R_Γ −submodule of M, then there exists injective R_Γ −submodule M_1 such that $N \leq_e M_1 \leq M$, so by Proposition(1.9) in [1] M_1 is a direct summand of M, hence $M = M_1 \oplus M_2$ for some R_Γ −submodule M_2 of M, since $M_2 \leq_e M_2$ by Lemma(3.3) in [1] $N \oplus M_2 \leq_e M_1 \oplus M_2 = M$.

Proposition 2.18. Let M be R_Γ −module. Then M is $I\Gamma$ −Extending if and only if either M is simple or M is injective.

Proof: Assume M is not simple, then there exists a nonzero R_Γ −submodule N of M with $M \neq N$, Also there exists injective R_Γ −submodule K such that $N \leq_e K \leq M$ but K is a direct summand of M by Proposition(1.9) in [1], then $M = K \oplus L$ for some R_Γ −submodule L of M, if $L = M$, then M is injective, if $L \neq M$, then L is proper closed, so by Proposition(2.6) L is injective, hence M is injective [1]. The other direction follows from Proposition(2.10).

Corollary 2.19. Every $I\Gamma$ −Extending R_Γ −module is quasi-injective.

Corollary 2.20. Let M be a not simple R_Γ −module. Then M is injective if and only if M is $I\Gamma$ −Extending.

3. Quasi-Injective Extending Gamma Modules

In this section we introduce the concept of quasi-injective extending gamma modules as a generalization of quasi-injective gamma modules. An R_Γ −module M is quasi-injective if it is M −injective, that is for any R_Γ −submodule N of M and R_Γ −homomorphism $f: N \to M$, there exists an R_Γ −endomorphism g of M such that $gi = f$ where i is the inclusion mapping of N into M [2].

Definition 3.1. An R_Γ −module M is called quasi-injective extending gamma (simply $QI\Gamma$ −Extending) if every proper R_Γ −submodule of M is essential in a quasi-injective R_Γ −submodule of M, that is, for each proper R_Γ −submodule N of M , there exists an quasi-injective R_Γ −submodule K of M such that $N \leq_e K$.

Proposition 3.2. If M is $QI\Gamma$ −Extending R_Γ −module, then each proper closed R_Γ −submodule of M is quasi-injective.

Proof: Assume N is proper closed R_Γ −submodule of M, then there exists an quasi-injective R_Γ −submodule K of such that $N \leq_e K \leq M$, since N is closed then $N = K$.

The converse of Proposition(3.2) is not true in general, for example $M = Z$ as Z_Z −module.

Proposition 3.3. If M is $QI\Gamma$ −Extending R_Γ −module, then $N \cap M$ is quasi-injective for each proper direct summand N of $E(M)$.

Proof: Let N be a proper direct summand of $E(M)$, then $E(M) = N \oplus B$ for some R_Γ −submodule B of $E(M)$, we claim that $N \cap M$ is closed in M, assume that $N \cap M \leq_e K$ where K is an R_Γ −submodule of M with $N \cap M \neq K$, let $k \in K$, then $k = n + b$ where $n \in N$ and $b \in B$. Now consider $k \notin N$, then $b \neq 0$. But $M \leq_e E(M)$ and $0 \neq b \in B \leq E(M)$, therefore there is $r_1, r_2, \dots, r_n \in R$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $\sum_{i=1}^{n} r_i\gamma_i b\ (\neq 0) \in M$, so $\sum_{i=1}^{n} r_i\gamma_i k = \sum_{i=1}^{n} r_i\gamma_i n + \sum_{i=1}^{n} r_i\gamma_i b$, and $\sum_{i=1}^{n} r_i\gamma_i n = \sum_{i=1}^{n} r_i\gamma_i k - \sum_{i=1}^{n} r_i\gamma_i b \in N \cap M \leq K$, thus $\sum_{i=1}^{n} r_i\gamma_i b = \sum_{i=1}^{n} r_i\gamma_i k - \sum_{i=1}^{n} r_i\gamma_i n \in B \cap K$, since $(N \cap M) \cap B = 0$, then $(N \cap M) \cap (B \cap K) = 0$, but $N \cap M \leq_e K$, so $B \cap K = 0$, hence $\sum_{i=1}^{n} r_i\gamma_i b = 0$ which is a contradiction, thus $N \cap M$ is closed in M, if $N \cap M = M$, then $M \leq N$, so $M \cap B = 0$ but $M \leq_e E(M)$, then $B = 0$ which is a contradiction, thus $N \cap M$ is a proper closed R_Γ −submodule of M, so by Proposition(3.2) $N \cap M$ is a quasi-injective.

Proposition 3.4. Every quasi-injective R_Γ −module is $QI\Gamma$ −Extending.

Proof: Assume M is quasi-injective and N is a proper R_Γ −submodule of M, then by Zorn's lemma N is essential in a maximal closed R_Γ −submodule K of M, by Corollary(3.11) in [2] K is a quasi-injective, hence M is $QI\Gamma$ −Extending.

The converse of Proposition(3.4) is not true in general, see Example(3.8)(5).

Corollary 3.5. Every semisimple R_Γ −module is $QI\Gamma$ −Extending.

The converse of Corollary(3.5) is not true in general, for Example $M = Q$ as Z_Z −module is injective, so $QI\Gamma$ −Extending but not semisimple.

An R_Γ −module M is called regular if for each $m \in M$, there exists $f \in Hom_{R_\Gamma}(M, R)$ and $\gamma \in \Gamma$ such that $m = f(m)\gamma m$ [3]. Every cyclic R_Γ −submodule of regular R_Γ −module is a direct summand [3].

Corollary 3.6. Every regular cyclic R_Γ −module is $QI\Gamma$ −Extending.

The converse of Corollary(3.6) is not true in general, for example $M = Q$ as Z_Z −module is $QI\Gamma$ −Extending but not regular.

It is proved in [2], that every gamma module has quasi-injective extension say quasi-injective hull (denote by $Q(M)$) which is unique up to isomorphism.

Corollary 3.7. Every R_Γ −module can be embedded in $QI\Gamma$ −Extending.

Examples 3.8.

1- If M is $QI\Gamma$ −Extending , then M contains quasi-injective hull of each it's proper R_Γ −submodules, the proof is essentially as in Proposition(2.15).

2- An R_Γ −module $M = Z_6$ as Z_Z −module is a semisimple [3], so M is $QI\Gamma$ −Extending by Corollary(3.5). Note that M is not $I\Gamma$ −Extending

3- Every $I\Gamma$ −Extending is $QI\Gamma$ −Extending. The converse is not true for example see Example(2).

4- The Z_Z −module $M = Z\oplus Z_2$ is not $QI\Gamma$ −Extending, if not the R_Γ −submodule $N = Z\oplus(0)$ is proper closed in M which is not quasi-injective [2], which is a contradiction by Proposition(3.2).

5- Let $M = Z_2\oplus Z_4$ as Z_Z − module. The only R_Γ −submodules of M are (0), $N_1 = (0)\oplus Z_4$, $N_2 = Z_2\oplus(0)$, $N_3 = Z_2\oplus\langle 2\rangle = \{(0,0),(1,2),(1,0),(0,2)\}$, $N_4 = \langle(1,1)\rangle = \{(0,0),(1,1),(0,2),(1,3)\}$, $N_5 = \langle(0,2)\rangle = \{(0,0),(0,2)\}$, $N_6 = \langle(1,2)\rangle = \{(0,0),(1,2)\}$ and M. Note that N_2, N_5 and N_6 are simple , N_3 semisimple , $N_4 \cong N_1$, so every proper R_Γ −submodule of M is quasi-injective, hence M is $QI\Gamma$ −Extending but M is not quasi-injective since the R_Γ −submodule $N_5 \cong N_2$ but N_2 is a direct summand while N_5 is not direct summand which is a contradiction see Corollary(3.9) in [2].

6- The Z_Z −module $M = Z_6\oplus Z_4$ is not $QI\Gamma$ −Extending, since $M = N\oplus B$ where $N = \langle 3\rangle\oplus Z_4$ and $B = \langle 2\rangle\oplus(0)$, so N is a proper closed of M but N is not quasi-injective since the R_Γ −submodule $K = (0)\oplus\langle 2\rangle \cong Z_3\oplus(0)$ of N is not direct summand of N which is contradiction see Corollary(3.9) in [2].

The concept quasi-injective extending gamma modules is a proper generalization of quasi-injective gamma modules, see Examples(3.8)(5).

We conclude from Proposition(2.10), Corollary(2.19) and Proposition(3.4) the following chart of implications for R_Γ −modules

$$\textbf{Injective} \Longrightarrow \textbf{I}\Gamma-\textbf{Extending} \Longrightarrow \textbf{Quasi}-\textbf{Injective} \quad \begin{matrix} \nearrow\ \textbf{QI}\Gamma-\textbf{Extending} \\ \searrow\ \Gamma-\textbf{Extending} \end{matrix}$$

An R_Γ −submodule of $QI\Gamma$ −Extending need not be $QI\Gamma$ −Extending, for example The Z_Z −module Q is injective [1] hence $QI\Gamma$ −Extending but the R_Γ −submodule Z is not $QI\Gamma$ −Extending by Examples(3.8)(1).

An R_Γ −submodule N of R_Γ −module M is called R_Γ −idempotent if $N = (N:_{R_\Gamma} M)\Gamma N$ and M is called fully R_Γ −idempotent if every R_Γ −submodule of M is R_Γ −idempotent [3].

Proposition 3.9. If M is $QI\Gamma$ −Extending. Then every proper R_Γ −idempotent R_Γ −submodule of M is quasi-injective ($QI\Gamma$ −Extending).

Proof: Let N be a proper R_Γ −idempotent R_Γ −submodule of $QI\Gamma$ −Extending M. Then M contains quasi-injective hull $Q(N)$ of N by Examples(3.8). Since N is R_Γ −idempotent, then $N = (N:_{R_\Gamma} Q(N))\Gamma N$. For each R_Γ −submodule X of N and R_Γ −homomorphism $f: X \to N$, there exists $g: Q(N) \to Q(N)$ which extends f, for each $n \in N$, $n = r\gamma m$ where $r \in (N:_{R_\Gamma} Q(N))$,

$\gamma \in \Gamma$ and $m \in N$, so $g(n) = g(r\gamma m) = r\gamma g(m) \in N$, thus N is quasi-injective [3].

Corollary 3.10. Let M be fully R_Γ −idempotent. Then M is $QI\Gamma$ −Extending if and only if every R_Γ −submodule of M is $QI\Gamma$ −Extending.

An R_Γ −module M is called duo if $f(N) \subseteq N$ for each R_Γ −submodule N of M and $f \in End_{R_\Gamma}(M)$ [3].

Corollary 3.11. If M is duo $I\Gamma$ −Extending R_Γ −module, then every R_Γ −submodule of M is quasi-injective.

Proof: By Corollary(2.19) M is quasi-injective. For any proper R_Γ −submodule N of M, let X be an R_Γ −submodule of N and $f: X \to N$ be an R_Γ −homomorphism, then there exists $\alpha: E(M) \to E(M)$ which extends to f, since M is quasi-injective, then $\alpha(M) \subseteq M$ [2], hence $\theta = \alpha|_M : M \to M$ is extends to f but M is duo therefore $\beta = \theta|_N : N \to N$ extends to f, thus N is quasi-injective [2].

It is proved in [3], that every fully R_Γ −idempotent R_Γ −module is duo.

Corollary 3.12. If M is fully R_Γ −idempotent $I\Gamma$ −Extending R_Γ −module, then every R_Γ −submodule of M is quasi-injective.

We need the following lemma to prove Proposition(3.14).

Lemma 3.13. Let M be an R_Γ −module. If A essential R_Γ −submodule of M and B is a closed R_Γ −submodule of M, then $A \cap B$ is closed in A.

Proof: Let B be a closed R_Γ −submodule of M and A essential R_Γ −submodule of M. By Lemma(3.5) in [2] B must be a complement of some R_Γ −submodule T of M, $(B \cap A) \cap (T \cap A) = (B \cap T) \cap A = 0$. Assume there is an R_Γ −submodule N of A contains $B \cap A$ properly , then $(B + N) \cap T \neq 0$, so there exists $0 \neq t = b + n$ where $t \in T$, $b \in B$ and $n \in N$, since A essential in M, then there exists $r_1, r_2, \dots, r_n \in R$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $0 \neq \sum_{i=1}^{n} r_i \gamma_i t = \sum_{i=1}^{n} r_i \gamma_i b + \sum_{i=1}^{n} r_i \gamma_i n \in T \cap A$, so $\sum_{i=1}^{n} r_i \gamma_i b = \sum_{i=1}^{n} r_i \gamma_i t - \sum_{i=1}^{n} r_i \gamma_i n \in B \cap A \subseteq N$, hence $0 \neq \sum_{i=1}^{n} r_i \gamma_i t \in (T \cap A) \cap N$, so $B \cap A$ is maximal R_Γ −submodule of A with respect to $(B \cap A) \cap (T \cap A) = 0$, hence $B \cap A$ is a complement of $T \cap A$ in A.

Proposition 3.14. Let M be a $QI\Gamma$ −Extending R_Γ −module and N be a nontrivial closed R_Γ −submodule of M. Then N and N^c are quasi-injective R_Γ −submodule of M.

Proof: Let N be a nontrivial closed R_Γ −submodule of an $QI\Gamma$ −Extending R_Γ −module M. Then N is a quasi-injective by Proposition(3.2). In case $M = N \oplus N^c$, then N^c is a proper closed in M and hence N^c is a quasi-injective by Proposition(3.2), in case $M \neq N \oplus N^c$, then there is a quasi-injective R_Γ −submodule Q of M such that $N \oplus N^c \leq_e Q \leq M$ since $N \oplus N^c \leq_e M$ by Lemma(3.4) in [2], then $Q \leq_e M$ but N is closed in M, so $N = N \cap Q$ is closed in Q by Lemma(3.13), hence N is a quasi-injective by Corollary(3.11) in [2].

Direct sum of two QIΓ −Extending need not be QIΓ −Extending, for example see Examples(3.8)(6).

Proposition 3.15. If the direct sum of every two $QI\Gamma$ −Extending is $QI\Gamma$ −Extending, then M is quasi-injective if and only if M is $QI\Gamma$ −Extending.

Proof: Let M be $QI\Gamma$ −Extending, since $Q(M)$ is quasi-injective, so $Q(M)$ is $QI\Gamma$ −Extending, hence $M\oplus Q(M)$ is $QI\Gamma$ −Extending, by Proposition(3.2) M is quasi-injective.

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مقاسات التوسع (شبه -) الاغمارية من نمط گاما

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المستخلص :

في هذا البحث نطرح مفهوم المقاسات الاغمارية (شبه الاغمارية) الموسعة من نمط كاما كتعميم الى مفهوم المقاسات الاغمارية (شبه الاغمارية) من نمط كاما . المقاس M يسمى مقاس اغماري (شبه اغماري) موسع من نمط كاما اذا كان كل مقاس جزئي فعلي في M يكون جوهريا" في مقاس جزئي اغماري (شبه اغماري) في M . مفهوم المقاسات الاغمارية الموسعة من نمط كاما تقع بين المقاسات الاغمارية من نمط كاما والمقاسات شبه الاغمارية من نمط كاما .