Journal of AL-Qadisiyah for computer science and mathematics Vol.9 No.2 Year 2017
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

$\mathcal{B}-C^*$ algebra Metric Space and Some Results Fixed point Theorems

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Recived : 29\8\2017 Revised : 7\9\2017 Accepted : 11\9\2017

Abstract:

On the aim and properties of C^*- algebra, in this paper we introduce the two concepts $\mathcal{B}-C^*$ algebra and $\mathcal{B}-C^*$algebra metric space as well as introduce concept convergent and Cauchy sequence in space and to study the existence of fixed point theorems with contraction condition and $\mathcal{B}-C^*$ algebra expansion on this space.

Keywords: $\mathcal{B}-C^*$ algebra, $\mathcal{B}-C^*$ algebra metric space, convergent, contraction function, fixed point theorem and $\mathcal{B}-C^*$ algebra expansion .

Mathematical subject classification: 03E72 , 46 S40.

1. Introduction:

At the beginning of the introduction here we must introduce the concept of C^*- algebra presented by which [2]. The main idea $\mathbb{B}$ is a unital algebra with unit I. An involution on $\mathbb{B}$ is a conjugate linear function $b \longrightarrow b^*$ satisfying

(i) $(b^*)^* = b$, (ii) $(bc)^* = c^*b^*$,

for all $b, c \in \mathbb{B}$. $(\mathbb{B},*)$ is called $*-$ algebra.

Banach $*-$ algebra is a $*-$ algebra $\mathbb{B}$ with complete submultiplicative norm such that $\|b^*\| = \|b\|$. A C^*- algebra is a Banach $*-$ algebra such that $\|b^*b\| = \|b\|^2$ [2,6].

Clearly that under the norm topology, $L(H)$, the set of all bounded linear operator on Hilbert space H, is a C^*- algebra.

As we previously knew, Banach contraction is a very useful, a simple and classical tool in modern analysis. Also, it is important tool for solving problems of presence in many branches of mathematics. In general, fixed point theory has been mainstreamed, with the usual contractive condition is replaced by a new contractive condition. On the other hand, work spaces are replaced by metric spaces with an order on closed and bounded Banach algebra. In recent years, O Reagan and Petrusel [5] began investigations on a fixed theoretical point in the required metric space. And that many authors mainstream their theories a fixed point on a different type of C^*- algebra metric algebra spaces [3,4]. in this paper We will present Based on the concept and property of $\mathcal{B}-C^*$ algebra we first introduce a concept of $\mathcal{B}-C^*$algebra metric space, some fixed point theorems for function the contractive condition and $\mathcal{B}-C^*$ algebra expansion on such space.

Journal of AL-Qadisiyah for computer science and mathematics Vol.9 No.2 Year 2017
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Now: Suppose the following two conditions:

$\mathcal{C} = \{\mathcal{M}: \mathcal{M}$ is a C^* − subalgebra *closed* subset of $\mathbb{B}\}$.

$\mathcal{B} = \{\mathcal{L}: \mathcal{L}$ is a C^* − subalgebra *closed and bounded* subset of $\mathbb{B}\}$.

In the item, $\mathcal{B}$ mean to us that $\mathcal{B} - C^*$ algebra.

2- Main Results

Definition (2.1): Suppose that the function $d_{\mathcal{B}}: X \times X \to \mathcal{B}$ is defined, with the following properties:

(1) $d_{\mathcal{B}}(x, y) \succcurlyeq 0_{\mathcal{B}}$, for all x, y in X.

(2) $d_{\mathcal{B}}(x, y) = 0_{\mathcal{B}} \Leftrightarrow x = y$.

(3) $d_{\mathcal{B}}(x, y) = d_{\mathcal{B}}(y, x)$ for all x, y in X.

(4) $d_{\mathcal{B}}(x, z) \preccurlyeq d_{\mathcal{B}}(x, y) + d_{\mathcal{B}}(y, z)$ for all $x, y, z \in X$.

Then $d_{\mathcal{B}}$ is said to be a $\mathcal{B} - C^*$ algebra metric on X, and $(X, \mathcal{B}, d_{\mathcal{B}})$ is said to be a $\mathcal{B} - C^*$ algebra metric space.

Definition (2.2): Suppose that $(X, \mathcal{B}, d_{\mathcal{B}})$ is a $\mathcal{B} - C^*$ algebra metric space. Let $\{x_n\}$ be a sequence in X and $x \epsilon X$. If $d_{\mathcal{B}}(x_n, x) \xrightarrow{\|.\|_{\mathcal{B}}} 0_{\mathcal{B}}$ $(n \to \infty)$, then it is said that $\{x_n\}$ is converge to x.

Definition (2.3): Suppose that $(X, \mathcal{B}, d_{\mathcal{B}})$ is a $\mathcal{B} - C^*$ algebra metric space. Let $\{x_n\}$ be a sequence in $\mathcal{B}$. If for any $n \in \mathbb{N}$, $d_{\mathcal{B}}(x_{m+n}, x_n) \xrightarrow{\|.\|_{\mathcal{B}}} 0_{\mathcal{B}}$ $(n \to \infty)$, then it is called a Cauchy sequence in $\mathcal{B}$

Note: A $\mathcal{B} - C^*$ algebra metric space $(X, \mathcal{B}, d_{\mathcal{B}})$ is complete if every Cauchy sequence in $\mathcal{B}$ is convergent.

Example (2.4): If X is a topological space, $CB(X)$ be a collection closed and bounded continuous subalgebra of algebra $\ell^\infty(X)$, then $CB(X)$ is C^* − algebra.

For $u, v \in CB(X)$, we define $d_{\mathcal{B}}: X \times X \longrightarrow CB(X)$ by

$d_{\mathcal{B}}(u, v) = \max(|u_1 - v_1|, |u_2 - v_2|)$

Where $u = [u_1, u_2]$, $v = [v_1, v_2]$

Then $d_{\mathcal{B}}$ is a $\mathcal{B} - C^*$ algebra metric space, and $(X, \mathcal{B}, d_{\mathcal{B}})$ is a complete $\mathcal{B} - C^*$ algebra metric space.

Definition (2.5),[3]: Let $(X, \mathcal{B}, d_{\mathcal{B}})$ be a complete metric space and $h: X \to X$ be a function such that for all $x, y \in X$,

$$d_{\mathcal{B}}(hx, hy) \leq \lambda\, d_{\mathcal{B}}(x, y),$$

where $\lambda \in [0, 1)$. Then h has a fixed point, i.e., there exists a point $x \in X$ such that $hx = x$.

Definition (2.6),[4]: An element $a \in \mathcal{B}$ is invertible if there is element $b \in \mathcal{B}$ such that $ab = ba = 1$

$$Inv(\mathcal{B}) = \{a \in \mathcal{B}: a \text{ is invertible}\}$$

Lemma (2.7): Let $\mathcal{B}$ be a unital $\mathcal{B} - C^*$algebra with a unit $1_{\mathcal{B}}$

1- If $b \in \mathcal{B}_+$, we have $b \preccurlyeq 1_{\mathcal{B}} \Leftrightarrow \|b\| \preccurlyeq 1_{\mathcal{B}}$.
2- If $b \in \mathcal{B}_+$ with $\|b\| \preccurlyeq \frac{1}{2}$, then $1_{\mathcal{B}} - b$ is invertible.
3- If $b, c \in \mathcal{B}_+$ and $cb = bc$, then $bc \succcurlyeq 0_{\mathcal{B}}$.
4- If $b, c \in \acute{\mathcal{B}}_h$ and $d \in \acute{\mathcal{B}}_+$ with $d \preccurlyeq b \preccurlyeq c$ and $1 - b \in \acute{\mathcal{B}}_+$ is an invertible, then $(1 - b)^{-1}c \succcurlyeq (1 - b)^{-1}d$.

Theorem (2.8): Let $(X, \mathcal{B}, d_{\mathcal{B}})$ be a complete $\mathcal{B} - C^*$algebra metric space. Suppose that the function $h: X \to X$ satisfies the following condition:

$$d_{\mathcal{B}}(hx, hy) \preccurlyeq \wp^* d_{\mathcal{B}}(x, y)\wp$$

where $\wp \in \acute{\mathcal{B}}_+$ with $\|\wp\| \prec 1$, then h has a unique fixed point in X.

Proof: If $\wp = 0_{\mathcal{B}}$, it is clear that h has fixed point in X.

Suppose that $\wp \neq 0_{\mathcal{B}}$

Let $x_0 \in X$ and $x_{n+1} = hx_n = \cdots = h^{n+1}x_0$

If $b, c \in \mathcal{B}$ and $b \preccurlyeq c$, then for any $d \in \mathcal{B}$ both d^*bd and d^*cd are positive element and $d^*bd \preccurlyeq d^*cd$, we get

$$d_{\mathcal{B}}(x_{n+1},x_n)=d_{\mathcal{B}}(hx_n,hx_{n-1})\preccurlyeq \wp^* d_{\mathcal{B}}(x_n,x_{n-1})\wp$$

$$\preccurlyeq (\wp^*)^2 d_{\mathcal{B}}(x_{n-1},x_{n-2})\wp^2$$

$$\preccurlyeq \cdots \preccurlyeq (\wp^*)^n d_{\mathcal{B}}(x_1,x_0)\wp^n$$

$$=(\wp^*)^n \mathcal{P} \wp^n$$

We have to prove $\{x_n\}$ is Cauchy sequence, for any $n,m \geqslant 1$

$$d_{\mathcal{B}}(x_{m+n},x_n)\preccurlyeq \wp[d_{\mathcal{B}}(x_{m+n},x_{m+n-1})+d(x_{m+n-1},x_m)]$$

$$=\wp d_{\mathcal{B}}(x_{m+n},x_{m+n-1})+\wp d_{\mathcal{B}}(x_{m+n-1},x_m)$$

$$\preccurlyeq (\wp^*)^n\mathcal{P}\wp^n+\cdots+(\wp^*)^m\mathcal{P}\wp^m$$

$$=\sum_{k=n}^{m}(\wp^*)^k\mathcal{P}\wp^k$$

$$=\sum_{k=n}^{m}(\wp^*)^k\mathcal{P}^{\frac{1}{2}}\mathcal{P}^{\frac{1}{2}}\wp^k$$

$$=\sum_{k=n}^{m}\left(\wp^k\mathcal{P}^{\frac{1}{2}}\right)^*(\mathcal{P}^{\frac{1}{2}}\wp^k)$$

$$=\sum_{k=n}^{m}\left|\mathcal{P}^{\frac{1}{2}}\wp^k\right|^2$$

$$\preccurlyeq\left\|\sum_{k=n}^{m}\left|\mathcal{P}^{\frac{1}{2}}\wp^k\right|^2\right\|I$$

$$\preccurlyeq\sum_{k=n}^{m}\left\|\mathcal{P}^{\frac{1}{2}}\right\|^2\|\wp^k\|^2 I$$

$$\preccurlyeq\|\mathcal{P}\|\sum_{k=n}^{m}\|\wp\|^{2k}I$$

$$\preccurlyeq\|\mathcal{P}\|\frac{\|\wp\|^{2n}}{1-\|\wp\|}I\to 0_{\mathcal{B}}$$

Therefore $\{x_n\}$ is Cauchy sequence. Since $(X,\mathcal{B},d_{\mathcal{B}})$ is complete, there exist an $x\in X$ such that $lim_{n\to\infty}x_n = lim_{n\to\infty}hx_{n-1}=x$.

Since

$$0_{\mathcal{B}}\preccurlyeq d_{\mathcal{B}}(hx,x)\preccurlyeq d_{\mathcal{B}}(hx,hx_n)+d_{\mathcal{B}}(hx_n,x)$$

$$\preccurlyeq \wp^* d_{\mathcal{B}}(x_n,x)\wp+d_{\mathcal{B}}(x_{n+1},x)\to 0_{\mathcal{B}}$$

hence $hx=x$, (x is fixed point of h)

Now, let z be another fixed point of h such that $x\neq z$

$$0_{\mathcal{B}}\preccurlyeq d_{\mathcal{B}}(x,z)=d_{\mathcal{B}}(hx,hz)\preccurlyeq \wp^* d_{\mathcal{B}}(x,z)\wp.$$

We have

$$0_{\mathcal{B}}\preccurlyeq\|d_{\mathcal{B}}(x,z)\|=\|d_{\mathcal{B}}(hx,hz)\|\preccurlyeq\|\wp^* d_{\mathcal{B}}(x,z)\wp\|$$

$$\preccurlyeq\|\wp^*\|\|d_{\mathcal{B}}(x,z)\|\|\wp\|$$

$$=\|\wp\|^2\|d_{\mathcal{B}}(x,z)\|\prec\|d_{\mathcal{B}}(x,z)\|$$

Hence $x=z$

This means that the fixed point is unique.

Theorem (2.9): Let $(X,\mathcal{B},d_{\mathcal{B}})$ be a complete $\mathcal{B}-C^*$ algebra metric space, and $h: X\to X$ satisfies the following condition for all $x,y\in X$

$$d_{\mathcal{B}}(hx,hy)\preccurlyeq\mathfrak{J}\big(d_{\mathcal{B}}(hy,x)+d_{\mathcal{B}}(hx,y)\big)$$

where $\mathfrak{J}\in\acute{\mathcal{B}}_+$ and $\|\mathfrak{J}\|\prec\frac{1}{2}$. Then there exists a unique fixed point in X.

Proof: If $\mathfrak{J}=0_{\mathcal{B}}$, it is clear that h has fixed point in X.

Suppose that $\mathfrak{J}\neq 0_{\mathcal{B}}$

Let $x_0\in X$ and $x_{n+1}=hx_n=\cdots=h^{n+1}x_0$

$$d_{\mathcal{B}}(x_n,x_{n+1})=d_{\mathcal{B}}(hx_{n-1},hx_n)$$

$$\preccurlyeq\mathfrak{J}\big(d_{\mathcal{B}}(hx_n,x_{n-1})+d_{\mathcal{B}}(hx_{n-1},x_n)\big)$$

$$=\mathfrak{J}\big(d_{\mathcal{B}}(hx_n,hx_{n-2})+d_{\mathcal{B}}(hx_{n-1},hx_{n-1})\big)$$

$$\preccurlyeq\mathfrak{J}\big(d_{\mathcal{B}}(hx_n,hx_{n-1})+d_{\mathcal{B}}(hx_{n-1},hx_{n-2})\big)$$

$$=\mathfrak{J}d_{\mathcal{B}}(hx_n,hx_{n-1})+\mathfrak{J}d_{\mathcal{B}}(hx_{n-1},hx_{n-2})$$

$$=\mathfrak{J}d_{\mathcal{B}}(x_{n+1},x_n)+\mathfrak{J}d_{\mathcal{B}}(x_n,x_{n-1})$$

Using lemma (2.7), we have

$$(I-\mathfrak{J})d_{\mathcal{B}}(x_{n+1},x_n)\preccurlyeq\mathfrak{J}d_{\mathcal{B}}(x_n,x_{n-1})$$

Since $b\in\acute{\mathcal{B}}_+$ with $\|\mathcal{B}\|\preccurlyeq\frac{1}{2}$, then $(1_{\mathcal{B}}-\mathfrak{J})^{-1}\in\acute{\mathcal{B}}_+$ and furthermore $\mathfrak{J}(1-\mathfrak{J})^{-1}\in\acute{\mathcal{B}}_+$ with $\|\mathfrak{J}(1-\mathfrak{J})^{-1}\|\prec 1$, therefore :

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ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

$$d_{\mathcal{B}}(x_{n+1},x_n) \preccurlyeq \mathfrak{J}(1-\mathfrak{J})^{-1} d_{\mathcal{B}}(x_n,x_{n-1}) = r\, d_{\mathcal{B}}(x_n,x_{n-1})$$

where $r = \mathfrak{J}(1-\mathfrak{J})^{-1}$

We have to prove $\{x_n\}$ is Cauchy sequence, for any $n,m \succcurlyeq 1$

$$d_{\mathcal{B}}(x_{m+n},x_n) \preccurlyeq \mathfrak{J}[d_{\mathcal{B}}(x_{m+n},x_{m+n-1}) + d_{\mathcal{B}}(x_{m+n-1},x_m)]$$

$$= \mathfrak{J} d_{\mathcal{B}}(x_{m+n},x_{m+n-1}) + \mathfrak{J} d_{\mathcal{B}}(x_{m+n-1},x_m)$$

$$\preccurlyeq \mathfrak{J} d_{\mathcal{B}}(x_{m+n},x_{m+n-1}) + \mathfrak{J}^2[d_{\mathcal{B}}(x_{m+n-1},x_{m+n-2}) + d_{\mathcal{B}}(x_{m+n-2},x_n)]$$

$$= \mathfrak{J} d_{\mathcal{B}}(x_{m+n},x_{m+n-1}) + \mathfrak{J}^2 d_{\mathcal{B}}(x_{m+n-1},x_{m+n-2}) + \mathfrak{J}^2 d_{\mathcal{B}}(x_{m+n-2},x_n)$$

$$\preccurlyeq \mathfrak{J} d_{\mathcal{B}}(x_{m+n},x_{m+n-1}) + \mathfrak{J}^2 d_{\mathcal{B}}(x_{m+n-1},x_{m+n-2}) + \mathfrak{J}^3 d_{\mathcal{B}}(x_{m+n-2},x_{m+n-3}) + \cdots + \mathfrak{J}^{m-1} d_{\mathcal{B}}(x_{n+1},x_n)$$

$$\preccurlyeq \mathfrak{J} r^{m+n-1} + \mathfrak{J}^2 r^{m+n-2} + \cdots + \mathfrak{J}^{m-1} r^n$$

$$= \sum_{k=1}^{m-1} \mathfrak{J}^k r^{n+m-k} + \mathfrak{J}^{m-1} r^n$$

$$= \sum_{k=1}^{m-1} \left|\mathfrak{J}^{\frac{k}{2}} r^{\frac{n+m-k}{2}}\right|^2 + \left|\mathfrak{J}^{\frac{m-1}{2}} r^{\frac{n}{2}}\right|^2$$

$$\preccurlyeq \sum_{k=1}^{m-1} \|\mathfrak{J}\|^k \|r\|^{n+m-k} I + \|\mathfrak{J}\|^{m-1}\|r\|^n$$

$$\preccurlyeq \frac{\|\mathfrak{J}\|^m \|r\|^{n+1}}{\|\mathfrak{J}\| - \|r\|} I + \|\mathfrak{J}\|^{m-1}\|r\|^n I \to 0_{\mathcal{B}}$$

Therefore $\{x_n\}$ is Cauchy sequence. Since $(X,\mathcal{B},d_{\mathcal{B}})$ is complete, there exist an $x \in X$ such that $lim_{n\to\infty} x_n = lim_{n\to\infty} hx_{n-1} = x$.

$$d_{\mathcal{B}}(hx,x) \preccurlyeq \mathfrak{J}[d_{\mathcal{B}}(hx,hx_n) + d_{\mathcal{B}}(hx_n,x)]$$

$$\preccurlyeq \mathfrak{J}[d_{\mathcal{B}}(hx,x_n) + d_{\mathcal{B}}(hx_n,x) + d_{\mathcal{B}}(x_{n+1},x)]$$

$$\preccurlyeq \mathfrak{J}[d_{\mathcal{B}}(hx,x_n) + d_{\mathcal{B}}(x_n,x) + d_{\mathcal{B}}(x_{n+1},x)] + d_{\mathcal{B}}(x_{n+1},x)$$

This is equivalent to

$$(I-\mathfrak{J}) d_{\mathcal{B}}(hx,x) \preccurlyeq \mathfrak{J}\big(d_{\mathcal{B}}(x,x_n) + d_{\mathcal{B}}(x_{n+1},x)\big) + d_{\mathcal{B}}(x_{n+1},x)$$

Then

$$\|d_{\mathcal{B}}(hx,x)\| \preccurlyeq \|\mathfrak{J}(1-\mathfrak{J})^{-1}\|(\|d_{\mathcal{B}}(x,x_n)\| + \|d_{\mathcal{B}}(x_{n+1},x)\|) + \|\mathfrak{J}(1-\mathfrak{J})^{-1}\|\|d_{\mathcal{B}}(x_{n+1},x)\|$$

Now, let z is another fixed point of h such that $x \neq z$

$$0_{\mathcal{B}} \preccurlyeq d_{\mathcal{B}}(x,z) = d_{\mathcal{B}}(hx,hz) \preccurlyeq \mathfrak{J}(d_{\mathcal{B}}(hx,z)+d_{\mathcal{B}}(hz,x))$$

$$= \mathfrak{J}(d_{\mathcal{B}}(x,z)+d_{\mathcal{B}}(z,x))$$

i.e. $$d_{\mathcal{B}}(x,z) \preccurlyeq \mathfrak{J}(1-\mathfrak{J})^{-1} d_{\mathcal{B}}(x,z)$$

Since $$\|\mathfrak{J}(1-\mathfrak{J})^{-1}\| \prec 1$$

$$0_{\mathcal{B}} \preccurlyeq \|d_{\mathcal{B}}(x,z)\| = \|d_{\mathcal{B}}(hx,hz)\|$$

$$\preccurlyeq \|\mathfrak{J}(1-\mathfrak{J})^{-1} d_{\mathcal{B}}(x,z)\|$$

$$\preccurlyeq \|\mathfrak{J}(1-\mathfrak{J})^{-1}\|\|d_{\mathcal{B}}(x,z)\| \prec \|d_{\mathcal{B}}(x,z)\|$$

Hence $x = z$.

Therefore h has a fixed point of X.

Corollary (2.10): (K-function) : Let $(X,\mathcal{B},d_{\mathcal{B}})$ be a complete $\mathcal{B}-C^*$ algebra metric space, and $h: X \to X$ satisfies the following condition for all $x,y \in X$

$$d_{\mathcal{B}}(hx,hy) \preccurlyeq \mathcal{G}\big(d_{\mathcal{B}}(hx,x) + d_{\mathcal{B}}(hy,y)\big)$$

where $\mathcal{G} \in \acute{\mathcal{B}}_+$ and $\|\mathcal{G}\| \prec \frac{1}{2}$. Then there exists a unique fixed point in X.

Theorem (2.11): Let $(X,\mathcal{B},d_{\mathcal{B}})$ be a complete $\mathcal{B}-C^*$ algebra metric space, and $h: X \to X$ be a function satisfies the following condition for all $x,y \in X$

$$d_{\mathcal{B}}(hx,hy) \preccurlyeq \gamma d_{\mathcal{B}}(hx,y) + \eta d_{\mathcal{B}}(hy,x)$$

s.t. $\eta,\gamma \in \acute{\mathcal{B}}_+$ and $\|\eta\| + \|\gamma\| < 1$. Then there exists a unique fixed point in X.

Journal of AL-Qadisiyah for computer science and mathematics Vol.9 No.2 Year 2017
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Proof :

If $\eta, \gamma = 0_{\mathcal{B}}$, it is clear that h has fixed point in X.

Suppose that $\wp \neq 0_{\mathcal{B}}$.

Let $x_0 \in X$ and $x_{n+1} = hx_n = \cdots = h^{n+1}x_0$

$$d_{\mathcal{B}}(x_n, x_{n+1}) = d_{\mathcal{B}}(hx_{n-1}, hx_n)$$

$$\preccurlyeq \eta d_{\mathcal{B}}(hx_{n-1}, x_{n-1}) + \gamma d_{\mathcal{B}}(hx_n, x_n)$$

$$\preccurlyeq \gamma d_{\mathcal{B}}(x_{n+1}, x_n)$$

$$\preccurlyeq \gamma\left(d_{\mathcal{B}}(hx_n, x_n) + d_{\mathcal{B}}(hx_{n-1}, x_{n-1})\right)$$

From which it follows $(1_{\mathcal{B}} - \gamma)d_{\mathcal{B}}(x_n, x_{n+1}) \preccurlyeq \gamma d_{\mathcal{B}}(x_n, x_{n-1})$ …(1)

By a similar way, we have

$$d_{\mathcal{B}}(x_{n+1}, x_n) = d_{\mathcal{B}}(hx_n, hx_{n-1})$$

$$\preccurlyeq \eta d_{\mathcal{B}}(hx_n, x_n) + \gamma d_{\mathcal{B}}(hx_{n-1}, x_{n-1})$$

$$\preccurlyeq \eta d_{\mathcal{B}}(x_{n+1}, x_n)$$

$$\preccurlyeq \eta\left(d_{\mathcal{B}}(hx_n, x_n) + d_{\mathcal{B}}(hx_{n-1}, x_{n-1})\right)$$

That is,

$(1_{\mathcal{B}} - \eta)d_{\mathcal{B}}(x_n, x_{n+1}) \preccurlyeq \eta d_{\mathcal{B}}(x_n, x_{n-1})$ …(2)

Now, from (1) and (2), we get

$$(1_{\mathcal{B}} - \frac{\eta+\gamma}{2})d_{\mathcal{B}}(x_n, x_{n+1}) \preccurlyeq \frac{\eta+\gamma}{2} d_{\mathcal{B}}(x_n, x_{n-1})$$

Since $\eta, \gamma \in \acute{\mathcal{B}}_+$ and $\|\eta + \gamma\| \preccurlyeq \|\eta\| + \|\gamma\| \prec 1$, then $(1_{\mathcal{B}} - \frac{\eta+\gamma}{2})^{-1} \in \acute{\mathcal{B}}_+$ which together with lemma (2.7) we obtain

$$d_{\mathcal{B}}(x_n, x_{n+1}) \preccurlyeq (1_{\mathcal{B}} - \tfrac{\eta+\gamma}{2})^{-1}\tfrac{\eta+\gamma}{2} d_{\mathcal{B}}(x_n, x_{n-1})$$

Let $\mathscr{r} = [\left(1_{\mathcal{B}} - \frac{\eta+\gamma}{2}\right)^{-1}\frac{\eta+\gamma}{2}]$, then $\|\mathscr{r}\| = \left\|(1_{\mathcal{B}} - \frac{\eta+\gamma}{2})^{-1}\frac{\eta+\gamma}{2}\right\| \prec 1$.

Thus, $\{x_n\}$ is Cauchy sequence in X and therefore by the completeness of X, there are $x \in X$ such that $lim_{n\to\infty} x_n = x$

$$d_{\mathcal{B}}(hx, x) \preccurlyeq d_{\mathcal{B}}(hx, x_{n+1}) + d_{\mathcal{B}}(x_{n+1}, x)$$

$$= d_{\mathcal{B}}(hx, hx_n) + d_{\mathcal{B}}(x_{n+1}, x)$$

$$\preccurlyeq \gamma d_{\mathcal{B}}(hx, x_n) + \eta d_{\mathcal{B}}(hx_n, x) + d_{\mathcal{B}}(x_{n+1}, x)$$

$$= \gamma d_{\mathcal{B}}(x, x_n) + \eta d_{\mathcal{B}}(x_{n+1}, x) + d_{\mathcal{B}}(x_{n+1}, x)$$

and then

$$\|d_{\mathcal{B}}(hx, x)\| \preccurlyeq \|\gamma\|\|d_{\mathcal{B}}(x, x_n)\| + \|\eta\|\|d_{\mathcal{B}}(x_{n+1}, x)\| + \|d_{\mathcal{B}}(x_{n+1}, x)\|$$

By the continuity of the metric and the norm, we know

$$\|d_{\mathcal{B}}(hx, x)\| \preccurlyeq \|\eta\|\|d_{\mathcal{B}}(hx, x)\|$$

Since the $\|\eta\| \prec 1$ that $lim_{n\to\infty}\|d_{\mathcal{B}}(hx, x)\| = 0_{\mathcal{B}}$, thus $hx = x$.

By the same logic of the theorem (2.8), we get $x = y$

Hence x is a fixed point of h.

Corollary(2.12): Let $(X, \mathcal{B}, d_{\mathcal{B}})$ be a complete $\mathcal{B} - C^*$ algebra metric space, and $h: X \to X$ satisfies the following condition for all $x, y \in X$

$$d_{\mathcal{B}}(hx, hy) \preccurlyeq \gamma d_{\mathcal{B}}(hx, x) + \eta d_{\mathcal{B}}(hy, y)$$

s.t. $\eta, \gamma \in \acute{\mathcal{B}}_+$ and $\|\eta\| + \|\gamma\| < 1$. Then there exists a unique fixed point in X.

3. $\mathcal{B} - C^*$ algebra expansion

Definition(3.1) : Let X be a nonempty set. We said that the function $\varphi: X \longrightarrow X$ is $\mathcal{B} - C^*$ algebra expansion on X, it is satisfies the following condition:

(1) $\varphi(X) = X$
(2) $d_{\mathcal{B}}(\varphi x, \varphi y) \succcurlyeq \mathscr{w}^* d_{\mathcal{B}}(x, y)\mathscr{w}$ for all $x, y \in X$

where $\mathscr{w} \in \mathcal{B}$ is an invertible element and $\|\mathscr{w}\| \prec 1$.

Theorem(3.2): Let $(X, \mathcal{B}, d_{\mathcal{B}})$ be a complete $\mathcal{B} - C^*$ algebra metric space, then for all algebra expansion the function φ, there exists a unique fixed point

Proof: Firstly, to prove that φ is 1-1. Assume that $\varphi x = \varphi y$, where $x, y \in X$

$$0_{\mathcal{B}} = d_{\mathcal{B}}(\varphi x, \varphi y) \succcurlyeq w^* d_{\mathcal{B}}(x, y) w$$

Since $w^* d_{\mathcal{B}}(x, y) w = 0_{\mathcal{B}}$. Also w is invertible, $d_{\mathcal{B}}(x, y) = 0_{\mathcal{B}}$, then $x = y$

Thus φ is 1-1.

Now, we will prove φ has a unique fixed point

Since φ is invertible and for any $x, y \in X$

$$d_{\mathcal{B}}(\varphi x, \varphi y) \succcurlyeq w^* d_{\mathcal{B}}(x, y) w$$

Replace the above formula, x, y with $\varphi^{-1}x, \varphi^{-1}y$, respectively, we get

$$d_{\mathcal{B}}(x, y) \succcurlyeq w^* d_{\mathcal{B}}(\varphi^{-1}x, \varphi^{-1}y) w$$

This mean

$$(w^*)^{-1} d_{\mathcal{B}}(x, y) w^{-1} \succcurlyeq d_{\mathcal{B}}(\varphi^{-1}x, \varphi^{-1}y)$$

$$\Longrightarrow \ (w^{-1})^* d_{\mathcal{B}}(x, y) w^{-1} \succcurlyeq d_{\mathcal{B}}(\varphi^{-1}x, \varphi^{-1}y)$$

By theorem (2.8) , there exist a unique fixed point x such that $\varphi^{-1}x = x$.

This means $\varphi x = x$.

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Journal of AL-Qadisiyah for computer science and mathematics Vol.9 No.2 Year 2017
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

الفضاء المتري الجبري من النوع $\mathcal{B}-C^*$ وبعض نتائج مبرهنات النقطة الصامدة

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المستخلص :

حول فكرة وخصائص C^* الجبري. في هذا الورقة البحثية سنقدم مفهوم $\mathcal{B}-C^*$ الجبري و كذلك الفضاء المتري الجبري من النوع $\mathcal{B}-C^*$ بالإضافة الى ذلك سنقدم مفهوم التقارب ومتتابعة كوشي في الفضاء و دراسة مبرهنات الاساسية للنقطة الصامدة وبالاخير سيتم التطرق الى التوسيع الجبري من النوع $\mathcal{B}-C^*$.