

On Coefficient Bounds of Bi-Univalent Functions Subordinated via Chebyshev Polynomials and Symmetric q -Derivatives

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ABSTRACT

This study presents and examines a novel subclass of bi-univalent functions linked to the symmetric q -derivative operator. For functions in this class, we calculate estimates for the initial Taylor–Maclaurin coefficients $|b_2|$ as well as $|b_3|$, and establish the associated Fekete–Szegő inequality. Additionally, various specific instances and implications of the principal findings are examined.

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1. Preliminaries and Basic Definitions

Consider $\mathcal{A}$ represent a collection of all analytic functions ℓ stipulated in the open unit disk $\mathfrak{U}$, according to the criteria $\ell(0) = 0$ as well as $\ell'(0) - 1 = 0$, where $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$. Consequently, any $\ell \in \mathcal{A}$ possesses a Taylor series expansion of the following form:

$$\ell(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathfrak{U}. \quad (1.1)$$

Consider $\mathcal{S}$ be a subset of $\mathcal{A}$ containing functions of the form (1.1) that are also univalent within $\mathfrak{U}$. If the functions ℓ and ψ are analytic in $\mathfrak{U}$, then ℓ is considered subordinate to ψ , denoted by $\ell(z) < \psi(z)$ for all $z \in \mathfrak{U}$, if there exists a Schwarz function $w(z)$ that is analytic in $\mathfrak{U}$ and satisfies $w(0) = 0$ with $|w(z)| < 1$ for $z \in \mathfrak{U}$, such that $\ell(z) = \psi(w(z))$ for $z \in \mathfrak{U}$.

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Each function containing $\ell \in \mathcal{S}$ possesses an inverse map ℓ^{-1} which fulfills the following conditions:

$$\ell^{-1}(\ell(z)) = z \quad (z \in \mathfrak{V}),$$

and

$$\ell(\ell^{-1}(w)) = w \quad (|w| < r_0(\ell); r_0(\ell) \geq \frac{1}{4}).$$

The inverse of the function can be expressed as

$$\ell^{-1}(w) = w - \mathfrak{d}_2 w^2 + (2\mathfrak{d}_2^2 - \mathfrak{d}_3)w^3 - (5\mathfrak{d}_2^3 - 5\mathfrak{d}_2\mathfrak{d}_3 + \mathfrak{d}_4)w^4 + \dots \quad (1.2)$$

Consult reference [10] for additional details. A function ℓ included within the class $\mathcal{A}$ is deemed bi-univalent in $\mathfrak{V}$ if each ℓ as well as its inverse ℓ^{-1} are univalent in $\mathfrak{V}$. The set of bi-univalent functions described in the unit disk $\mathfrak{V}$ is represented by Σ . For a succinct history and significant examples of functions in Σ , consult [31]. Lewin [22] explored the class of bi-univalent functions, obtaining a bound of 1.51 for the modulus of the second coefficient $|\mathfrak{d}_2|$. Later, Brannan and Clunie [7] proposed that $|\mathfrak{d}_2| \leq \sqrt{2}$, for functions ℓ in Σ . [26] subsequently showed that the maximum value of $|\mathfrak{d}_2|$ is $4/3$ for $\ell(z) \in \Sigma$. Brannan and Taha [8] developed subclasses of the bi-univalent function class Σ analogous to the subclasses $\mathcal{S}^*(\beta)$ respectively $\mathbb{K}(\beta)$ of starlike and convex functions of order β (where $0 \leq \beta < 1$) as referenced in [26]. They additionally defined the classes $\mathcal{S}_{\Sigma}^*(\beta)$ along with $\mathbb{K}_{\Sigma}(\beta)$ for bi-starlike functions of order α and bi-convex functions of order β , which correspond to $\mathcal{S}^*(\beta)$ as well as $\mathbb{K}(\beta)$. They supplied imprecise estimates for the starting coefficients of these categories. Recently, numerous writers have explored bounds for different subclasses of bi-univalent functions [4, 12, 23, 29, 30, 31]. Nonetheless, there is scant understanding of the limits on the general coefficient $|\mathfrak{d}_n|$ for $n \geq 4$. Only a few studies have addressed the general coefficient bounds $|\mathfrak{d}_n|$ for analytic bi-univalent functions [4, 5, 15, 16, 20]. The problem of estimating the coefficient $|\mathfrak{d}_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \dots\}$ remains an open question. Chebyshev polynomials have gained prominence in numerical analysis, both in theory and application. There exist four varieties of Chebyshev polynomials. Most literature on orthogonal polynomials of the Chebyshev family focuses primarily on the first and second kinds, $T_n(x)$ and $\mathfrak{U}_n(x)$, and their various applications. See [9] and [24] for more details.

The Chebyshev polynomials of the first and second sorts are well-known in mathematics. For a genuine variable t inside the interval $(-1, 1)$, they are formulated as:

$$T_n(x) = \cos n\theta$$

$$\mathfrak{U}_n(x) = \frac{\sin(n+1)\theta}{\sin \theta},$$

where n represents the degree of the polynomial, and $x = \cos \theta$. In the domain of Geometric Function Theory, numerous subclasses of analytic functions are currently examined from diverse viewpoints.

Fractional q -calculus is a crucial instrument in the investigation of subclasses of analytic functions. The theory of univalent functions can be augmented through the application of q -calculus. Additionally, fractional q -integral and fractional q -derivative operators are employed to formulate various subclasses of analytic functions, as referenced in [1, 6, 14, 27]. A recent work [28] concentrated on utilizing fractional q -calculus operators to establish new categories of functions that are analytic within the open disk. Subsequently, [25] investigated the approximation and geometric characteristics of these q -operators inside specific subclasses of analytic functions in the compact disk. To facilitate comprehension, we provide essential definitions and ideas of q -calculus utilized in this research. We assume that $0 < q < 1$ and adhere to the notation and language from [13]. The definitions of fractional q -calculus operators for complex-valued functions $\ell(z)$ are reexamined.

In 1909, Jackson proposed the q -derivative operator $\mathbb{D}_q$ of a function l , defined by equation (1.1):

$$\mathbb{D}_q \ell(z) = \begin{cases} \frac{\ell(z) - \ell(qz)}{(1-q)z}, & \text{if } z \neq 0 \\ \ell'(z), & \text{if } z = 0 \end{cases}, \quad (3.1)$$

for $q \in (0,1)$. It is clear that $\lim_{q \rightarrow 1^-} \mathbb{D}_q \ell(z) = \ell'(z)$ for ℓ is given by equation (1.1).

The information that follows is the corresponding formula for the function z^n as q -derivative:

$$\mathbb{D}_q z^n = [n]_q z^{n-1}, n \in \mathbb{N}, \quad (3.2)$$

where

$$[n]_q = \sum_{\ell=1}^n q^{\ell-1}, [0]_q = 0 \quad (3.3)$$

The q -analogue of natural numbers is referred to as the basic number n . It is derived from (3.3) that $[n]_q = 1 - q^n / 1 - q$ and $\lim_{q \rightarrow 1^-} [n]_q = n$ for $q \in (0,1)$.

and on a product of functions, the general Leibniz rule for action of powers of the q -derivative is

$$\mathbb{D}_q^{(n)}(\ell \psi)(z) = \sum_{\ell=0}^n \binom{n}{\ell} D_q^{(\ell)} \ell(q^{n-\ell} z) D_q^{(n-\ell)} \psi(q^\ell z), \quad (3.7)$$

while $\binom{n}{\ell}_q$ denotes the q -binomial coefficients defined by

$$\binom{n}{\ell}_q = \frac{[n]_q!}{[\ell]_q! [n-\ell]_q!},$$

for $\ell = 0, 1, 2, \dots, n$ such that

$$[n]_q! = \prod_{\ell=1}^n [\ell]_q, [0]_q! = 1$$

is q -factorial function. Note that $\lim_{q \rightarrow 1^-} [n]_q! = n!$ is factorial function, and $\lim_{q \rightarrow 1^-} \binom{n}{\ell}_q = \binom{n}{\ell}$ are binomial coefficients. We refer to [11, 17, 19] about some properties of difference operator $\mathbb{D}_q$.

It follows from (3.7) that $\mathbb{D}_q^{(2)} \ell(z) = \mathbb{D}_q(\mathbb{D}_q \ell(z))$, and $\mathbb{D}_q^{(n)} \ell(z) = \mathbb{D}_q(\mathbb{D}_q^{(n-1)} \ell(z))$

Definition 1. See [21]: The symmetric q -derivative $\widetilde{\mathbb{D}}_q \ell$ of a function ℓ , as delineated in (1), can be articulated in the following manner:

$$\widetilde{\mathbb{D}}_q \ell(z) = \begin{cases} \frac{\ell(qz) - \ell(q^{-1}z)}{(q - q^{-1})z}, & \text{if } z \neq 0 \\ \ell'(0), & \text{if } z = 0 \end{cases}$$

Form (4) we that $\widetilde{\mathbb{D}}_q z^n = [\widetilde{n}]_q z^{n-1}$. And a power series of $\widetilde{\mathbb{D}}_q \ell$ is

$$\tilde{\mathbb{D}}_q \ell(z) = 1 + \sum_{n=2}^{\infty} [\widetilde{n}]_q \mathfrak{d}_n z^{n-1}$$

When ℓ assumes the configuration (1) as well as the representation $[\widetilde{n}]_q$ signifies the quantity

$$[\widetilde{n}]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

It is straightforward to verify if the subsequent attributes are valid.

$$\tilde{\mathbb{D}}_q [\ell(z) + \mathfrak{y}(z)] = \tilde{\mathbb{D}}_q \ell(z) + \tilde{\mathbb{D}}_q \mathfrak{y}(z)$$

$$\tilde{\mathbb{D}}_q [\ell(z) \mathfrak{y}(z)] = \ell(q^{-1}z) \tilde{\mathbb{D}}_q \mathfrak{y}(z) + \mathfrak{y}(qz) \tilde{\mathbb{D}}_q \ell(z)$$

Finally, we obtain the relation given below

$$\tilde{\mathbb{D}}_q \ell(z) = \mathbb{D}_{q^2} \ell(q^{-1}z)$$

And

$$\tilde{\mathbb{D}}_q \mathfrak{y}(w) = \frac{\mathfrak{y}(qw) - \mathfrak{y}(q^{-1}w)}{(q - q^{-1})w} = 1 - [\widetilde{2}]_q \mathfrak{d}_2 w + [\widetilde{3}]_q (2\mathfrak{d}_2^2 - \mathfrak{d}_3) w^2 + \dots$$

Definition 2. A function $\ell \in \Sigma$ stipulated by (1) is classified within the class $\tilde{\mathbb{H}}_{\Sigma}^q(m, \delta)$, if the subsequent requirements are met:

$$\left(\tilde{\mathbb{D}}_q \ell(z) \right)^{1/\delta} < H(z, m) := \frac{1}{1 - 2mz + z^2}, \quad \left(\frac{1}{2} < m < 1, z \in \mathfrak{V} \right)$$

and

$$\left(\tilde{\mathbb{D}}_q \mathfrak{y}(w) \right)^{1/\delta} < H(w, m) := \frac{1}{1 - 2mw + w^2}, \quad \left(\frac{1}{2} < m < 1, w \in \mathfrak{V}, \text{ and } \mathfrak{y} = \ell^{-1} \right)$$

We note that if $m = \cos \alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3} \right)$, then

$$H(z, m) = \frac{1}{1 - 2mz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin \alpha} z^n, \quad (z \in \mathfrak{V})$$

Thus

$$H(z, m) = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha) z^2 + \dots, \quad (z \in \mathfrak{V})$$

Next, we write

$$H(z, m) = 1 + \mathfrak{U}_1(m)z + \mathfrak{U}_2(m)z^2 + \dots, \quad (z \in \mathfrak{V}, \quad m \in (-1, 1))$$

Where $\mathfrak{U}_{n-1} = \frac{\sin(n \arccos m)}{\sqrt{1-m^2}}$ ($n \in \mathcal{N}$) represent the Chebyshev polynomials of the second category. It is also recognized that

$$\mathfrak{U}_n(m) = 2m\mathfrak{U}_{n-1}(m) - \mathfrak{U}_{n-2}(m)$$

and

$$\mathfrak{U}_1(m) = 2m, \quad \mathfrak{U}_2(m) = 4m^2 - 1, \quad \mathfrak{U}_3(m) = 8m^3 - 4m, \dots$$

The Chebyshev polynomials $T_n(m)$, $m \in (-1, 1)$, of the first kind have a generating function defined by the following expression

$$\sum_{n=0}^{\infty} T_n(m) z^n = \frac{1 - mz}{1 - 2mz + z^2}, \quad z \in \mathfrak{Y}$$

This study examines the application of Chebyshev polynomial rotations to calculate the initial coefficients of specific subclasses of bi-univalent functions constructed with the symmetric q -derivative operator. The Fekete-Szegő inequalities applicable to the class $\mathfrak{H}_{\Sigma}^q(m, \delta)$ are likewise established.

2. Initial Lemma

Lemma 5.[10] Suppose that p denote the class of functions possessing a positive real part, encompassing all analytic functions $p: \mathfrak{Y} \rightarrow \mathbb{C}$ the fact that fulfill $v(0) = 1$ and exhibit a positive real part $Re(v(z)) > 0$. This class is referred to as the Caratheodory class. The subsequent outcome is necessary for substantiating our findings.

Lemma 6 [10] If the function v belonging to p is given by

$$v(z) = 1 + v_1 z + v_2 z^2 + v_3 z^3 + \dots$$

then $|v_n| \leq 2$, ($n \in \mathbb{N} = \{1, 2, \dots\}$).

3. Estimating Coefficients for the Class $\mathfrak{H}_{\Sigma}^q(t, \delta)$

Theorem 7 Suppose ℓ defined by (1.1) is within the class $\mathfrak{H}_{\Sigma}^q(m, \delta)$. Thus

$$|d_2| \leq \frac{4m\sqrt{m}}{\sqrt{\left(\frac{2}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1\right) ([\widetilde{2}]_q)^2 - \frac{2}{\delta^2} ([\widetilde{2}]_q)^2\right) 4m^2 + 2 \left(\frac{2}{\delta^2} ([\widetilde{2}]_q)^2\right) m + \left(\frac{2}{\delta^2} ([\widetilde{2}]_q)^2\right)}}$$

and

$$|d_3| \leq \frac{2m\delta}{[\widetilde{3}]_q} + \frac{4m^2\delta^2}{([\widetilde{2}]_q)^2}$$

proof. Assume ℓ belong to $\mathfrak{H}_{\Sigma}^q(m, \delta)$, and denote y as the analytic extension of ℓ^{-1} to Y . There are two functions h and v , both analytic in $\mathfrak{Y}$, fulfilling $h(0) = v(0) = 0$, $|h(z)| < 1$, along with $|v(w)| < 1$ across all $z, w \in \mathfrak{Y}$, corresponding to

$$\left(\mathfrak{D}_q \ell(z)\right)^{1/\delta} = H(h(z), m) \tag{3.1}$$

$$\left(\mathfrak{D}_q y(w)\right)^{1/\delta} = H(v(w), m) \tag{3.2}$$

Subsequently, define the functions $\mathfrak{b}$ as well as $\mathfrak{b}$ in p as

$$v(z) = \frac{1 + h(z)}{1 - h(z)} = 1 + v_1 z + v_2 z^2 + \dots$$

and

$$\ell(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + \ell_1 z + \ell_2 z^2 + \dots$$

The following allows us to derive

$$h(z) = \frac{v(z) - 1}{v(z) + 1} = \frac{1}{2} v_1 z + \frac{1}{2} \left(v_2 - \frac{1}{2} v_1^2 \right) z^2 + \dots \quad (3.3)$$

and

$$v(w) = \frac{\ell(w) - 1}{\ell(w) + 1} = \frac{1}{2} \ell_1 w + \frac{1}{2} \left(\ell_2 - \frac{1}{2} \ell_1^2 \right) w^2 + \dots \quad (3.4)$$

By combining equations (3.1), (3.2), (3.3), and (3.4),

$$\left(\mathbb{D}_q \ell(z) \right)^\delta = 1 + \frac{1}{2} \mathfrak{U}_1(m) v_1 z + \left(\frac{1}{4} \mathfrak{U}_2(m) v_1^2 + \frac{1}{2} \mathfrak{U}_1(m) \left(v_2 - \frac{1}{2} v_1^2 \right) \right) z^2 + \dots \quad (3.5)$$

and

$$\left(\mathbb{D}_q \ell(w) \right)^\delta = 1 + \frac{1}{2} \mathfrak{U}_1(m) \ell_1 w + \left(\frac{1}{4} \mathfrak{U}_2(m) \ell_1^2 + \frac{1}{2} \mathfrak{U}_1(m) \left(\ell_2 - \frac{1}{2} \ell_1^2 \right) \right) w^2 + \dots \quad (3.6)$$

Based on (3.5) and (3.6), we obtain

$$\frac{1}{\delta} [\widetilde{2}]_q \mathfrak{d}_2 = \frac{\mathfrak{U}_1(m)}{2} v_1 \quad (3.7)$$

$$\frac{1}{\delta} [\widetilde{3}]_q \mathfrak{d}_3 + \frac{1}{2\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 \mathfrak{d}_2^2 = \frac{\mathfrak{U}_1(m)}{2} \left(v_2 - \frac{v_1^2}{2} \right) + \frac{\mathfrak{U}_2(m)}{4} v_1^2 \quad (3.8)$$

$$-\frac{1}{\delta} [\widetilde{2}]_q \mathfrak{d}_2 = \frac{\mathfrak{U}_1(m)}{2} \ell_1 \quad (3.9)$$

$$\frac{1}{\delta} [\widetilde{3}]_q (2\mathfrak{d}_2^2 - \mathfrak{d}_3) + \frac{1}{2\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 \mathfrak{d}_2^2 = \frac{\mathfrak{U}_1(m)}{2} \left(\ell_2 - \frac{\ell_1^2}{2} \right) + \frac{\mathfrak{U}_2(m)}{4} \ell_1^2 \quad (3.10)$$

From equations (3.7) and (3.9), we derive

$$v_1 = -\ell_1 \quad (3.11)$$

$$\frac{2}{\delta^2} ([\widetilde{2}]_q)^2 \mathfrak{d}_2^2 = \frac{\mathfrak{U}_1^2(m)}{4} (v_1^2 + \ell_1^2) \quad (3.12)$$

By adding (3.8) to (3.10), we obtain

$$\left(2 \frac{1}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 \right) \mathfrak{d}_2^2 = \frac{\mathfrak{U}_1(m)}{2} (v_2 + \ell_2) + \frac{\mathfrak{U}_2(m) - \mathfrak{U}_1(m)}{4} (v_1^2 + \ell_1^2) \quad (3.13)$$

Substituting (3.12) into equation (3.13),

$$\left(2 \frac{1}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 - \frac{\mathfrak{U}_2(m) - \mathfrak{U}_1(m)}{\mathfrak{U}_1^2(m)} (2\delta^2 ([\widetilde{2}]_q)^2) \right) \mathfrak{d}_2^2 = \frac{\mathfrak{U}_1(m)}{2} (v_2 + \ell_2) \quad (3.14)$$

From Lemma 5 and (3.14) we get

$$|d_2| \leq \frac{4m\sqrt{m}}{\sqrt{\left(\frac{2}{\delta}[\widetilde{3}]_q + \frac{1}{\delta}\left(\frac{1}{\delta} - 1\right)([\widetilde{2}]_q)^2 - \frac{2}{\delta^2}([\widetilde{2}]_q)^2\right)4m^2 + 2\left(\frac{2}{\delta^2}([\widetilde{2}]_q)^2\right)m + \left(\frac{2}{\delta^2}([\widetilde{2}]_q)^2\right)}}$$

The result of subtracting (3.10), from (3.8)

$$\frac{1}{\delta}[\widetilde{3}]_q(2d_3 - 2d_2^2) = \frac{\Theta_1(m)}{2}(v_2 - \ell_2) + \frac{\Theta_2(m)}{4}(v_1^2 - \ell_1^2) \quad (3.15)$$

In view of (3.11) and (3.12), we obtain (3.15).

$$\begin{aligned} d_3 &= \frac{\delta \Theta_1(m)}{4[\widetilde{3}]_q}(v_2 - \ell_2) + d_2^2 \\ &= \frac{\delta \Theta_1(m)}{4[\widetilde{3}]_q}(v_2 - \ell_2) + \frac{\delta^2 \Theta_1^2(m)(v_1^2 + \ell_1^2)}{8([\widetilde{2}]_q)^2} \end{aligned} \quad (3.16)$$

As indicated by Lemma 5,

$$|d_3| \leq \frac{2\delta m}{[\widetilde{3}]_q} + \frac{4\delta^2 m^2}{([\widetilde{2}]_q)^2}$$

By substituting $\delta = 1$ into Theorem 7 yields the subsequent consequence.

Corollary 8. Suppose ℓ defined by (1.1) is in the class $\mathbb{H}_{\Sigma}^q(m, 1)$. Subsequently

$$|d_2| \leq \frac{2m\sqrt{2m}}{\sqrt{\left([\widetilde{3}]_q - ([\widetilde{2}]_q)^2\right)4m^2 + 2([\widetilde{2}]_q)^2m + ([\widetilde{2}]_q)^2}}$$

and

$$|d_3| \leq \frac{2m}{[\widetilde{3}]_q} + \frac{4m^2}{([\widetilde{2}]_q)^2}$$

Remark 9. The estimates for coefficients $|d_2|$ and $|d_3|$ in the **Theorem 7** and the previous Corollary are the general case. When we take $\lim_{q \rightarrow 1^-}$ for these estimates, we obtain the estimates presented by [2, 32].

4. Fekete-Szegő inequalities for the function class $\mathbb{H}_{\Sigma}^q(\ell, \delta)$

Theorem 10. Assume ℓ , as specified by equation (1.1), be a member of the class $\mathbb{H}_{\Sigma}^q(m, \delta)$ while $\gamma \in \mathbb{R}$. Subsequently

$$|d_3 - \gamma d_2^2| \leq \begin{cases} \frac{2\delta m}{[\widetilde{3}]_q}; & \text{or } |1 - \gamma| \leq \frac{\left(\frac{4}{\delta}[\widetilde{3}]_q + \frac{2}{\delta}\left(\frac{1}{\delta} - 1\right)([\widetilde{2}]_q)^2 - \frac{4}{\delta^2}([\widetilde{2}]_q)^2\right)m^2 + \frac{2}{\delta^2}([\widetilde{2}]_q)^2m + \frac{1}{\delta^2}([\widetilde{2}]_q)^2}{\frac{4}{\delta}[\widetilde{3}]_q m^2} \\ 8m|h(\gamma)|; & \text{or } |1 - \gamma| \geq \frac{\left(\frac{4}{\delta}[\widetilde{3}]_q + \frac{2}{\delta}\left(\frac{1}{\delta} - 1\right)([\widetilde{2}]_q)^2 - \frac{4}{\delta^2}([\widetilde{2}]_q)^2\right)m^2 + \frac{2}{\delta^2}([\widetilde{2}]_q)^2m + \frac{1}{\delta^2}([\widetilde{2}]_q)^2}{\frac{4}{\delta}[\widetilde{3}]_q m^2} \end{cases}$$

Proof. From (3.14) and (3.16)

$$d_3 - \gamma d_2^2 = (1 - \gamma)d_2^2 + \frac{\delta \Theta_1(m)}{4[\widetilde{3}]_q}(v_2 - \ell_2)$$

$$d_3 - \gamma d_2^2 = (1 - \gamma) \frac{\Theta_1^3(m)(v_2 + \ell_2)}{2\Theta_1^2(m) \left(\frac{2}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 \right) - (\Theta_2(m) - \Theta_1(m)) \left(\frac{4}{\delta^2} ([\widetilde{2}]_q)^2 \right)} + \frac{\Theta_1(m)}{4\delta [\widetilde{3}]_q} (v_2 - \ell_2)$$

$$d_3 - \gamma d_2^2 = \Theta_1(m) \left[\left(\left(h(\gamma) + \frac{\delta}{4[\widetilde{3}]_q} \right) v_2 + \left(h(\gamma) - \frac{\delta}{4[\widetilde{3}]_q} \right) \ell_2 \right) \right]$$

Where

$$h(\gamma) = \frac{(1 - \gamma) \Theta_1^2(m)}{2\Theta_1^2(m) \left(\frac{2}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 \right) - (\Theta_2(m) - \Theta_1(m)) \left(\frac{4}{\delta^2} ([\widetilde{2}]_q)^2 \right)}$$

$$|d_3 - \gamma d_2^2| \leq \begin{cases} \frac{2\delta m}{[\widetilde{3}]_q}; & 0 \leq |h(\gamma)| \leq \frac{\delta}{4[\widetilde{3}]_q} \\ \frac{8m^3 |1 - \gamma|}{2m^2 \left(\frac{2}{\delta} [\widetilde{3}]_q + \frac{1}{\delta} \left(\frac{1}{\delta} - 1 \right) ([\widetilde{2}]_q)^2 - \frac{2}{\delta^2} ([\widetilde{2}]_q)^2 \right) + \frac{2}{\delta^2} ([\widetilde{2}]_q)^2 m + \frac{1}{\delta^2} ([\widetilde{2}]_q)^2}; & |h(\gamma)| \geq \frac{\delta}{4[\widetilde{3}]_q} \end{cases}$$

By substituting $\delta = 1$ into Theorem 7 yields the subsequent consequence.

Corollary 11 . Assume ℓ , as specified by problem (1.1), be a member of the class $\mathbb{H}_{\Sigma}^q(m, 1)$ with $\gamma \in \mathbb{R}$. Subsequently

$$|d_3 - \gamma d_2^2| \leq \begin{cases} \frac{2m}{[\widetilde{3}]_q}; & \ell \text{ or } |1 - \gamma| \leq \frac{(4[\widetilde{3}]_q - 4([\widetilde{2}]_q)^2)m^2 + 2([\widetilde{2}]_q)^2 m + ([\widetilde{2}]_q)^2}{4[\widetilde{3}]_q m^2} \\ \frac{8m^3 |1 - \gamma|}{2(2[\widetilde{3}]_q - 2([\widetilde{2}]_q)^2)m^2 + 2([\widetilde{2}]_q)^2 m + ([\widetilde{2}]_q)^2}; & \ell \text{ or } |1 - \gamma| \geq \frac{(4[\widetilde{3}]_q - 4([\widetilde{2}]_q)^2)m^2 + 2([\widetilde{2}]_q)^2 m + ([\widetilde{2}]_q)^2}{4[\widetilde{3}]_q m^2} \end{cases}$$

Remark 12. The estimates for coefficients $|d_2|$ and $|d_3|$ in the Theorem 10 and the previous Corollary are the general case. When we take $\lim_{q \rightarrow 1^-} [\cdot]_q$ for these estimates, we obtain the estimates presented by [2, 32].

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