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The Condition Order Spectrum

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Abstract

In this paper, we will study the relation between complex character f and the condition order spectrum $\mathcal{SP}_\varepsilon(a)$ in real and complex ordered Banach algebra A .We prove many properties on it's . afterwards, we prove that $\mathcal{SP}_\varepsilon(a)$ is upper semi continuous Finally we will show the relation between condition Quasinilpotent $\mathcal{QN}_\varepsilon(A)$ and the radical .

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1 Introduction and Preliminaries .

The study of spectrum has been started and developed turn of the century in[G. Frobenius,1908] for the canonical order of $\mathbb{R}^n$, [Krien-Rutman,1939] for Banach space, [Helmut H. Schaefer, 1986] for order topological linear space, [H. Raubenheimer and S. Rode,1996] for ordered Banach algebra and [E. B. Daies introduce(Spectral Theory and Differential Operators) ,1996] , [K. Yosida introduce (On the Theory of Spectra),1940].
in [D. Sukumar 2007] introduce the concept of condition spectrum in real and complex Banach algebra. In this chapter we introduce four sections
Section one, focuses upon the introduction and Preliminaries of the paper
Section two, concentrates on a new concept of a spectrum of elements in real and complex unital order Banach algebra is the condition order spectrum and condition order radius. We will prove some properties different from the properties to the usual spectrum which is a subset of condition order spectrum.
Section three, involves the properties of condition order spectrum and the concept of the condition of Quisinilpotent. This section will prove some of the properties to shows the relation between Quisinilpotent and Radical.
Section four, shown the relation between character and condition order spectrum. Also, we will introduce the concept of pseudo spectrum in a condition different from condition spectrum in [4].

Definitions 1.1[5]:

- A partially ordered set $(A, \succcurlyeq)$ is called ***direct set*** if for all α, β in A such that $\alpha \neq \beta$,then there exists μ such that $\mu \succcurlyeq \alpha, \mu \succcurlyeq \beta$.
- A function f from a direct set A to a non-empty set X is called a ***net*** on X and it is denoted by $\{f_\lambda\}_{\lambda \epsilon \Lambda}$.

Definition 1.2[6]: Let A be a real or complex Banach algebra with identity and C non-empty subset of A .We call C a cone if it satisfies the following

1. $a + b\epsilon C$ for all $a , b \epsilon C$

2. $\beta a \epsilon C$ for all $a \epsilon C$ and $\beta \geq 0$.

In addition, if C satisfies $C \cap -C = \{0\}$, then C will be called a proper cone induced an ordering $(\preccurlyeq)$ on A by $a \preccurlyeq b$,if and only if $b - a\epsilon C$ for all $a, b \epsilon A$ We say that C is algebra cone if it is satisfies the following:

1. $a \cdot b\epsilon C$ for all $a, b \epsilon C$,

2. $e \epsilon C$.

Definition 1.3(Ordered Banach Algebras)[6]: Every complex Banach algebra A with identity generated by C is called ordered Banach algebra (OBA) if A is ordered by a relation $(\preccurlyeq)$ such that for every $n, m, c \epsilon A$ and $\beta \geq 0$

$a , b \succcurlyeq 0 \Longrightarrow a + b \succcurlyeq 0$

1. $a \succcurlyeq 0, \lambda \geq 0 \Longrightarrow \lambda a \succcurlyeq 0$

2. $a, b \succcurlyeq 0 \Longrightarrow a.b \succcurlyeq 0$.

so if A is ordered by an algebra cone C, we will obtain (A, C) which is an ordered Banach algebra.

Example 1.4[7]: Let A be Banach algebra and $C = \{\alpha e : \alpha \geq 0\}$ then C is closed, algebra cone of A, then (A, C) be an ordered Banach algebra.

Example 1.5 [7]: Let $\mathbb{C}$ be Banach algebra of all complex numbers with norm $\|x\| = |x|$ and the subset of $\mathbb{C}$ of all non-negative real numbers $\mathbb{R}^+$. Then $\mathbb{R}^+$ is a closed, algebra cone of $\mathbb{C}$ and $(\mathbb{C}, \mathbb{R}^+)$ be an order Banach algebra.

Definition 1.6(Character)[8]: Let A be Banach algebra for all $n \epsilon A$ and $\beta \epsilon F$, we call the functional f is ***Character*** if it satisfies:

i- Linear

that is $f(a + b) = f(a) + f(b)$

$and\ \beta f(a) = f(\beta a)$

ii- Multiplicative

that is $f(ab) = f(a)f(b)$

iii- $f(e) = 1$

If $F = \mathbb{R}$, we say that f is real character, and $\mathcal{M}_A$ denote the set of all real character by

If $F = \mathbb{C}$, we say that f is complex character, $Ch(A)$ denotes the set of all complex character .

Theorem 1.7(Gelfand–Mazur)[5]: Let A is Banach algebra which is every non-zero element is invertible, and then there is a unique isomorphism from A into $\mathbb{C}$.

Theorem 1.7(Automatic Continuity of Character)[5]: let A be a Banach algebra over F and let $f: A \to F$ be a character then f is continuous and has the norm $\|f\| \leq 1$.

2 Condition order spectrum properties order in Banach algebra

Definition 2.1: Let A be OBA with algebra cone C, we say that C is invertible algebra cone if every element in C has inverse. We denote that the set of all invertible elements in A by $IN(C)$.

Definition 2.2 (Spectrum): Let A be OBA and $a\epsilon A$, the order spectrum of a which denoted by $\mathcal{SP}(a)$ is defined by:

$\mathcal{SP}(a)=\{\lambda\epsilon\mathbb{C}: \lambda - a \notin IN(C)\}$.

Definition 2.3: Let A be OBA and $a\ \epsilon A$, the order spectrum radius of a which denoted by $r(a)$ is defined by :

$r(a) = sup\{|\lambda| : \lambda\epsilon\mathcal{SP}(a)\}$.

Proposition2.4: Let (A, C)be OBA .If C is inverse-closed then $\mathcal{SP}(a, A) = \mathcal{SP}(a, C)$ for $a\epsilon A$.

Proof:-Since $C \subseteq A$ is inverse – closed so $IN(c) \subseteq IN(a)$

let $\lambda\epsilon\mathcal{SP}(a, A)$. Then $\lambda - a \notin IN(a)$ therefore $\lambda - a \notin IN(c)$ and $\lambda\epsilon\mathcal{SP}(a, C)$.
Conversely, let $\lambda\epsilon\ \mathcal{SP}(a, C)$ then $\lambda - a \notin IN(c)$

If $\lambda - a\epsilon IN(a)$ so $\lambda - a$ has inverse, since C is inverse –closed then $\lambda - a\epsilon IN(c)$ this contradiction

□

Definition2.5 (Spectrum Order Banach algebra) : Let A be OBA with identity and $a\epsilon A$, the spectrum which denoted by $\mathcal{SP}_o(a)$ is define by

$\mathcal{SP}_o(a) = \{\alpha + it\epsilon\mathbb{C}: (\alpha - a)^2 + t^2 \notin IN(C)\}$.

The complexification of a real order Banach algebra A denoted by $A_{\mathbb{C}}$ such that $A_{\mathbb{C}} = A \times A$,

Let $(a, b), (c, d)\ \epsilon A_{\mathbb{C}}$ and $\alpha + i\beta\epsilon\mathbb{C}$

$(a, b) + (c, d) = (a + c, b + d)$

$\alpha + i\beta(a, b) \quad = (\alpha a - \beta b, \alpha b + \beta a)$

$(a, b), (c, d) \quad = (ac - bd, ad + bc)$.

With these operations $A_{\mathbb{C}}$ become complex order Banach algebra ,if A has identity e , then $(e, 0)$ becomes the identity of $A_{\mathbb{C}}$.

And the complexification to C is $C_{\mathbb{C}} = C \times C$.

Definition 2.6(Strong algebra cone in OBA): Let A be a OBA with identity, we say that C is Strong algebra cone in OBA if it satisfies the following:

- If C is an invertible algebra cone .
- $\lambda a\epsilon C$ for all $a\epsilon C$ and $\lambda\epsilon F^* = F\backslash\{0\}$

Definition 2.7: Let $0 < \varepsilon < 1$, define

$$C_\varepsilon = \{a\epsilon IN(a) : \|a\|\,\|a^{-1}\| < \tfrac{1}{\varepsilon}\}$$

In complexification,

$$C'_\varepsilon = \{(a,0)\epsilon IN((a,0)): \|((\alpha,0) - (a,0))^2 + t^2\|\,\|((\alpha,0) - (a,0))^2 + t^2)^{-1}\| < \tfrac{1}{\varepsilon}\}$$

Definition 2.8 (Condition order spectrum): Let A be a complex OBA and $a\epsilon A$ with $0 < \varepsilon < 1$ the condition order spectrum of a for ε is defined by $\mathcal{SP}_\varepsilon(a) = \{\lambda\epsilon\mathbb{C}: \|\lambda - a\|\,\|(\lambda - a)^{-1}\| \geq \frac{1}{\varepsilon}\}$

Remark 2.9 : $\mathcal{SP}_\varepsilon(a^{-1}) = \{\lambda^{-1}: \lambda\epsilon\mathcal{SP}_\varepsilon(a)\}$

If A be a real OBA the condition order spectrum is defined by $\mathcal{SP}_\varepsilon(a) =\{\alpha + it\ \epsilon\mathbb{C}: (\alpha - a)^2 + t^2 \notin C_\varepsilon\}$

$= \{\alpha + it\epsilon\mathbb{C}: \|(\alpha - a)^2 + t^2\|\,\|((\alpha - a)^2 + t^2)^{-1}\| \geq \frac{1}{\varepsilon}\}$.

In complexification,

$$\mathcal{SP}_\varepsilon((a,0)) = \{\alpha + it\epsilon\mathbb{C}: \|((\alpha,0) - (a,0))^2 + t^2\|\,\|((\alpha,0) - (a,0))^2 + t^2)^{-1}\| \notin C'_\varepsilon\}$$

$$= \{\alpha + it\epsilon\mathbb{C}: \|((\alpha,0) - (a,0))^2 + t^2\|\,\|((\alpha,0) - (a,0))^2 + t^2)^{-1}\| \geq \tfrac{1}{\varepsilon}\}$$

The condition order spectral radius which denoted by $r_\varepsilon(a)$ is define by $r_\varepsilon(a) = sup\{|\lambda| : \lambda\ \epsilon\ \mathcal{SP}_\varepsilon(a)\}$.

and $r_\varepsilon(a^{-1}) = sup\{|\lambda^{-1}| : \lambda\ \epsilon\ \mathcal{SP}_\varepsilon(a^{-1})\}$

Proposition 2.10:

1. $a \epsilon IN(C)$ if and only if $(a,0)\epsilon IN(C_{\mathbb{C}})$.
2. $(a,b)\epsilon IN(C)$ if and only if $(a,-b)\epsilon IN(C_{\mathbb{C}})$.

Proof:- It is clear from definition.

Proposition 2.11 : $\mathcal{SP}_{\varepsilon}\left((a,0),A_{\mathbb{C}}\right) = \mathcal{SP}_{\varepsilon}(a)$.

Proof:-Let $\alpha + it\ \epsilon\ \mathcal{SP}_{\varepsilon}(a)$

Then there is $a \epsilon IN(C_{\mathbb{C}})$ such that

$\|(\alpha-a)^2+t^2\|\,\|(\alpha-a)^2+t^2\|^{-1} \geq \frac{1}{\varepsilon}$

then by proposition(4.2.10(1))

$\|((\alpha,0)-(a,0))^2+t^2\|\,\|((\alpha,0)-(a,0))^2+t^2\|^{-1} \geq \frac{1}{\varepsilon}$

then $(\alpha-a,t)(\alpha-a,-t) \notin C'_{\varepsilon}$

and by proposition (4.2.10(2)) $(\alpha-a,t) \notin C'_{\varepsilon}$

then $(\alpha+it)(1,0)-(a,0) \notin C'_{\varepsilon}$

then $\alpha+it\ \epsilon\ \mathcal{SP}_{\varepsilon}\left((a,0),A_{\mathbb{C}}\right)$.

□

Theorem 2.12: $\mathcal{SP}(a) \subseteq \mathcal{SP}_{\varepsilon}(a)$.

Proof:-Let $\lambda \epsilon \mathcal{SP}(a)$ then $\lambda - a \notin IN(C)$

so $\lambda - a \notin C_{\varepsilon}$.

Then $\|\lambda - a\|\,\|\lambda-\alpha\|^{-1} \geq \frac{1}{\varepsilon}$,so $\lambda \epsilon \mathcal{SP}_{\varepsilon}(a)$.

□

Proposition 2.13: Let (A,C) with algebra cone C and B be subalgebra of A with algebra cone $C \cap B$ and $e \in B \subset A$ such that the condition spectrum radius in B is monotone . If $a,b \epsilon A$ such that $o \leq a \leq b$ relative to C and

$r_{\varepsilon}{}^{B}(b) = r_{\varepsilon}{}^{A}(b)$ then $r_{\varepsilon}{}^{A}(a) \leq r_{\varepsilon}{}^{A}(b)$.

Proof:-Since B is subalgebra of A, then $(B, C\cap B)$is ordered Banach algebra by complex (theorem 2.3.8[9]), the condition spectrum radius is monotone then $r_{\varepsilon}{}^{B}(a) \leq r_{\varepsilon}{}^{B}(b)$, since $B \subseteq A$ and B is subalgebra, then $\mathcal{SP}_{\varepsilon}{}^{A}(a) \subseteq \mathcal{SP}_{\varepsilon}{}^{B}(a)$

so $r_{\varepsilon}{}^{A}(a) \leq r_{\varepsilon}{}^{B}(a)$ and we have $r_{\varepsilon}{}^{A}(b) = r_{\varepsilon}{}^{B}(b)$, then we obtain

$r_{\varepsilon}{}^{A}(a) \leq r_{\varepsilon}{}^{B}(a) \leq r_{\varepsilon}{}^{B}(b) = r_{\varepsilon}{}^{A}(b)$

□

Theorem 2.14: $r(a) \leq r_{\varepsilon}(a) \leq \frac{1+\varepsilon}{1-\varepsilon}\|a\|$

Proof:-Since $\mathcal{SP}(a) \subseteq \mathcal{SP}_{\varepsilon}(a)$, then $r(a) \leq r_{\varepsilon}(a)$

Let $\alpha + it \epsilon \mathbb{C}$ $a \epsilon A$

If $|\alpha+it| \leq \|a\|$ since (1- ε)$\|a\| \leq (1+\varepsilon)\|a\|$, then $|\alpha+it| \leq \frac{1+\varepsilon}{1-\varepsilon}\|a\|$.

If $|\alpha+it| \geq \|a\|$, then $(\alpha-a)^2+t^2$ is invertible by proposition(2.2.22) and $\|((\alpha-a)^2+t^2)^{-1}\| \leq \frac{1}{|\alpha+it|-\|a\|}$

Then $\|(\alpha-a)^2+t^2\|\,\|((\alpha-a)^2+t^2)^{-1}\| \geq \frac{1}{\varepsilon}$ so $\frac{|\alpha+it|+\|a\|}{|\alpha+it|-\|a\|} \geq \frac{1}{\varepsilon}$

and $\frac{\varepsilon|\alpha+it|+\|a\|}{|\alpha+it|-\|a\|} \geq 1$

$\varepsilon(|\alpha+it| + \|a\| \geq |\alpha+it| - \|a\|$

$\varepsilon|\alpha+it| + \varepsilon\|a\| \geq |\alpha+it| - \|a\|$

$\|a\| + \varepsilon\|a\| \geq |\alpha+it| - \varepsilon|\alpha+it|$

$(1+\varepsilon)\|a\| \geq (1-\varepsilon)|\alpha+it|$

$|\alpha+it| \leq \frac{1+\varepsilon}{1-\varepsilon}\|a\|$.

□

In [2], we have the next example which explains that the radius is not a necessary condition radius and the spectrum is not a necessary condition spectrum.

Example(Bilateral Shift) 2.15: Let $f: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ define by $f(e_i) = e_{i+1}$ for all $i\epsilon\mathbb{Z}$,such that e_i is the standard basis. Let $\beta\epsilon\mathbb{C}$, consider $f - \beta$ define by $b(t) = -\beta + e^{it}$

Such that $\| f - \beta \| = \|b(t)\|_{\infty}$. Hence $\| f - \beta \| = |\beta| + 1$.When $\beta \neq 1$ we have $\beta \notin \mathcal{SP}(a)$ and $(f - \beta)^{-1} = \frac{1}{b(t)}$, then

$$\| (f - \beta)^{-1} \| = \frac{1}{\|b(t)\|_{\infty}} = \frac{1}{\inf_{0\le t\le 2\pi} |b(t)|}$$

$$= \begin{cases} \frac{1}{|\beta|-1} & if\ |\beta| > 1 \\ \frac{1}{1-|\beta|} & if\ |\beta| < 1 \end{cases}$$

So, $$\|(f - \beta)\| \, \| (f - \beta)^{-1} \| = \begin{cases} \frac{|\beta|+1}{|\beta|-1} & if\ |\beta| > 1 \\ \frac{1+|\beta|}{1-|\beta|} & if\ |\beta| < 1 \end{cases}$$

From this $\|(f - \beta)\| \, \| (f - \beta)^{-1} \| \geq \frac{1}{\varepsilon}$ if and only if $1 > |\beta| \geq \frac{1-\varepsilon}{1+\varepsilon}$ or $\frac{1+\varepsilon}{1-\varepsilon} \geq |\beta| > 1$. It is known that $\mathcal{SP}(f) = \{\beta: |\beta| = 1\}$which is the Range of f. Hence $\mathcal{SP}_\varepsilon(a) = \{\beta: \frac{1-\varepsilon}{1+\varepsilon} \leq |\beta| \leq \frac{1+\varepsilon}{1-\varepsilon}\}$.

Theorem 2.15: Let A be a real OBA with identity and let f be a complex character from A to $\mathbb{C}$ with $f(e) = 1$, then $f(a)\epsilon\, \mathcal{SP}_\varepsilon(a)$ for all $a\epsilon A$.

Proof :- Let $a\epsilon A$ and $f(a) = \alpha + it$ and $(\alpha - a)^2 + t^2$ is invertible, then

$1 = |f(e)| = |f((\alpha - a)^2 + t^2)((\alpha - a)^2 + t^2)^{-1}| \leq \varepsilon\|(\alpha - a)^2 + t^2\| \, \|(\alpha - a)^2 + t^2)^{-1}\|$

so $\|(\alpha - a)^2 + t^2\| \, \|(\alpha - a)^2 + t^2)^{-1}\| \geq \frac{1}{\varepsilon}$

Then $f(a) = \alpha + it\ \epsilon\ \mathcal{SP}_\varepsilon(a)$.

□

Theorem 2.16: Let A be a real OBA with identity , let $0 < \varepsilon < 1$ and f be a linear functional from A to $\mathbb{C}$ .If $f(a)\epsilon\mathcal{SP}_\varepsilon(a)$,then f is character.

Proof :-Since $\mathcal{SP}_\varepsilon(e) = \{1\}$, then $f(e) = 1$

let $a\epsilon A$ and $\|a\| = 1$, by theorem(4.2.14) we have $\|f\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|a\|$

define $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ by $\varphi(\alpha + it) = f(exp(\alpha + it)(a)$ (1.1)

then φ is entire function since $f(a)\epsilon\mathcal{SP}_\varepsilon(a)$, $\varphi(\alpha + it) \neq 0$

we have that $|\varphi(\alpha + it)| \leq \|f\| \, \|exp(|\alpha + it|a)\| \leq \frac{1+\varepsilon}{1-\varepsilon} exp(|\alpha + it|.1)$.

We can write $\varphi(\alpha + it)$ by the way: $\varphi(\alpha + it)=exp(g(\alpha + it))$ for some Entire function g which define by $g(\alpha + it) = \delta(\alpha + it) + \beta$ by [37] since $f(exp(0)) = 1, \varphi(exp(0)) = 1$ and $\varphi(0) = exp(g(\alpha + it)) = 1$ since $1 = exp(0)$, then$f(a)(\alpha + it) + \beta = 0$, so $\beta = 0$ and

$g(\alpha + it)= f(a)(\alpha + it)$, we obtain

$\varphi(\alpha + it) = exp(f(a)(\alpha + it))$

$$=\sum_{i=0}^{n} \frac{f(a)^n}{n!}(\alpha + it)^n \qquad (1.2)$$

by(1.1) we have

$\varphi(\alpha + it) = f[\sum_{i=0}^{n} \frac{a^n}{n!}(\alpha + it)^n] = \sum_{i=0}^{n} \frac{f(a^n)}{n!}(\alpha + it)^n$ by comparing with (1.2) $f(a^n) = f(a)^n$.

□

3 further properties of condition order spectrum

Definition 3.1(condition Quasinilpotent) : Let A order Banach algebra. Then A is called condition Quasi nilpotent if $\mathcal{SP}_\varepsilon(a) = \{0\}$and denoted by $\mathcal{QN}_\varepsilon(A)$.

Definition 3.2 (Radical): Let A be OBA with identity , then the radical which denoted by $Rad\ (A)$ is define as$Rad\ (A) = \{x\epsilon A: e - xz\epsilon\ C: z\epsilon A\ \}$

Proposition3.3: $Rad\ (A) = \{a\epsilon A: aA \subset \mathcal{QN}_\varepsilon(A)\}$ =$[a\epsilon A: Aa \subset \mathcal{QN}_\varepsilon(A)\}$

Proof:-Let $a\epsilon\{a\epsilon A: aA \subset \mathcal{QN}_\varepsilon(A)\}$. Then $\mathcal{SP}_\varepsilon(az) = \{0\}$

For all $z\epsilon A$, so $1 - za\epsilon C$ which implies that $a\epsilon Rad\ (A)$ conversely, let $a\epsilon Rad\ (A)$. Then $1 - za\epsilon C$ for all $z\epsilon A$ this implies $1 - za\epsilon C$ is invertible, so $\mathcal{SP}_\varepsilon(az) = \{0\}$ for all $z\epsilon A$ we obtain $az\epsilon\mathcal{QN}_\varepsilon(A)$, then $aA \subset \mathcal{QN}_\varepsilon(A)$.

□

Corollary 3.4: $Rad\ (A) \subset \mathcal{QN}_\varepsilon(A)$.

Proof:-It is clear .

Proposition 3.5: Let $(A\ , C)$ be a OBA and the condition spectral radius is monotone. Then the following are satisfies:

1. If b is Quasi nilpotent, then a is Quasi nilpotent

2. If $b\epsilon\ Rad\ (A)$, then a is Quasi nilpotent

3. If $b\epsilon Rad\ (A)$ and a in the center of A then $a\epsilon Rad\ (A)$.

Proof:-

1. Since $b\epsilon\mathcal{QN}_\varepsilon(A)$ then $r_\varepsilon(b) = 0$,since $0 \le r_\varepsilon(a) \le 0$ then we give $\mathcal{SP}_\varepsilon(a) = \{0\}$.

2. Since $b\epsilon Rad\ (A)$, then $b\epsilon\mathcal{QN}_\varepsilon(A)$ by corollary(4.3.4) so by (1) above we have $a\epsilon\mathcal{QN}_\varepsilon(A)$.

3. Since $b\epsilon Rad\ (A)$ then $b\epsilon\mathcal{QN}_\varepsilon(A)$ so $r_\varepsilon(a) = 0$ let $x\epsilon A$, since $xa = ax$ then $r_\varepsilon(ax) \le r_\varepsilon(a)r_\varepsilon(x) \le 0.r_\varepsilon(x) = 0$ we obtain $aA \subseteq \mathcal{QN}_\varepsilon(A)$,then $a\epsilon Rad\ (A)$.

□

Theorem 3.6: Let A be OBA and the condition spectrum radius is monotone such that $0 \le a \le b$ .If $b\epsilon Rad\ (A)$ then $aC \subset \mathcal{QN}_\varepsilon(A)$.

Proof:- Let $b\epsilon Rad\ (A)$. Then $bc \subset Rad\ (A) \subset \mathcal{QN}_\varepsilon(A)$ so $r_\varepsilon(bc) = 0$, since $o \le a \le b$,then $0 \le ac \le bc$ for all $c\epsilon C$ and $r_\varepsilon(ac) \le r_\varepsilon(bc)$ because of the condition spectrum radius is monotone
we obtain that $r_\varepsilon(ac) = 0$ then $aC \subset \mathcal{QN}_\varepsilon(A)$.

□

4 The Relation Between Character And Condition Order Spectrum

Theorem 4.1: Let A be OBA and $a\ \epsilon A$ then $\mathcal{SP}_\varepsilon(a)$ is upper semi continuous on A, that is: If $a\epsilon A$ then for every open set U containing $\mathcal{SP}_\varepsilon(a)$there exists $\delta > 0$ such that $\|a - b\| < \delta$ implies $\mathcal{SP}_\varepsilon(b) \subset U$

Proof:-Suppose that there exists sequence $\{b_n\}$and $\{\alpha_n\}$ such that

$a = \lim_{n\to\infty} b_n$, $\alpha_n\epsilon\mathcal{SP}_\varepsilon(b_n) \cap U^c$from theorem (4.2.14[9]). Then $|\alpha_n| \le \frac{1+\varepsilon}{1-\varepsilon}\|b_n\|$ so $\{\alpha_n\}$ is bounded sequence in $\mathbb{C}$ and therefore has a convergent subsequence .(without loss a generality) we may assume that $\{\alpha_n\}$ convergent say to α that is $(\alpha_n - b_n)^2 + t^2 \to (\alpha - a)^2 + t^2$ as $n \to \infty$,since U^c is closed, so $\overline{U} = U^c$ then $\alpha \notin U$ we obtain that $\alpha - a$ is invertible ,since the set of all invertible is open so for n large enough $\alpha_n - b_n$ is invertible which is contradiction.

□

Theorem 4.2: Let A be OBA and $a \epsilon A$,let U, V are two disjoint open set such that $\mathcal{SP}_\varepsilon(a) \subset U \cup V$ and $\mathcal{SP}_\varepsilon(a) \cap U \neq 0$. then there exists $r > 0$ such that $\|a - b\| < r$ implies $\mathcal{SP}_\varepsilon(b) \cap U \neq 0$

Proof:- Since the spectrum is upper semi continuous there exists $\delta > 0$ such that $\|a - b\| < \delta$ imply $\mathcal{SP}_\varepsilon(b) \subset U \cup V$

Suppose that there exists a sequence $b_n \to a$ such that $\mathcal{SP}_\varepsilon(b) \subset V$ for n large enough let f be a function on $U \cup V$ define by $f(a)$ =1 if $a \epsilon U$ and $f(a) = 0$ if $a \epsilon \mathcal{V}$, then f is holomorphic on $U \cup V$, since $b_n \to a$ and the definition of the functional culculas we see that $\lim_{n\to\infty} f(b_n) = f(a)$ and $f(b_n) = 0$ for n large enough

,since $\mathcal{SP}(f(a)) \subseteq \mathcal{SP}_\varepsilon(f(a)) = \mathcal{SP}_\varepsilon(0) = \{0\}$, then $\mathcal{SP}(f(a)) \subseteq \{0\}$

since $\mathcal{SP}(f(a)) = f(\mathcal{SP}(a))$ by spectral mapping theory [1], then $f(\mathcal{SP}(a)) \subseteq \{0\}$ but $f(\mathcal{SP}(a)) \subseteq f(\mathcal{SP}_\varepsilon(a))$, then $f(\mathcal{SP}_\varepsilon(a)) = \{0\}$ so we obtain $f(\mathcal{SP}_\varepsilon(a)) = \mathcal{SP}_\varepsilon(f(a)) = \{0\}$

but $\mathcal{SP}_\varepsilon(f(a))$ contain 1 and this contradiction Then $\mathcal{SP}_\varepsilon(b) \cap U \neq 0$.

□

Theorem 4.3: let A be OBA and f a linear functional from A into $\mathbb{C}$ such that $f(e) = 1$,then the two following conditions are equivalent

1. $f(a) = i\alpha$ then $f(a^n) = -\alpha^n$

2. f is Character

Proof:- (1) $\longrightarrow$ (2)To prove that f is character we have to prove that $f(ab) = f(a)f(b)$

if $f(a) = \alpha + i\beta$,then $f(a - \alpha) = i\beta$, by(1) $f(a^n) = -\alpha^n$

$f((a - \alpha)^n) = -\beta^n$,then $(f(a - \alpha))^n = (i\beta)^n = -\beta^n$

if $n = 2k + 1$, $k = 0,2,4,\ldots$

that is $f((a - \alpha)^n) = (f(a - \alpha))^n$

$\sum_{n=0}^{\infty}(-1)^n\binom{n}{k} f(a^{n-k})\alpha^k = \sum_{n=0}^{\infty}(-1)^n\binom{n}{k} f(a)^{n-k}\alpha^k$, then $f(a^{n-k}) = f(a)^{n-k}$

so $f(a^{k+1}) = f(a)^{k+1}$ that is $f(a^m) = f(a)^m$ such that $m = k + 1$, then f is character

(2) $\to$ (1) since $f(a) = i\alpha$, then $f(ab) = i\alpha f(b)$ for all $a , b\epsilon A , \alpha\epsilon\mathbb{R}$

since $f(ab) = f(a)f(b)$, then $f(a.a) = f(a).f(a)$

then $f(a^n) = f(a)^n$

since $f(a) = i\alpha$ we have $(f(a))^n = (i\alpha)^n = i^n\alpha^n = -\alpha^n$ such that $n = 2k + 1$, $k = 0,2,4, \ldots$

Theorem 4.4: let A be a complex OBA with identity and let f be a character functional on A

i- If $f(1) = 0$, then $\|f\| \leq \varepsilon$, $0 < \varepsilon < 1$

ii- If $f(1) \neq 0$, then $f(a) \,\epsilon \mathcal{SP}_\epsilon(a)$, for every $a\epsilon A$, where $\varepsilon = \frac{\varepsilon'}{\|f(1)\|}$

Proof:- (i) let $a, b\epsilon A$, since f is character. Then $f(ab) = f(a)f(b)$ so $f(ab) - f(a)f(b) = 0$ if $b = 1$, we obtain $\|f(a)\| = 0 < \varepsilon$, then $\|f\| < \varepsilon$.

(ii) suppose $f(a) = \beta$ and $\beta - a$ is invertible since f is character then f is continuous then $\|f(\beta - a)f(\beta - a)^{-1}\| = \|f((\beta - a)(\beta - a)^{-1}\| \leq \varepsilon$ such that $\|\beta - a\| \|(\beta - a)^{-1}\| \| \geq \frac{\|f(1)\|}{\varepsilon}$, then $\beta\epsilon\mathcal{SP}_\epsilon(a)$

Definition 4.5(condition pseudo spectrum): Let A be OBA and $a\epsilon A$, $\lambda\epsilon\mathbb{C}$. The condition pseudo spectrum of a which denoted by $\mathcal{PSP}_{\varepsilon}(a)$ is define by $\mathcal{PSP}_{\varepsilon}(a)=\{\mathcal{SP}_{\varepsilon}(a)\cap\{\lambda\epsilon\mathbb{C}:|\lambda|=r_{\varepsilon}(a)\}$ for all $\lambda\epsilon\mathbb{C}$.

In the following theorem we prove that the condition radius is convergent

Theorem 4.6: Let A be a real OBA and $\{a_n\}$ be a sequence, such that $a_n\to a$ as $n\to\infty$ .If $\mathcal{PSP}_{\varepsilon}(a)$ contains at least one isolated point in $\mathcal{SP}_{\varepsilon}(a)$ then $r_{\varepsilon}(a_n)\to r_{\varepsilon}(a)$ as $n\to\infty$.

Proof:-Let $\lambda\epsilon\ \mathcal{PSP}_{\varepsilon}(a)$ isolated point in $\mathcal{SP}_{\varepsilon}(a)$ and let $\mathcal{B}(\lambda,r)$open set such that $\mathcal{B}(\lambda,r)\cap\mathcal{SP}_{\varepsilon}(a)=\lambda$

let V open set such that $\mathcal{B}(\lambda,r)\cap\ V=\emptyset$ that mean $\lambda\notin\ V$ and

$\mathcal{SP}_{\varepsilon}(a)\backslash\lambda\subset V$ let $0\leq k\leq r$. Then $sp_{\varepsilon}(a)\subset\mathcal{B}(k,r_{\varepsilon}(a))$ from the upper semi continuous of the spectral radius and the fact $a_n\to a$ it follows that there exists $\mathcal{N}_{k_1},\mathcal{N}_{k_1}\epsilon\mathbb{N}$ such that $r_{\varepsilon}(a_n)\leq r_{\varepsilon}(a)+k$ and$r_{\varepsilon}(a_n)\geq r_{\varepsilon}(a)-k$ for all $n\epsilon\mathbb{N}$, we have $\mathcal{SP}_{\varepsilon}(a)\subset\mathcal{B}(\lambda,k)\cup V$ and $\mathcal{SP}_{\varepsilon}(a)\cap\mathcal{B}(\lambda,k)\neq\emptyset$ so by using theorem(4.4.2[9]) $\mathcal{SP}_{\varepsilon}(a_n)\cap\mathcal{B}(\lambda,k)\neq\emptyset$ suppose $\alpha_n\epsilon\mathcal{SP}_{\varepsilon}(a)$ and $|\alpha_n-\lambda|<k_1$ for all $n\geq\mathcal{N}_{k_1}$, then $r_{\varepsilon}(a)\geq|\alpha_n|>r_{\varepsilon}(a)-\varepsilon$

let $\mathcal{N}=max\{\mathcal{N}_{k_1},\mathcal{N}_{k_2}\}$, so $r_{\varepsilon}(a)-k<r_{\varepsilon}(a_n)<r_{\varepsilon}(a)+k$ that mean $-k<r_{\varepsilon}(a_n)-r_{\varepsilon}(a)<k$

$|\ r_{\varepsilon}(a_n)-\ r_{\varepsilon}(a)\ |<k$ that mean$r_{\varepsilon}(a_n)\to r_{\varepsilon}(a)$.

□

Definition 4.7: let A be complex OBA with identity, we define the distance between $\lambda\epsilon\mathbb{C}$ and the $\mathcal{SP}_{\varepsilon}(a)$ by

$d(\lambda,\mathcal{SP}_{\varepsilon}(a))=inf\{|\lambda-\beta|:\beta\epsilon\mathcal{SP}_{\varepsilon}(a)\}$. And if $\lambda=0$ the distance was between 0 and $\mathcal{SP}_{\varepsilon}(a)$ will denoted by $\delta(a)=d(0,\mathcal{SP}_{\varepsilon}(a))$

Theorem 4.8: let A be OBA and C is closed normal and closed-inverse algebra cone, if $a\epsilon A$,then $\delta(a)\epsilon\mathcal{SP}_{\varepsilon}(a)$.

Proof:- if $a\notin IN(a)$,then $\delta(a)=0\epsilon\mathcal{SP}_{\varepsilon}(a)$ suppose that $a\epsilon IN(a)$,since $a\epsilon C$ and C is inverse-closed then $a^{-1}\epsilon C$

since C is normal and closed then

$r_{\varepsilon}(a^{-1})\epsilon\mathcal{SP}_{\varepsilon}(a^{-1})$

(*)

so that $r_{\varepsilon}(a^{-1})=\frac{1}{\alpha+it}$,for some $\alpha+it\epsilon\mathcal{SP}_{\varepsilon}(a)$

since $r_{\varepsilon}(a^{-1})=\frac{1}{\delta(a)}$,then by comparing with (*) obtain $\delta(a)\epsilon\mathcal{SP}_{\varepsilon}(a)$. □

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المستخلص :

في هذا البحث سندرس العلاقة بين التوصيف المعقد f والطيف المرتب الشرطي $\mathcal{SP}_\varepsilon(a)$ في بناخ الجبرا المرتب المعقد والحقيقي A. سنبرهن بعض الخواص عليه

بالاضافة الى ذلك، سنبرهن $\mathcal{SP}_\varepsilon(a)$ شبه مستمرة عليا

اخيرا، سنبين العلاقة بين حالة شبه nilpotent و الرادكالية .