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T-semimaximal submodules

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Abstract
In this paper, we define and study the notions of t-semimaximal submodule as a generalization of semimaximal submodule. We provided many properties and characterizations of this concept are provided.

Key words: maximal submodule, semimaximal submodule, t-semimaximal submodule and t-semisimple modules.

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1. Introduction

Throughout this paper R is a ring with unity and M unitary a right R-module. The second singular (or Goldi torsion) of M is denoted by $Z_2(M)$ and defined as $Z(M/Z(M)) = Z(M)/Z_2(M)$ where Z(M) is the singular submodule of M[5]. A module M is called Z_2-torsion if $Z_2(M) = M$. A submodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \cap W \neq (0)$ for every non-zero submodule W of M[7].

The concept of t-essential submodules is introduced as a generalizations of essential submodules [2]. A submodule N of M is said to be t-essential in M (denoted by ($N \leq_{tes} M$) if for every submodule B of M, $N \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. A submodule N of a module M is called small in M and denoted by $N \ll M$ if for every $K \leq M$ the equality $M = N + K$ implies $M = K$. A module M is called hollow if every proper submodule of M is small in M "[10].

Asgari and Haghany in [3] introduced the concept of t-semisimple modules and t-semisimple rings; A module M is called t-semisimple if every submodule N of M contains a direct summand K of M such that K is t-essential in N. A submodule N of a module M is called semimaximal if M/N is a semisimple module [9].

In this paper, we introduce a generalization of semimaximal submodule, namely t-semimaxmal. A submodule N of a module M is called t-semimaxmal if M/N is t-semisimple module. This paper consists of two sections, in section two of this paper, we define and study the concept of t- maximal submodules and give some properties and charerizations of it.

Proposition (1.1)[2]:" The following statements are equivalent for a submodule A of an R-module M:

(1) A is t-essential in M.

(2) $(A + Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$;

(3) $A+Z_2(M)$ is essential in M;

(4) M/A is Z_2 -torsion [3]".

Corollary (1.2) [3]:" Let M be a t-semisimple module. Then:
(1) Every submodule of M is t-semisimple.
(2) Every homomorphic image of M is t-semisimple".

Corollary (1.3) [3]: "A module M is t-semisimple if and only if M has no proper t-essential submodule which contains $Z_2(M)$".

Corollary (1.4) [3]: Every direct sum of t-semisimple modules is t-semisimple.

2. t-semimaxmal submodules

in this section, we will introduce and study the concept of t-semimaximal submodule

Definition (2.1): A submodule N of module M is called t-semimaximal if M/N is a t-semisimple module.

Proposition (2.2): Let M be an R-module. $Z_2(M)$ is semimaximal submodule of M if and only if $Z_2(M)$ is t-semimaximal submodule of M.

Proof: $\Rightarrow$ It is clear.

$\Leftarrow$ Since $Z_2(M)$ is t-semimaximal submodule of M, then $M/Z_2(M)$ is a t-semisimple module . Hence $\dfrac{M/Z_2(M)}{Z_2(M/Z_2(M))}$ is semisimple by [3, Theorem 2.3], but $Z_2\left(M/Z_2(M)\right) = (0)$. Hence $M/Z_2(M)$ is semisimple module . Thus $Z_2(M)$ is semimaximal submodule of M.

Remarks and Examples (2.3):

(1) It is clear that every semimaxmal submodule of a right R-module is t-semimaxmal submodule but not conversely, for example: $4Z$ is a t-semimaximal submodule of Z as Z-module (because $Z/4Z$ is t-semisimple Z-module [3])

(2) Every t-essential (or essential) submodule N of M is t-semimaxmal (by [3, Example 2.2(i)])

(3) Let $N \leq M$ and W be the complement of N, then $N \oplus W$ is t-semimaxmal of M.

(4) Let $N \leq W \leq M$ and N be a t-semimaxmal submodule, then W is a t-semimaxmal submodule of M.

Proof: Let $f: M/N \mapsto M/W$ defined by $f(m+N) = m+W$, for all $m \in M$.
It is clear that f is a well-defined and epimorphism. Since M/N is t-semisimple it follows from Hence $\frac{M}{W}$ is t-semisimple by Corollary 1.2(2) that M/W is t-semisimple and hence W is a t-semimaxmal submodule of M.

(5) If N is t-semimaximal of M and $N \leq K \leq M$, then N is t-semimaximal of K.

Proof: Since N is t-semimaxmal of M it follows that M/N is t-semisimple. But $K/N \leq M/N$, hence by Corollary 1.2(1) K/N is t-semisimple. Thus N is t-semimaxmal of K.

(6) Let $\{M_i, i \in I\}$ be a family of R-modules and let $= \oplus_{i\in I} M_i$. If A_i is t-semimaximal of M_i, then $\oplus_{i\in I} A_i$ is t-semimaxmal of $\oplus_{i\in I} M_i$.

Proof: Since A_i is t-semimaxmal of M_i, it follows that M_i/A_i is t-semisimple and hence $\oplus_i M_i/A_i$ is t-semisimple by Corollary 1.4. Thus $\oplus_{i\in I} A_i$ t-semimaxmal of $\oplus_{i\in I} M_i$.

(7) Let $\leq K \leq M$. Then Ka t-semimaximal submodule if and only if of M, K/B is a t-semimaxmal submodule of M/B .

Proof: $\Rightarrow$Since K is a t-semimaxmal submodule of M, it follows that M/K t-semisimple. But $\frac{M/B}{K/B} \simeq M/K$, it follows that $\frac{M/B}{K/B}$ is a t-semisimple module and hence K/B is a t-semimaximal submodule of M/B.
$\Leftarrow$ By similarly way of first direction.

(8) $Rad\,(M)$ is t-semimaxmal submodule of M if and only if $M = M_1 \oplus M_2$ such that M_1 is semisimple and $Rad\, M \leq_{tes} M_2$ [3, Proposition 2.10].

(9) If (0) is a t-semimaxmal submodule of a module M then N is t-semimaxiaml , for each non-zero submodule N of M.

Proof: suppose that (0) is t-semimaxmal submodule of a module M, thus $M/(0) \simeq M$ is t-semisimple. Hence M/N is t-semisimple by [3, Corollary 2.4(2)]. Thus N is t-semimaxmal.

(10) If N is a nonzero t-semimaxmal submodule (0) need not be t-semimaxmal, for example: $6Z$ in Z-module is t-semimaxmal. But (0) is not t-semimaxmal since $Z/(0) \simeq Z$ is not t-semisimple

(11) M is t-semisimple R-module if and only if M is t-semisimple $R/ann(M)(0)$ -module.

Proof: Since every submodule of M R-module if and only if every submodule of M $R/ann(M)$ module [10].

Proposition (2.4): Every submodule of t-semisimple R-module is t-semimaxmal submodule.

Proof: Let $U \leq M$ and $\pi: M \mapsto M/U$ be the natural epiomorphism. Hence M/U is t-semisimple by Corollary 1.2(2). Thus U is t-semimaxmal.

Proposition (2.5): The intersection of any two t-semimaximal submodules of an R-module is t-semimaximal submodule.

Proof: Let U_1, U_2 be two t-semimaximal submodules of M. Thus M/U_1 and M/U_2 are t-semisimple modules and hence $M/U_1 \oplus M/U_2$ is t-semisimple by Corollary 1.4. Since $M/U_1 \cap U_2$ is an isomorphism to a submodule of $M/U_1 \oplus M/U_2$ it follows that $M/U_1 \cap U_2$ is t-semisimple. Thus $U_1 \cap U_2$ is a t-semimaxmal submodule of M.

Proposition (2.6): Let U_1 be a t-semimaximal submodule of an R-module M_1 and U_2 be a t-semimaxmal submodule of an R-module M_2. Then $U_1 \oplus U_2$ is a t-semimaximal submodule of $M_1 \oplus M_2$.

Proof: By hypothesis, M/U_1 and M/U_2 are t-semisimple R-module and hence from Corollary 1.4 we have that $M/U_1 \oplus M/U_2$ is t-semisimple Since $M_1 \oplus M_2/U_1 \oplus U_2 \simeq M/U_1 \oplus M/U_2$. It follows that $M_1 \oplus M_2/U_1 \oplus U_2$ is t-semisimple and hence $U_1 \oplus U_2$ is t-semimaximal in $M_1 \oplus M_2$.

Proposition (2.7): Let M be an R-module and $N \leq M$. Then N is a t-semimaxmal if and only if M/W is semisimple, for each t-closed submodule W of M and $W \supseteq N$.

Proof: $\Rightarrow$ Let W be a t-closed submodule of M with $W \supseteq N$. Hence M/W is a t-closed in M/N by [4, Lemma 2.5]. But N is a t-semimaxmal by hypothesis, so M/N is t-semisimple. Then by [3, Corollary 2.17], $\frac{M/N}{W/N}$ is semisimple and hence $\frac{M}{W}$ is semisimple.

$\Leftarrow$ To prove N is a t-semimaximal submodule of M. Let C/N be a t-closed in M/N, hence C is a t-closed of M, and $C \supseteq N$. So that M/C is semisimple by hypothesis, but $M/C \simeq \frac{M/N}{C/N}$ so that $\frac{M/N}{C/N}$ is semisimple for each t-closed submodule C/N of M/N, which implies M/N is t-semisimple by [3, Corollary 2.17]. Thus N is t-semimaxmal submodule of M.

Proposition (2.8): If $Rad(M)$ is a t-semimaxmal and M is hollow then $M/Rad(M)$ is Z_2-torsion.

Proof: Since $Rad(M)$ is t-semimaxmal, $M/Rad(M)$ is t-semisimple. By [3, Proposition 2.10] we have that $M = M_1 \oplus M_2$ where M_1 is semisimple and $Rad(M) \leq_{tes} M_2$. Let $A \ll M$, then $A \leq Rad(M) \leq_{tes} M_2$, so if M is hollow, every submodule of M contain in M_2. Hence $M = M_2$ and thus $M/Rad(M)$ is Z_2-torsion.

Proposition (2.9): Let $N \leq M$. If $(N:_R M)$ is t-semimaxmal ideal in R,then N is t-semimaximal.

Proof: Since $(N:_R M)$ is t-semimaxmal ideal in R, $R/(N:M)$ is a t-semisimple R-module. Since M/N is an R-module, $\frac{M}{N}$ is an $\bar{R}$-module where ($\bar{R} = R/ann(M/N)$) that is M/N is an $R/(N:_R M)$-module. Hence M/N is a t-semisimple $\bar{R}$-module. Hence M/N is a t-semisimple R-module (by Remarks and Examples 2.3(13)). Thus N is t-semimaxaml.

Remark (2.10): If R is t-semisimple ring and M is an R-module, then every submodule of M is t-semimaxaml.

Proof: Since R is a t-semisimple, every R-module M is t-semisimple[3,Theorem 3.2]. Hence by Proposition 2.3 every submodule of M is t-semimaxiaml.

Proposition (2.11): Let $N \leq M$. Then N is a t-semimaxiamal submodule in M if and only if for each submodule A of M with $A \supseteq N$, there exist $K, K' \leq N$ such that $A = K + K'$ and $M = K + L$ for some $L \leq M$ and $N \leq_{tes} K'$, $K \cap L = N, K \cap K' = N$.

Proof: $\Rightarrow$ Let N be a t-semimaxmal submodule in M, then M/N is t-semisimple. For each $A \supseteq N$, $\frac{A}{N} \leq \frac{M}{N}$. Hence by [3, Proposition 2.13(3)] $A/N = K/N \oplus K'/N$ for each $K, K' \leq M$ with $K/N \leq^{\oplus} M/N$ and K'/N is Z_2-torsion. Hence $N \leq_{tes} K'$ by Proposition 1.1. $K/N \leq^{\oplus} M/N$, then $K/N \oplus L/N = M/N$ for some $L \leq M$ with $N \leq L$, then $K + L = M$ with $K \cap L = N$.

$\Leftarrow$ For any $A/N \leq M/N$. As $A = K + K', K \cap K' = N$, then $A/N = K/N \oplus K'/N$. Also, $K + L = M, K \cap L = N$, then $K/N \oplus L/N = M/N$, so $K/N \leq^{\oplus} M/N$ But $N \leq_{tes} K'$ implies K'/N is Z_2-torsion. Hence $A/N = K/N \oplus K'/N$ with $K/N \leq M/N$ and K'/N is Z_2-torsion implies M/N is t-semisimple by [3, Proposition 2.13(3)]. Thus N is t-semimaxmal submodule in M.

Proposition (2.12): An R-module M is t-semisimple if and only if $\forall\ N \leq M, N + Z_2(M)$ is semimaxmal.

Proof: $\Rightarrow$ Suppose that M is t-semisimple, then $N + Z_2(M)$ is closed in M, $\forall\ N \leq M$ by [3,Corollary 2.8]. But $N + Z_2(M)$ contains $Z_2(M)$, so $N + Z_2(M)$ is t-closed [2,Proposition 2.6(4)]. Hence by [3, Corollary 2.17], $M/N + Z_2(M)$ is semisimple.

$\Leftarrow$ Since $\forall$ $N \leq M, N + Z_2(M)$ is semimaxmal, so that $M/N + Z_2(M)$ is semisimple. Hence $M/N + Z_2(M)$ is semisimple (if $N = 0$). This implies M is t-semisimple [3, Theorem 2.3].

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