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SS-Flat Modules

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Abstract. In this paper, we introduce the dual notion of ss-injective module, namely ss-flat module. The connection between ss-injectivity and ss-flatness is given. Min-Coherent rings, FS-rings, PS-rings, and universally mininjective rings are characterized in terms of ss-flat modules and ss-injectivite modules.

Key Words: min-coherent ring; ss-coherent ring; ss-flat module; ss-injective module; PS ring; FS ring; universally mininjective ring.

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1. Introduction

In [1], the notion of ss-injectivity was introduced and studied. A right R-module M is called ss-injective if any right R-homomorphism $f: S_r \cap J \longrightarrow M$ extends to R; equivalently, if $\operatorname{Ext}^1(R/(S_r \cap J), M) = 0$. L. Mao [2] introduced the notion min-flat, for any left R-module N, N is called min-flat if $\operatorname{Tor}_1(R/I, N) = 0$ for every simple right ideal I.

In this paper, we introduce and investigate the notion of ss-flat modules as a generalization of flat modules. A left R-module M is said to be ss-flat if $\operatorname{Tor}_1(R/(S_r \cap J), M) = 0$. Examples are established to show that the notion of ss-flatness is distinct from that of min-flatness and flatness. several properties of the class of flat modules are given, for example, we prove that a left R-module M is ss-flat iff $M^+ = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is ss-injective iff the sequence $0 \longrightarrow (S_r \cap J) \otimes M \longrightarrow R \otimes M$ is exact. Also, we prove that the class of all left is closed under pure submodule and direct limits. In Theorem 2.9, we prove that a ring R is right min-coherent iff the class of ss-flat modules is closed under direct products iff ${}_R R^S$ is ss-flat, for any index set S iff every left R-module has (SSF)-preenvelope, where SSF is the class of all left ss-flat modules. Also, we introduce the concept of ss-coherent ring as a proper generalization of coherent ring. Many characterization of ss-coherent rings are given, for example, we prove that a ring R is right ss-coherent iff (a right R-module M is ss-injective iff M^+ is ss-flat) iff the class of all ss-injective right R-modules is closed under direct limits. We study ss-flat modules and ss-injective modules over commutative ring. For example, we prove that a commutative ring R is min-coherent iff $\operatorname{Hom}(M, N)$ is ss-flat for all projective R-modules M and N. Also, we prove that if R is a commutative ss-coherent ring, then an R-module M is ss-injective iff $\operatorname{Hom}(M, N)$ is ss-flat for any injective R-module N. In

Proposition 2.22, we prove that if M is a simple module over a commutative ring R, then M is ss-flat iff M is ss-injective. As a corollary, we prove that if R is a commutative ring, then R is a universally mininjective iff R is PS-ring iff R is an FS-ring,

Next, we recall some facts and notions needed in the sequel. An exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of right R-modules is called pure if every finitely presented right R-module P is projective with respect to this sequence and we called that $f(A)$ is a pure submodule of B [3]. A right R-module M is called pure injective if M is injective with respect to every pure exact sequence [3]. Let R be a ring and $\mathcal{F}$ be a class of right R-modules. An R-homomorphism $f: M \longrightarrow N$ is said to be $\mathcal{F}$-preenvelope of M where $N \in \mathcal{F}$ if, for every R-homomorphism $g: M \longrightarrow F$ with $F \in \mathcal{F}$, there is an R-homomorphism $h: N \longrightarrow F$ such that $hf = g$. An R-homomorphism $f: N \longrightarrow M$ is said to be $\mathcal{F}$-precover of M where $N \in \mathcal{F}$ if, for every R-homomorphism $g: L \longrightarrow M$ with $L \in \mathcal{F}$, there is an R-homomorphism $h: L \longrightarrow N$ such that $fh = g$ [4]. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a class of left (resp. right) R-modules. The pair ($\mathcal{F}$,$\mathcal{G}$) is said to be almost dual pair if for any left R-module M, $M \in \mathcal{F}$ if and only if $M^+ \in \mathcal{G}$; and $\mathcal{G}$ is closed under direct summands and direct products [4, p. 66].

Throughout this paper, R is an associative ring with identity and all modules are unitary. By J (resp., S_r) we denote the Jacobson radical (resp., the right socle) of R. If X is a subset of R, the right annihilator of X in R is denoted by $r(X)$. Let M and N be R-modules. The character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The symbol $\mathrm{Hom}(M,N)$ (resp., $\mathrm{Ext}^n(M,N)$) means $\mathrm{Hom}_R(M,N)$ (resp., $\mathrm{Ext}_R^n(M,N)$), and similarly $M \otimes N$ (resp., $\mathrm{Tor}_n(M,N)$) means $M \otimes_R N$ (resp., $\mathrm{Tor}_n^R(M,N)$) for an integer $n \geq 1$.

We can find the general background materials, for example in [1, 2, 5].

2. ss-Flat Modules

Definition 2.1. A left R-module M is said to be ss-flat if $\mathrm{Tor}_1(R/(S_r \cap J), M) = 0$.

Examples 2.2.

(1) Any flat module is ss-flat, but the converse is not true. For example the $\mathbb{Z}$-module $\mathbb{Z}_n$ is not flat for all $n \geq 2$ (see [5, Examples (2), p. 155]), but it is clear that $\mathbb{Z}_p$ as $\mathbb{Z}$-module is ss-flat for any prime number p.

(2) Every ss-flat module is min-flat, since if M is an ss-flat left R-module, then M^+ is an ss-injective right R-module (by Lemma 2.3) and hence from [1, Lemma 2.6] we have that M^+ is right mininjective. By [2, Lemma 3.2], M is min-flat.

(3) The Björk Example [6, Example 4.15]. Let F be a field and let $a \mapsto \bar{a}$ be an isomorphism $F \longrightarrow \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. By [1, Example 4.4], R is right mininjective ring but not right ss-injective ring. If $\dim({}_{\bar{F}}F)$ is finite, then R right artinian by [6, Example 4.15]. Therefore, R is a right coherent ring. Thus R^+ is a left min-flat R-module by [2, Theorem 4.5], but the left R-module R^+ is not ss-flat by Theorem 2.10 below.

Lemma 2.3. The following statements are equivalent for a left R-module M:

(1) M is ss-flat.

(2) M^+ is ss-injective.

(3) $\mathrm{Tor}_1(R/A, M) = 0$, for every semisimple small right ideal A of R.

(4) $\mathrm{Tor}_1(R/B, M) = 0$ for every finitely generated semisimple small right ideal B of R.

(5) The sequence $0 \longrightarrow (S_r \cap J) \otimes M \longrightarrow R_R \otimes M$ is exact.

(6) The sequence $0 \longrightarrow A \otimes M \longrightarrow R_R \otimes M$ is exact for every finitely generated semisimple small right ideal A of R.

Proof. (1) $\Leftrightarrow$ (2) This follows from $\mathrm{Ext}^1(R/(S_r \cap J)\ , M^+) \cong \mathrm{Tor}_1(R/(S_r \cap J)\ , M)^+$ (see the dual version of [7, Theorem 3.2.1]).
(2)$\Rightarrow$(3) By the dual version of [7, Theorem 3.2.1] and [1, Proposition 2.7], $\mathrm{Tor}_1(R/A\ ,\ M)^+ \cong \mathrm{Ext}^1(R/A\ , M^+) = 0$ for every semisimple small right ideal A of R.
(3)$\Rightarrow$(1) Clear.
(4)$\Rightarrow$(3) Let I be a semisimple small right ideal of R, so $I = \varinjlim I_i$, where I_i is a finitely generated semisimple small right ideal of R, $f_{ij}: I_i \longrightarrow I_j$ is the inclusion map, and (I_i, f_{ij}) is a direct system (see [7, Example 1.5.5 (2)]). Clearly, $(R/I_i\ , h_{ij})$ is a direct system of R-modules, where $h_{ij}: R/I_i \longrightarrow R/I_j$ is defined by $h_{ij}(a + I_i) = a + I_j$ with direct limit $(h_i\,, \varinjlim R/I_i)$. Since the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & I_i \arrow[r, "i_i"] \arrow[d, "f_{ij}"'] & R \arrow[r, "\pi_i"] \arrow[d, equal] & R/I_i \arrow[r] \arrow[d, "h_{ij}"] & 0 \\
0 \arrow[r] & I_j \arrow[r, "i_j"] & R \arrow[r, "\pi_j"] & R/I_j \arrow[r] & 0
\end{tikzcd}$$

where i_i and π_i are the inclusion and natural maps, respectively, thus the sequence $0 \longrightarrow I \xrightarrow{i} R \xrightarrow{u} \varinjlim R/I_i \longrightarrow 0$ is exact by [3, 24.6]. It follows from [3, 24.4] that the following diagram is commutative:

$$\begin{tikzcd}
R \arrow[r, "\pi_i"] \arrow[d, equal] & R/I_i \arrow[r] \arrow[d, "h_i"] & 0 \\
R \arrow[r, "u"] & \varinjlim R/I_i \arrow[r] & 0
\end{tikzcd}$$

Thus the family of mappings $\left\{g_i: R/I_i \longrightarrow R\big/\varinjlim I_i\ , \text{where } g_i(a + I_i) = a + \varinjlim I_i\right\}$ forms a direct system of homomorphisms, since for $i \le j$, we get $g_j h_{ij}(a + I_i) = g_j\big(a + I_j\big) = a + \varinjlim I_i = g_i(a + I_i)$ for all $a + I_i \in R/I_i$. Thus, there is an R-homomorphism α such that the following diagram is commutative with short exact rows (see [3, 24.1]):

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i"] \arrow[d, equal] & R \arrow[r, "\pi_i"] \arrow[d, equal] \arrow[rr, bend left, "u"] & R/I_i \arrow[r, "h_i"] \arrow[d, equal] & \varinjlim R/I_i \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & I \arrow[r, "i"] & R \arrow[r, "\pi_i"] \arrow[rr, bend right, "\pi"'] & R/I_i \arrow[r, "g_i"] & R\big/\varinjlim I_i \arrow[r] & 0
\end{tikzcd}$$

where π is the natural map, so it follows from [8, Exercise 11 (1), p. 52] that $\varinjlim R/I_i \cong R\big/\varinjlim I_i$. Therefore,

$$\mathrm{Tor}_1(R/I\,, M) = \mathrm{Tor}_1\left(R\big/\varinjlim I_i\,, M\right)$$

$\cong \mathrm{Tor}_1\left(\varinjlim R/I_i\,, M\right)$ (by [9, Theorem XII.5.4 (4)])

$\cong \varinjlim \mathrm{Tor}_1(R/I_i\,, M) = 0$ (by [10, Proposition 7.8]).
(3)$\Rightarrow$(4) Clear.
(1)$\Leftrightarrow$(5) By [9, Theorem XII.5.4 (3)], we have the exact sequence $0 \longrightarrow \mathrm{Tor}_1(R/(S_r \cap J)\,, M) \longrightarrow (S_r \cap J) \otimes M \longrightarrow R_R \otimes M$. Thus the equivalence between (1) and (5) is true.
(4)$\Leftrightarrow$(6) is similar to ((1)$\Leftrightarrow$(5)). ■

In following, we will use the symbol SSI (resp. SSF) to denote the classes of ss-injective right (resp. ss-flat left) R-modules.

Corollary 2.4. The pair (SSF, SSI) is an almost dual pair.

Proof. By Lemma 2.3 and [1, Theorem 2.4]. ■

Lemma 2.5. For a ring R, the following statements hold:

(1) If $S_r \cap J$ is finitely generated, then every pure submodule of ss-injective right R-module is ss-injective.
(2) Every pure submodule of ss-flat left R-module is ss-flat.
(3) Every direct limits (direct sums) of ss-flat left R-modules is ss-flat.
(4) If M, N are left R-modules, $M \cong N$, and M is ss-flat, then N is ss-flat.

Proof. (1) Let M be an ss-injective right R-module and N be a pure submodule of M. Since $R/(S_r \cap J)$ is finitely presented, thus the sequence $\mathrm{Hom}(R/(S_r \cap J)\,, M) \longrightarrow \mathrm{Hom}(R/(S_r \cap J)\,, M/N) \longrightarrow 0$ is exact. By [9, Theorem XII.4.4 (4)], we have the exact sequence
$\mathrm{Hom}(R/(S_r \cap J)\,, M) \longrightarrow$
$\mathrm{Hom}(R/(S_r \cap J)\,, M/N) \longrightarrow$
$\mathrm{Ext}^1(R/(S_r \cap J)\,, N) \longrightarrow$
$\mathrm{Ext}^1(R/(S_r \cap J)\,, M) = 0$ which leads to $\mathrm{Ext}^1(R/(S_r \cap J)\,, N) = 0$. Hence N is an ss-injective right R-module.
(2), (3) and (4) By Corollary 2.4 and [4, Proposition 4.2.8, p. 70]. ■

Recall that a right R-module M is said to be FP-injective (or absolutely pure) if $\mathrm{Ext}^1(N, M) = 0$ for every finitely presented right R-module N (see [11, 12]). A right R-module M is called n-presented, if there is an exact sequence $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ such that each F_i is a finitely generated free right R-module (see [13]). A ring R is called min-coherent, if every simple right ideal of R is finitely presented (see [2]); equivalently, if every finitely generated semisimple small right ideal is finitely presented. In the following definition, we will introduce the concept of ss-coherent ring as a generalization of coherent ring

Definition 2.6. A ring R is said to be right ss-coherent ring, if R is a right min-coherent and $S_r \cap J$ is finitely generated; equivalently, if $S_r \cap J$ is finitely presented.

Example 2.7.
(1) Every coherent ring is ss-coherent.
(2) Every ss-coherent ring is min-coherent.
(3) Let R be a commutative ring, then the polynomial ring $R[x]$ is not coherent ring with zero socle by [2, Remark 4.2 (3)]. Hence $R[x]$ is an ss-coherent ring but not coherent.

Corollary 2.8. A right ideal $S_r \cap J$ of a ring R is finitely generated if and only if every FP-injective right R-module is ss-injective.

Proof. By [11, Proposition, p. 361]. ■

Theorem 2.9. The following statements are equivalent for a ring R:
(1) R is a right min-coherent ring.
(2) If M is an ss-injective right R-module, then M^+ is ss-flat.
(3) If M is an ss-injective right R-module, then M^{++} is ss-injective.
(4) A left R-module N is ss-flat if and only if N^{++} is ss-flat.
(5) SSF is closed under direct products.
(6) ${}_R R^S$ is ss-flat for any index set S.
(7) $\mathrm{Ext}^2(R/I\,, M) = 0$ for every FP-injective right R-module M and every finitely generated semisimple small right ideal I.
(8) If $0 \longrightarrow N \longrightarrow M \longrightarrow H \longrightarrow 0$ is an exact sequence of right R-modules with N is FP-injective and M is ss-injective, then $\mathrm{Ext}^1(R/I\,, H) = 0$ for every finitely generated semisimple small right ideal I.
(9) Every left R-module has an (SSF)-preenvelope.
(10) If $\alpha: M \longrightarrow N$ is an (SSI)-preenvelope of a right R-module M, then $\alpha^+: N^+ \longrightarrow M^+$ is an (SSF)-precover of M^+.
(11) For any positive integer n and any $b_1, \ldots, b_n \in S_r \cap J$, then the right ideal $\{r \in R \mid b_1\, r + b_2\, r_2 + \cdots + b_n\, r\_n = 0 \text{ for some } r_1, \cdots, r_n \in R\}$ is finitely generated.
(12) For any finitely generated semisimple small right ideal A of R and any $x \in S_r \cap J$, then $\{r \in R: xr \in A\}$ is finitely generated.
(13) $r(x)$ is finitely generated for any simple right ideal xR.
(14) Every simple submodule of a projective right R-module is finitely presented.

Proof. (1)⇒(2) Let I be a finitely generated semisimple small right ideal of R, thus there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} I \longrightarrow 0$ in which F_i is a finitely generated free right R-module, $i = 1{,}2$ by hypothesis. Therefore, the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/I \longrightarrow 0$ is exact, where $i: I \longrightarrow R$ and $\pi: R \longrightarrow R/I$ are the inclusion and the natural maps, respectively and $\beta = i\alpha_1$. Thus R/I is 2-presented and hence [13, Lemma 2.7] implies that $\mathrm{Tor}_1(R/I\,, M^+) \cong \mathrm{Ext}^1(R/I\,, M)^+ = 0$. Therefore, M^+ is an ss-flat left R-module.
(2)⇒(3) By (2) and Lemma 2.3.
(3) ⇒ (4) Assume that N is an ss-flat left R-module, thus N^+ is ss-injective by Lemma 2.3 and this implies that N^{+++} is ss-injective by (3). So N^{++} is ss-flat by Lemma 2.3 again. The converse is obtained by [3, 34.6 (1)] and Lemma 2.5 (2).

(4)⇒(5) By (4), $(SSF)^{++} \subseteq SSF$. Since (SSF, SSI) is an almost dual pair (by Corollary 2.4), thus [4, Proposition 4.3.1 and Proposition 4.2.8 (3)] implies that SSF is closed under direct products.

(5)⇒(6) Clear.

(6)⇒(1) By Example 2.2 (2) and [2, Theorem 4.5].

(1) ⇒ (7) Let I be a finitely generated semisimple small right ideal of R and let M be a FP-injective right R-module. By [9, Theorem XII.4.4 (3)], we have the exact sequence $\mathrm{Ext}^1(I,M) \longrightarrow \mathrm{Ext}^2(R/I,M) \longrightarrow \mathrm{Ext}^2(R,M)$. But $\mathrm{Ext}^1(I,M) = 0$ (since M is FP-injective and I is finitely presented) and $\mathrm{Ext}^2(R,M) = 0$ (since R is projective). Thus $\mathrm{Ext}^2(R/I,M) = 0$.

(7)⇒(8) Let $0 \longrightarrow N \longrightarrow M \longrightarrow H \longrightarrow 0$ be an exact sequence of right R-modules, where N is FP-injective and M is ss-injective and let I be a finitely generated semisimple small right ideal of R. By [9, Theorem XII.4.4 (4)], we have an exact sequence $0 = \mathrm{Ext}^1(R/I,M) \longrightarrow \mathrm{Ext}^1(R/I,H) \longrightarrow \mathrm{Ext}^2(R/I,N) = 0$. Thus $\mathrm{Ext}^1(R/I,H) = 0$ for every finitely generated semisimple small right ideal I of R.

(8) ⇒ (1) Let N be a FP-injective right R-module, thus we have the exact sequence $0 \longrightarrow N \longrightarrow E(N) \longrightarrow E(N)/N \longrightarrow 0$. Let I be a finitely generated semisimple small right ideal of R, thus $\mathrm{Ext}^1(R/I,E(N)/N) = 0$ by hypothesis. So it follows from [9, Theorem XII.4.4 (4)] that the sequence $0 = \mathrm{Ext}^1(R/I,E(N)/N) \longrightarrow \mathrm{Ext}^2(R/I,N) \longrightarrow \mathrm{Ext}^2(R/I,E(N)) = 0$ is exact, and so $\mathrm{Ext}^2(R/I,N) = 0$. Hence we have the exact sequence $0 = \mathrm{Ext}^1(R,N) \longrightarrow \mathrm{Ext}^1(I,N) \longrightarrow \mathrm{Ext}^2(R/I,N) = 0$ (see [9, Theorem XII.4.4 (3)]). Thus $\mathrm{Ext}^1(I,N) = 0$ and this implies that I is finitely presented (see [11]). Therefore R is a right min-coherent.

(5)⇔(9) By Corollary 2.4 and [4, Proposition 4.2.8 (3), p. 70].

(2)⇒(10) Since $(SSI)^+ \subseteq SSF$ (by hypothesis) and $(SSF)^+ \subseteq SSI$ (by Lemma 2.3), thus the result follows from [14, 3.2, p. 1137].

(10)⇒(2) By taking M is an ss-injective right R-module in (10).

(1) ⇒ (11) Let $b_1, b_2, \dots, b_n \in S_r \cap J$. Put $K_1 = b_1R + b_2R + \cdots + b_nR$ and $K_2 = b_2R + \cdots + b_nR$. Thus $K_1 = b_1R + K_2$. Define $f: R \longrightarrow K_1/K_2$ by $f(r) = b_1r + K_2$ which is a well-define R -epimorphism, because if $r_1 = r_2 \in R$, then $b_1r_1 - b_1r_2 = 0 \in K_2$, that is $b_1r_1 + K_2 = b_1r_2 + K_2$. Now we have $\ker(f) = \{r \in R | b_1r + K_2 = K_2\} = \{r \in R| \, b_1r \in K_2\} = \{r \in R| \, b_1\, r + b_2\, r_2 + \cdots + b_n\, r_n = 0 \text{ for some } r_2,\cdots,r_n \in R\}$. By (1) and using [15, Lemma 4.54 (1)], we have that K_1/K_2 is finitely presented. But $R/\ker(f) \cong K_1/K_2$, so $\ker(f)$ is finitely generated.

(11)⇒(12) Let $x \in S_r \cap J$ and A be any finitely generated semisimple small right ideal of R, then $A = \bigoplus_{i=1}^{n} a_iR$, so we have that $\{r \in R | xr \in A\} = \{r \in R | x\, r + a_1\, r_1 + \cdots + a_n\, r_n = 0 \text{ for some } r_1,\cdots,r_n \in R\}$ if finitely generated by hypothesis.

(12)⇒(13) By taking $A = 0$.

(13)⇒(1) Let xR be a simple right ideal. Since $r(x)$ is finitely generated and $xR \cong R/r(x)$, thus xR is finitely presented.

(1)⇒(14) Let $S_r = \bigoplus_{i \in I} a_iR$, where a_iR is a simple right ideal for each $i \in I$. If P is a projective right R-module, then P is isomorphic to a direct summand of $R^{(S)}$ for some index set S. Let A be any simple submodule of P, then $A \cong B \leq^{\oplus} \bigoplus_S S_r = \bigoplus_S \bigoplus_{i \in I} a_iR$. Since A is finitely generated, then there are finite index sets $S_0 \subseteq S$ and $I_0 \subseteq I$ such that $A \cong B \leq^{\oplus} = \bigoplus_{S_0} \bigoplus_{i \in I_0} a_iR$, so it follows from [15, Lemma 4.54 (3)] that A is finitely presented.

(14)⇒(1) Clear. ■

Recall that a subclass $\mathcal{F}$ of Mod-R is said to be definable if it is closed under direct products, direct limits and pure submodules (see [4, Definition 2.4.1, p. 29]).

Theorem 2.10. The following statements are equivalent for a ring R:

(1) R is a right ss-coherent ring.

(2) A right R-module M is ss-injective if and only if M^+ is ss-flat.

(3) A right R-module M is ss-injective if and only if M^{++} is ss-injective.

(4) SSI is closed under direct limits.

(5) $S_r \cap J$ is finitely generated and every pure quotient of ss-injective right R-module is ss-injective.

(6) The following two conditions hold:
  (a) Every right R-module has an (SSI)-cover.
  (b) Every pure quotient of ss-injective right R-module is ss-injective.

Proof. (1)⇒(2) Let M^+ be ss-flat. Then M^{++} is ss-injective by Lemma 2.3, so it follows from [3, 34.6 (1)] and Lemma 2.5 (1) that M is ss-injective. The converse is obtained by Theorem 2.9.

(2)⇒(3) Let M^{++} be ss-injective, thus M^+ is ss-flat by Lemma 2.3 and hence M is ss-injective by hypothesis. The converse is true by Theorem 2.9.

(3) ⇒ (1) Let M be an FP-injective right R-module, then the exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$ is pure by [16, Proposition 2.6 (c)], so it follows from [3, 34.5] that the sequence $0 \longrightarrow M^{++} \longrightarrow E(M)^{++} \longrightarrow (E(M)/M)^{++} \longrightarrow 0$ is split. Since $E(M)^{++}$ is ss-injective by hypothesis, thus M^{++} is ss-injective and hence M is ss-injective by hypothesis again. Therefore, $S_r \cap J$ is finitely generated by Corollary 2.8, and so $S_r \cap J$ is finitely presented by Theorem 2.9. Thus R is a right ss-coherent ring.

(1)⇒(4) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a direct system of ss-injective right R-modules. Since $S_r \cap J$ is finitely presented, then $R/S_r \cap J$ is 2-presented, so it follows from [13, Lemma 2.9 (2)] that $\mathrm{Ext}^1\left(R/(S_r \cap J), \varinjlim M_\lambda\right) \cong \varinjlim \mathrm{Ext}^1(R/(S_r \cap J), M_\lambda) = 0$. Hence $\varinjlim M_\lambda$ is ss-injective.

(4)⇒(2) Let $\{E_i : i \in I\}$ be a family of injective right R-modules. Since $\bigoplus_{i\in I} E_i = \varinjlim \{\bigoplus_{i\in I_0} E_i : I_0 \subseteq I, I_0 \text{ finite}\}$ (see [3, p. 206]), then $\bigoplus_{i\in I} E_i$ is ss-injective and hence $S_r \cap J$ is finitely generated by [1, Corollary 2.25]. By Lemma 2.5, SSI is closed under pure submodules. Since SSI is closed under direct products (by [1, Theorem 2.4]) and since SSI is closed under direct limits (by hypothesis), thus SSI is a definable class. By [4, Proposition 4.3.8, p. 89], (SSI, SSF) is an almost dual pair and hence a right R-module M is ss-injective if and only if M^+ is ss-flat

(2)⇒(5) By the equivalence between (1) and (2), we have that $S_r \cap J$ is finitely generated. Now, let $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ be a pure exact sequence of right R-modules with M is ss-injective, so it follows from [3, 34.5] that the sequence $0 \longrightarrow (M/N)^+ \longrightarrow M^+ \longrightarrow N^+ \longrightarrow 0$ is split. By hypothesis, M^+ is ss-flat, so $(M/N)^+$ is ss-flat. Thus M/N is ss-injective by hypothesis again.

(5)⇒(4) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a direct system of ss-injective right R-modules. By [3, 33.9 (2)], there is a pure exact sequence $\bigoplus_{\lambda\in\Lambda} M_\lambda \longrightarrow \varinjlim M_\lambda \longrightarrow 0$. Since $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is ss-injective by [1, Corollary 2.25], thus $\varinjlim M_\lambda$ is ss-injective by hypothesis.

(5) ⇔ (6) By [1, Corollary 2.25] and [17, Theorem 2.5]. ∎

Corollary 2.11. A ring R is ss-coherent if and only if it is min-coherent and the class SSI is closed under pure submodules.

Proof. (⇒) Suppose that R is ss-coherent ring, thus R is min-coherent and $S_r \cap J$ is a finitely generated right ideal of R. By Lemma 2.5 (1), SSI is closed under pure submodules.

(⇐) Let M be any ss-injective right R-module. Since R is min-coherent, thus Theorem 2.9 implies that M^+ is ss-flat. Conversely, let M be any right R-module with such that M^+ is ss-flat. By Lemma 2.3, M^{++} is ss-injective. Since M is a pure submodule of M^{++} (by [3, 34.6 (1)]) and since SSI is closed under pure submodule (by hypothesis) it follows that M is ss-injective. Hence for any right R-module M, we have that M is ss-injective if and only if M^+ is ss-flat. Thus Theorem 2.10 implies that R is ss-coherent. ∎

Corollary 2.12. The following statements are equivalent for a right min-coherent ring R:

(1) Every ss-flat left R-module is flat.

(2) Every ss-injective right R-module is FP-injective.

(3) Every ss-injective pure injective right R-module is injective.

Proof. (1) ⇒ (2) For any ss-injective right R-module M, then M^+ is ss-flat by Theorem 2.9, and so M^+ is flat by hypothesis. Thus M^{++} is injective by [10, Proposition 3.54]. Since M is pure submodule of M^{++}, then M is FP-injective by [20, 35.8].

(2)⇒(3) By [16, Proposition 2.6 (c)] and [3, 33.7].

(3) ⇒ (1) Assume that N is an ss-flat left R-module, thus N^+ is ss-injective pure injective by Lemma 2.3 and [3, 34.6 (2)]. Thus N^+ is injective, and so N is flat by [10, Proposition 3.54]. ■

Proposition 2.13. The following statements are equivalent for a right ss-coherent ring R:

(1) R is a right ss-injective ring.
(2) Every left R-module has a monic ss-flat preenvelope.
(3) Every right R-module has epic ss-injective cover.
(4) Every injective left R-module is ss-flat.
(5) Every flat right R-module is ss-injective.

Proof. (1)⇒(2) Let N be a left R-module, then there is an R-epimorphism $\alpha: R_R^{(S)} \longrightarrow N^+$ for some index set S by [10, Theorem 2.35], and so there is an R-monomorphism $g: N \longrightarrow (R_R^+)^S$ by applying [9, Proposition XI.2.3], [3, 11.10 (2) (ii)] and [3, 34.6 (1)], respectively. On the other hand, N has ss-flat preenvelope $f: N \longrightarrow F$ by Theorem 2.9. Since $(R_R^+)^S$ is ss-flat by Theorem 2.9 again, thus there is an R-homomorphism $h: F \longrightarrow (R_R^+)^S$ such that $hf = g$, so this implies that f is an R-monomorphism.
(2)⇒(4) Let N be an injective left R-module, then there is an R-monomorphism $f: N \longrightarrow F$ with F is ss-flat. But $N \cong f(N) \subseteq^{\oplus} F$, so we have that N is ss-flat by Lemma 2.5 (4).
(4)⇒(5) Let M be a flat right R-module, then M^+ is injective and hence ss-flat. Thus M is ss-injective by Theorem 2.10.
(5)⇒(1) Obvious, since R_R is flat.
(1)⇒(3) Let M be any right R-module, then M has ss-injective cover, say, $g: N \longrightarrow M$ by Theorem 2.10. By [10, Theorem 2.35], there is an R-epimorphism $f: R_R^{(S)} \longrightarrow M$ for some index set S. Since $R_R^{(S)}$ is ss-injective by [1, Corollary 2.25], then there is an R-homomorphism $h: R_R^{(S)} \longrightarrow N$ such that $gh = f$, so g is an R-epimorphism.
(3)⇒(1) Let $f: N \longrightarrow R_R$ be an epic ss-injective cover. Since R_R is projective, then there is an R-homomorphism $g: R_R \longrightarrow N$ such that $fg = I_R$, thus f is split, and so $N = \ker(f) \oplus B$ for some ss-injective submodule B of N. Therefore, $R_R \cong N/\ker(f) \cong B$ is ss-injective. ■

Proposition 2.14. The class SSI is closed under cokernels of homomorphisms if and only if $\operatorname{coker}(\alpha)$ is ss-injective for every ss-injective right R-module M and $\alpha \in \operatorname{End}(M)$.

Proof. (⇒) Clear.
(⇐) Let A and B be any ss-injective right R-modules and f be any R-homomorphism from A to B. Define $\alpha: A \oplus B \longrightarrow A \oplus B$ by $\alpha\big((x,y)\big) = \big(0, f(x)\big)$. Thus, we have that $(A \oplus B)/\operatorname{im}(\alpha) \cong (A \oplus B)/(0 \oplus \operatorname{im}(f)) \cong A \oplus (B/\operatorname{im}(f))$ is ss-injective. Thus $B/\operatorname{im}(f)$ is ss-injective. ■

Proposition 2.15. The class SSF is closed under kernels of homomorphisms if and only if $\ker(\alpha)$ is ss-flat, for every ss-flat left R-module M and $\alpha \in \operatorname{End}(M)$.

Proof. (⇒) Clear.
(⇐) Let $g: N \longrightarrow M$ be any R-homomorphism with N and M are ss-flat left R-modules. Define $\alpha: N \oplus M \longrightarrow N \oplus M$ by $\alpha\big((a,b)\big) = \big(0, g(b)\big)$. Thus $\ker(\alpha) = \ker(g) \oplus M$ is ss-flat by hypothesis and hence $\ker(g)$ is ss-flat. ■

Theorem 2.16. If R is a commutative ring, then the following statements are equivalent:

(1) R is a min-coherent ring.
(2) $\operatorname{Hom}(M,N)$ is ss-flat for all ss-injective R-modules M and all injective R-modules N.
(3) $\operatorname{Hom}(M,N)$ is ss-flat for all injective R-modules M and N.
(4) $\operatorname{Hom}(M,N)$ is ss-flat for all projective R-modules M and N.
(5) $\operatorname{Hom}(M,N)$ is ss-flat for all projective R-modules M and all ss-flat R-modules N.

Proof. (1)⇒(2) If I is a finitely generated semisimple small ideal of R, then I is finitely presented. By [9, Theorem XII.4.4 (3)], we have the exact sequence $0 \longrightarrow \operatorname{Hom}(R/I, M) \longrightarrow \operatorname{Hom}(R,M) \longrightarrow \operatorname{Hom}(I,M) \longrightarrow 0$. Thus the sequence $0 \longrightarrow \operatorname{Hom}(\operatorname{Hom}(I,M),N) \longrightarrow \operatorname{Hom}(\operatorname{Hom}(R,M),N) \longrightarrow \operatorname{Hom}(\operatorname{Hom}(R/I,M),N) \longrightarrow 0$ is exact by [9, Theorem XII.4.4 (3)] again. So we have the exact sequence $0 \longrightarrow \operatorname{Hom}(M,N) \otimes I \longrightarrow \operatorname{Hom}(M,N) \otimes R \longrightarrow \operatorname{Hom}(M,N) \otimes (R/I) \longrightarrow 0$ by [7, Theorem 3.2.11] and this implies that $\operatorname{Hom}(M,N)$ is ss-flat.
(2)⇒(3) Clear.

(3) ⇒ (1) By [5, Proposition 2.3.4] and [10,Theorem 2.75], we have that $(R^{++})^S \cong (\mathrm{Hom}(R^+ \otimes R, \mathbb{Q}/\mathbb{Z}))^S \cong (\mathrm{Hom}(R^+, R^+))^S$ for any index set S. Thus $(R^{++})^S \cong \mathrm{Hom}(R^+, (R^+)^S)$ is ss-flat for any index set S by [3, 11.10 (2)] and since R^+ and $(R^+)^S$ are injective. Since R^S is a pure submodule of $(R^{++})^S$ by [3, 34.6 (1)] and [18, Lemma 1 (2)], so it follows from Lemma 2.5 (2) that R^S is ss-flat for any index set S. Thus (1) follows from Theorem 2.9.

(1)⇒(5) Since M is a projective R-module, thus there is a projective R-module P such that $M \oplus P \cong R^{(S)}$ for some index set S. Therefore, $\mathrm{Hom}(M, N) \oplus \mathrm{Hom}(P, N) \cong \mathrm{Hom}(R^{(S)}, N) \cong (\mathrm{Hom}(R, N))^S \cong N^S$ by [3, 11.10 and 11.11]. But N^S is ss-flat by Theorem 2.9, thus $\mathrm{Hom}(M, N)$ is ss-flat.

(5)⇒(4) Clear.

(4)⇒(1) For any index set S, by [3, 11.10 and 11.11], we have that $R^S \cong \mathrm{Hom}(R^{(S)}, R)$. Thus R^S is ss-flat by (4), so it follows from Theorem 2.9 that (1) holds. ■

Corollary 2.17. The following statements are equivalent for a commutative ss-coherent ring R:

(1) M is an ss-injective R-module.

(2) $\mathrm{Hom}(M, N)$ is ss-flat for any injective R-module N.

(3) $M \otimes N$ is ss-injective for any flat R-module N.

Proof. (1)⇒(2) By Theorem 2.16.

(2)⇒(3) By [10, Theorem 2.75], we have that $(M \otimes N)^+ \cong \mathrm{Hom}(M, N^+)$ for any R-module N. If N is flat, then N^+ is injective by [10, Proposition 3.54], so $(M \otimes N)^+$ is ss-flat by hypothesis. Therefore, $M \otimes N$ is ss-injective by Theorem 2.10.

(3)⇒(1) This follows from [5, Proposition 2.3.4], since R is flat. ■

Corollary 2.18. Let R be a commutative ss-coherent ring and SSF is closed under kernels of homomorphisms. Then the following conditions hold for any R-module N:

(1) $\mathrm{Hom}(M, N)$ is ss-flat for any ss-injective R-module M.

(2) $\mathrm{Hom}(N, M)$ is ss-flat for any ss-flat R-module M.

(3) $M \otimes N$ is ss-injective for any ss-injective R-module M.

Proof. (1) Let M be an ss-injective R-module. It is clear that the exact sequence $0 \longrightarrow N \longrightarrow E_0 \longrightarrow E_1$ induces the exact sequence $0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, E_0) \longrightarrow \mathrm{Hom}(M, E_1)$ where E_0 and E_1 are injective R-modules. By Theorem 2.16, we have that $\mathrm{Hom}(M, E_0)$ and $\mathrm{Hom}(M, E_1)$ are ss-flat, thus $\mathrm{Hom}(M, N)$ is ss-flat by hypothesis.

(2) Let M be an ss-flat R-module, so we have the exact sequence $0 \longrightarrow \mathrm{Hom}(N, M) \longrightarrow \mathrm{Hom}(F_0, M) \longrightarrow \mathrm{Hom}(F_1, M)$ where F_0 and F_1 are free R-modules. By Theorem 2.16, the modules $\mathrm{Hom}(F_0, M)$ and $\mathrm{Hom}(F_1, M)$ are ss-flat. Therefore $\mathrm{Hom}(N, M)$ is ss-flat by hypothesis.

(3) Let M be any ss-injective R-module, then $(M \otimes N)^+ \cong \mathrm{Hom}(M, N^+)$ is ss-flat by [10, Theorem 2.75] and applying (1), and hence $M \otimes N$ is ss-injective by Theorem 2.10. ■

Theorem 2.19. Let R be a commutative ss-coherent ring, then the following conditions are equivalent:

(1) R is an ss-injective ring.

(2) $\mathrm{Hom}(M, N)$ is ss-injective for any projective R-module M and any flat R-module N.

(3) $\mathrm{Hom}(M, N)$ is ss-injective for any projective R-modules M and N.

(4) $\mathrm{Hom}(M, N)$ is ss-injective for any injective R-modules M and N.

(5) $\mathrm{Hom}(M, N)$ is ss-flat for any flat R-module M and any injective R-module N.

(6) $M \otimes N$ is ss-flat for any flat R-module M and any injective R-module N.

Proof. (1)⇒(2) Since R is ss-injective, thus every flat R-module is ss-injective by Proposition 2.13. Let M be a projective R-module, then $M \oplus P \cong R^{(S)}$ for some projective R-module P and for some index set S. Thus for all flat R-module N, we have $\mathrm{Hom}(M, N) \oplus \mathrm{Hom}(P, N) \cong \mathrm{Hom}(R^{(S)}, N) \cong N^S$ by [3, 11.10 and 11.11]. Since N^S is ss-injective, thus $\mathrm{Hom}(M, N)$ is ss-injective.

(2)⇒(3) Clear.

(3)⇒(1) Since $R \cong \mathrm{Hom}(R, R)$ by [3, 11.11], thus R is ss-injective ring.

(1)⇒(4) By the dual version of [7, Theorem 3.2.1], $\mathrm{Ext}^1\big(R/(S_r\cap J)\ ,\mathrm{Hom}(M,N)\big) \cong \mathrm{Hom}(\mathrm{Tor}_1(R/(S_r\cap J)\ ,M),N)$ for all injective R-modules M and N. By Proposition 2.13, M is ss-flat. Thus $\mathrm{Tor}_1(R/(S_r\cap J)\ ,M)=0$ and hence $\mathrm{Hom}(M,N)$ is ss-injective.

(4)⇒(1) To prove R is an ss-injective ring, we need to prove that every injective R-module is ss-flat (see Proposition 2.13). Now, let M be any injective R-module, then $\mathrm{Hom}(M,R^+)$ is ss-injective, so $0=\mathrm{Ext}^1\big(R/(S_r\cap J)\ ,\mathrm{Hom}(M,R^+)\big)\cong \mathrm{Hom}(\mathrm{Tor}_1(R/(S_r\cap J),M),R^+)\cong (\mathrm{Tor}_1(R/(S_r\cap J),M)\otimes R)^+ \cong \mathrm{Tor}_1(R/(S_r\cap J),M)^+$ by applying the dual version of [7, Theorem 3.2.1], [10, Theorem 2.75] and [5, Proposition 2.3.4]. Therefore, $\mathrm{Tor}_1(R/(S_r\cap J),M)=0$, since $\mathbb{Q}/\mathbb{Z}$ is injective cogenerator. Thus M is ss-flat.

(5)⇒(1) and (6)⇒(1) By taking $M=R$ and using [3, 11.11] and [5, Proposition 2.3.4].

(1)⇒(5) Let M be a flat R-module and N be an injective R-module,then $\mathrm{Hom}(M,N)$ is injective. Therefore $\mathrm{Hom}(M,N)$ is ss-flat by Proposition 2.13.

(1)⇒(6) Let M be a flat R-module and let N be an injective R-module. Then N is ss-flat by Proposition 2.13, so the sequence $0\longrightarrow N\otimes(S_r\cap J)\longrightarrow N$ is exact. Since M is flat, then the sequence $0\longrightarrow M\otimes N\otimes(S_r\cap J)\longrightarrow M\otimes N$ is exact and this implies that $M\otimes N$ is ss-flat. ■

Proposition 2.20. Let R be a commutative ring, then the following statements are equivalent:

(1) M is ss-flat.

(2) $\mathrm{Hom}(M,N)$ is ss-injective for all injective R-module N.

(3) $M\otimes N$ is ss-flat for all flat R-module N.

Proof. (1)⇒(2) Let N be any injective R-module. Since $\mathrm{Ext}^1\big(R/(S_r\cap J)\ ,\mathrm{Hom}(M,N)\big) \cong \mathrm{Hom}(\mathrm{Tor}_1(R/(S_r\cap J)\ ,M),N)=0$ by the dual version of [7, Theorem 3.2.1], then $\mathrm{Hom}(M,N)$ is ss-injective.

(2)⇒(3) Let N be a flat R-module, then N^+ is injective by [10, Proposition 3.54]. So it follows from [10, Theorem 2.75] that $(M\otimes N)^+\cong \mathrm{Hom}(M,N^+)$ is ss-injective. Thus $M\otimes N$ is ss-flat by Lemma 2.3.

(3)⇒(1) Follows from [5, Proposition 2.3.4]. ■

Proposition 2.21. Let R be a commutative ring and M be a semisimple R-module. If M is ss-flat, then $\mathrm{End}(M)$ is ss-injective as R-module.

Proof. By [5, p. 157], there is a group epimorphism $\varphi:(S_r\cap J)\otimes M\longrightarrow (S_r\cap J)M$ given by $a\otimes x\longmapsto ax$ for each generator $a\otimes x\in(S_r\cap J)\otimes M$. Thus we have the commutative diagram:

$$\begin{array}{ccccc} 0\longrightarrow & (S_r\cap J)\otimes M & \xrightarrow{i_1\otimes I_M} & R\otimes M \\ & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f} \\ 0\longrightarrow & (S_r\cap J)M & \xrightarrow{i_2} & M \end{array}$$

where I_M is the identity map, i_1 and i_2 are the inclusion maps, and f is an isomorphism defined by [5, Proposition 2.3.4]. Since $f\circ(i_1\otimes I_M)$ is $\mathbb{Z}$-monomorphism, then φ is isomorphism. Therefore $(S_r\cap J)\otimes M\cong (S_r\cap J)M\subseteq J(M)=0$ by [19, Theorem 9.2.1]. So it follows from [10, Theorem 2.75] that $0=\mathrm{Hom}((S_r\cap J)\otimes M,M)\cong \mathrm{Hom}\big(S_r\cap J,\mathrm{End}(M)\big)$. But the sequence $0=\mathrm{Hom}\big(S_r\cap J,\mathrm{End}(M)\big)\longrightarrow \mathrm{Ext}^1\big(R/(S_r\cap J),\mathrm{End(M)}\big)\longrightarrow \mathrm{Ext}^1\big(R,End(M)\big)=0$ is exact by [9, Theorem XII.4.4 (3)]. Thus $\mathrm{Ext}^1\big(R/(S_r\cap J),\mathrm{End}(M)\big)=0$ and hence $\mathrm{End}(M)$ is an ss-injective as R-module. ■

Proposition 2.22. Let R be a commutative ring and M be a simple R-module. Then M is ss-flat if and only if M is ss-injective.

Proof. (⇒) Let $M=mR$ be a simple R-module. Define $f:\mathrm{Hom}(mR,mR)\longrightarrow mR$ by $f(\alpha)=\alpha(m)$. We assert that f is a well define R-homomorphism. Let $\alpha_1=\alpha_2$, then $\alpha_1(m)=\alpha_2(m)$, so $f(\alpha_1)=f(\alpha_2)$. Now, let $\alpha_1,\alpha_2\in\mathrm{End}(M)$ and $r_1,r_2\in R$, then $f(r_1\alpha_1+r_2\alpha_2)=(r_1\alpha_1+r_2\alpha_2)(m)=(r_1\alpha_1)(m)+(r_2\alpha_2)(m)=r_1\alpha_1(m)+r_2\alpha_2(m)=r_1f(\alpha_1)+r_2f(\alpha_2)$ proving the assertion. Since $f\big(\mathrm{End}(M)\big)=M$ and $\ker(f)=\{\alpha\in\mathrm{End}(M):f(\alpha)=0\}=\{\alpha\in\mathrm{End}(M):\alpha(m)=0\}=\{\alpha\in\mathrm{End}(M):0\neq m\in\ker(\alpha)\}=0$, then $\mathrm{End}(M)\cong M$ and hence M is ss-injective by Proposition 2.21.

(⇐) Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of all simple R-modules and $E=E(\bigoplus_{\lambda\in\Lambda}S_\lambda)$. Then $\mathrm{Hom}(M,E)\cong M$ by the proof of [12, Lemma 2.6], so it follows from the dual version of [7, Theorem 3.2.1] that $\mathrm{Ext}^1(R/(S_r\cap J),M)=\mathrm{Hom}(\mathrm{Tor}_1(R/(S_r\cap J),M),E)$. Since M is ss-injective,then $\mathrm{Hom}(\mathrm{Tor}_1(R/(S_r \cap J), M), E) = 0$. But E is injective cogenerator by [8, Corollary 18.19], thus $\mathrm{Tor}_1(R/(S_r \cap J), M) = 0$ (see [7, definition 3.2.7]) and hence M is ss-flat. ■

Recall that a ring R is called PS-ring (resp., FS-ring) if S_r is projective (resp., flat) (see [20]); equivalently, if $S_r \cap J$ is projective (resp., flat). The following corollary extends a result of [20, Proposition 8 (1)] that a commutative FS-ring is PS-ring.

Corollary 2.23. The following statements are equivalent for a commutative ring R:

(1) R is a universally mininjective.
(2) R is a PS-ring.
(3) R is an FS-ring.
(4) S_r is ss-flat.

Proof. By [1, Corollary 1.19] and Proposition 2.22. ■

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المقاسات المسطحة من النمط -SS

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المستخلص :

في هذا البحث، تم تقديم ودراسة المقاسات المصطحة من النمط -ss كمفهوم رديف للمقاسات الاغمارية من النمط -ss. الحلقات المتماسكة من النمط - min، الحلقات من النمط -FS، الحلقات من النمط -PS، والحلقات الاغمارية كليا من النمط -min قد شخصت باستخدام المقاسات المسطحة من النمط -ss والمقاسات الاغمارية من النمط -ss.