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On second-order differential subordination and superordination of analytic functions involving the Komatu integral operator

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Abstract
In the present paper by using properties of the Komatu integral operator, we derive some properties of subordinations and superordinations associated with the Hadamard product concept.

Key words: Differential subordination, Differential superordination, Univalent function, Convex function, Komatu integral operator, Hadamard product.

Mathematics Subject Classification: 30C45, 30A10, 30C80.

1. Introduction and Definitions

Let $U = \{z \in C : |z| < 1\}$ be an open unit disc in C (complex plane) and $\overline{U} = \{z \in C : |z| \leq 1\}$.

Let H(U) be the class of analytic functions in U and let H[a, k] be the subclass of H(U) of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

where $a \in C$ and $n \in N$ with $H_0 \equiv H[0, 1]$ and $H \equiv H[1, 1]$.Let A_p be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , \quad (z \in U) \qquad (1.1)$$

in the open unit disk U. For functions $f \in A_p$ given by equation (1.1) and $g \in A_p$ defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n , \quad (z \in U)$$

The Hadamard product(convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z) .$$

Let f and F be members of H(U).The function f is said to be subordinate to a function F or F is said to be superordinate to f, if there exists a Schwarz function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = F(w(z))$.

We denote this subordination by

$$f(z) \prec F(z) \text{ or } f \prec F.$$

Furthermore, if the function F is univalent in U, then we have the following equivalence [6, 12]

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

The method of differential subordinations (also known as the admissible functions method) was first introduced by Miller and Mocanu in 1978 [3] and the theory started to develop in 1981 [4]. For more details see [5].

Let Ω and Δ be sets in C, let ψ : $C^3 \times U \to C$ and h be univalent in U. If p is analytic in U with p(0) = a with generalizations of implication

$$\{\psi(p(z), zp'(z), zp''(z); z)\} \subset \Omega \Longrightarrow p(U) \subset \Delta,$$

with satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), zp''(z); z) \prec h(z), \quad (1.2)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination or more simply dominant, if p ≺ q for all p satisfying (1.2). A dominant $\tilde{q}$ satisfying $\tilde{q} \prec q$ for all dominants (1.2) is said to be the best dominant of (1.2). And if p and Ψ(p(z), zp' (z), zp"(z); z) are univalent in U with satisfies the second-order differential subordination

$$h(z) \prec \psi(p(z), zp'(z), zp''(z); z), \quad (1.3)$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination or more simply dominant, if q ≺ p for all p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies q ≺ $\tilde{q}$ for all subordinants q of (1.3) is said to be the best subordinant.

by using (1.3) we get

$$\Omega \subset \{\psi(p(z), zp'(z), zp''(z); z)\}.$$

For functions f and g ∈ A(p), The Komatu integral operator $D_{m,p}^{\lambda}$:A(p) → A(p) ($\lambda \geq 0, m \in N \cup \{0\}$ and $N = \{1, 2, 3, \dots\}$) defined as follows:[10].

$$D_{m,p}^{\lambda}(f * g)(z) = \frac{m^{\lambda}}{\Gamma(\lambda)} \int_0^1 t^{m-2} \left(\log\frac{1}{t}\right)^{\lambda-1} (f * g)(z)\ (tz)dt, \quad (1.4)$$

where the symbol Γ stands for the gamma function.

Thus, we get

$$D_{m,p}^{\lambda}(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{m}{m+n-1}\right)^{\lambda} a_n b_n z^n. \quad (1.5)$$

For λ, α ≥ 0, we obtain

$$D_{m,p}^{\lambda}(D_m^{\alpha}(f * g)(z)) = D_{m,p}^{\lambda+\alpha}(f * g)(z) .$$

From (1.5) we have

$$\frac{z}{p}\left(D_{m,p}^{\lambda}(f * g)(z)\right)' = mD_{m,p}^{\lambda+1}(f * g)(z) - (m-1)D_{m,p}^{\lambda}(f * g)(z). \quad (1.6)$$

The operator $D_m^{\lambda}(f * g)(z)$ is related to the transformation of the multiplier studied by Flett [8] Several interesting proposals were examined by the operator D_m^{λ} have been studied by Jung et al. [9] and Liu [11].

In order to prove our main results , we need the following definitions and lemmas.

Definition 1.1. ([13]) We denote by Q the set of functions q that are analytic and injective on $\overline{U}/E(q)$, where

$$E(q) = \{x \in \partial U ;\ \lim_{z \to x} q(z) = \infty\},$$

and are such that $q'(x) \neq 0$ for $x \in \partial U / E(q)$. The subclass of Q for which $q(0) = a$ is denoted by Q(a).

Definition 1.2. ([13]) Let Ω be a set in C, q(z) ∈ Q and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions ψ: $C^3 \times U \to C$ that satisfy the admissibility condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever r = q(x), s = yxq(x),

$$\Re e\left\{1 + \frac{t}{s}\right\} \geq y \Re e\left\{1 + \frac{xq''(x)}{q'(x)}\right\},$$

where z ∈ U, $x \in \partial U / E(q)$ and y ≥ n.

we get $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

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In particular, when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in Q$.
In this case, we set $\Psi_n[\Omega, M, a] = \Psi[\Omega, q]$, and in the special case when $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.3. ([14]) Let Ω be a set in C and $q \in H[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'[\Omega, q]$ consist of this functions $\psi : C^3 \times U \to C$ that satisfy the admissibility condition

$$\psi(r, s, t; x) \in \Omega,$$

whenever $r = q(z)$, $s = \frac{zq'(z)}{j}$ for $z \in U$ and

$$\Re e\left\{1 + \frac{t}{s}\right\} \leq \frac{1}{j}\Re e\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

for $z \in U$, $x \epsilon \partial U$ and $j \geq n \geq 1$. We write $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Lemma 1.4. ([13]) Let $\psi \in \Psi_n[\Omega, q]$ wtih $q(0) = a$. If the analytic function

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,$$

$(z \in U)$ satisfies the following inclusion relationship $\psi(p(z), zp'(z), zp''(z);\ z) \in \Omega$,

then

$$p(z) \prec q(z) \quad (z \in U).$$

Lemma 1.5. ([14]) Let $\psi \in \Psi_n[\Omega, q]$ wtih $q(0) = a$. If $p \in Q(a)$ and $\psi(p(z), zp'(z), zp''(z);\ z)$ is univalent in U, then

$$\Omega \subset \psi(p(z), zp'(z), zp''(z);\ z),$$

implies

$$q(z) \prec p(z).$$

In the present work, we get some results of differential subordination and superordination of Oros [15], [16] , we shall the study of the class of admissible functions involving the Komatu integral operator $D^{\lambda}_{m,p}(f * g)(z)$ de fined by (1.5) . We remark in passing that some interesting developments on differential subordination and superordination for various operators in connection with the Komatu integral operator were obtained by Ali *et al.* [1],[2] and Cho *et al.* [7].

2. Differential subordination results

Definition 2.1. Let Ω be a set in C,$q \in Q_0 \cap H[0,p]$. The class of admissible functions $\Phi_n[\Omega, q]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition

$$\phi(u, v, w; z; x) \notin \Omega,$$

Whenever

$$u = q(x), \quad v = \frac{yxq'(x) + p(m-1)q(x)}{pm}$$

and

$$\Re e\left\{\frac{m^2p^2w - p^2(m-1)^2u}{pcv - p(m-1)u} - 2p(m-1)\right\} \geq y\Re e\left\{1 + \frac{xq''(x)}{q'(x)}\right\},$$

for $z \in U$, $x \epsilon \partial U/E(q)$, $\lambda \geq 1$ and $y \geq p$.

Theorem 2.2. Let $\phi \in \Phi_n[\Omega, q]$. If $f \in A(p)$ satisfies

$$\left\{\phi\left(\begin{matrix} D^{\lambda}_{m,p}(f * g)(z), D^{\lambda+1}_{m,p}(f * g)(z), \\ D^{\lambda+2}_{m,p}(f * g)(z); z \end{matrix}\right); z \in U\right\} \subset \Omega, \qquad (2.1)$$

then

$$D^{\lambda}_{m,p}(f * g)(z) \prec q(z).$$

Proof. Let $g(z) \in U$ define by

$$g(z) = D^{\lambda}_{m,p}(f * g)(z).$$
(2.2)

In view of relation (1.6) with from (2.2), we have

$$D^{\lambda+1}_{m,p}(f * g)(z) = \frac{zg'(z) + p(m-1)g(z)}{pm}. \qquad (2.3)$$

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Based on that

$$D_{m,p}^{\lambda+2}(f * g)(z) = \frac{z^2 g''(z)+2p(m-1)zg'(z)+p^2(m-1)^2 g(z)}{p^2m^2}. \quad (2.4)$$

Define the transformation from C^3 to C by

$$u(r,s,t) = r, v(r,s,t) = \frac{s+p(m-1)r}{pm}, w(r,s,t) = \frac{t+2p(m-1)s+p^2(m-1)^2 r}{p^2m^2}. \quad (2.5)$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z)$$

$$= \phi\left(r, \frac{s+p(m-1)r}{pm}, \frac{t+2p(m-1)s+p^2(m-1)^2 r}{p^2m^2}; z\right). \quad (2.6)$$

The proof shall get use of Lemma 1.4 .Using equations (2.2), (2.3) amd (2.4), from (2,6), we have

$$\psi(g(z), zg'(z), z^2g''(z); z) = \phi(D_{m,p}^{\lambda}(f*g)(z), D_{m,p}^{\lambda+1}(f*g)(z), D_{m,p}^{\lambda+2}(f*g)(z); z). \quad (2.7)$$

Therefore, (2.1) we have

$$\psi(g(z), zg'(z), z^2g''(z); z) \in \Omega. \quad (2.8)$$

See that

$$1 + \frac{t}{s} = \frac{m^2p^2w - p^2(m-1)^2u}{pcv - p(m-1)u} - 2p(m-1),$$

and since the admissibility condition for $\psi \in \Psi_n[\Omega, q]$. By Lemma 1.4,

$$g(z) \prec q(z), \quad or \quad D_{m,p}^{\lambda}(f * g)(z) \prec q(z).$$

Theorem 2.3. Let $\phi \in \Phi_n[h, q]$ with q(0) = 1. If $f \in A(p)$ satisfies

$$\phi\begin{pmatrix} D_{m,p}^{\lambda}(f * g)(z), D_{m,p}^{\lambda+1}(f * g)(z), \\ D_{m,p}^{\lambda+2}(f * g)(z); z \end{pmatrix} \prec h(z) \quad (2.9)$$

then

$$D_{m,p}^{\lambda}(f * g)(z) \prec q(z). (z \in U)$$

The next is an extension of Theorem 2.2 to the case where the behavior of q(z) on ∂U is unknown.

Corollary 2.4. $\Omega \in C$ and let q(z) be univalent in U with q(0) =1. Let $\phi \in \Phi_n[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f(z) \in A(P)$ and

$$\phi\begin{pmatrix} D_{m,p}^{\lambda}(f * g)(z), D_{m,p}^{\lambda+1}(f * g)(z), \\ D_{m,p}^{\lambda+2}(f * g)(z); z \end{pmatrix} \in \Omega,$$

then

$$D_{m,p}^{\lambda}(f * g)(z) \prec q(z).$$

Proof. By Theorem 2.2 we have $D_{m,p}^{\lambda}(f * g)(z) \prec q(\rho z)$. The result now deduced from the following subordination relationship $q_\rho(z) \prec q(z)$.

Theorem 2.5. Let h(z) and q(z) be univalent in U, with q(0) = 1 and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : C^3 \times U \to C$ satisfy one of the following conditions:
(1) $\phi \in \Phi_n[h, q_\rho]$, for some $\rho \in (0,1)$, or
(2) there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.If $f(z) \in A(P)$ satisfy (2.9),
then

$$D_{m,p}^{\lambda}(f * g)(z) \prec q(z).$$

Proof. Case (1). By applying Theorem 2.2, we get $D_{m,p}^{\lambda}(f * g)(z) \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$, we deduce $D_{m,p}^{\lambda}(f * g)(z) \prec q(z)$.

Case (2). If we let $g_\rho(z) = D_{m,p}^{\lambda}(f * g)_\rho(z) = D_{m,p}^{\lambda}(f * g)(\rho z) = g(\rho z)$,

then

$$\phi(g_\rho(z), zg'_\rho(z), z^2g''_\rho(z); \rho z) = \phi(g(\rho z), zg'(\rho z), z^2g''(\rho z); \rho z) \in h_\rho(U).$$

By using Theorem 2.2 and the comment associated with (2.8) where $w(z) = \rho z$ is any mapping U in to U, we get

$g_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we obtain $g(z) \prec q(z)$.

Hence,

$$D_{m,p}^{\lambda}(f * g)(z) \prec q(z).$$

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Now, the next result we need the best dominant of the differential subordination (2.9).

Theorem 2.6. Let $h(z)$ be univalent in U and let $\phi : C^3 \times U \to C$. Suppose that the differential equation

$$\phi\left(\begin{matrix} q(z), \frac{zq'(z)+p(m-1)q(z)}{pm}, \\ \frac{z^2q''(z)+2p(m-1)zq'(z)+p^2(m-1)^2q(z)}{p^2m^2}; z \end{matrix}\right) = h(z). \quad (2.10)$$

has a solution q(z) with q(0) = 0 and satisfy one of the following conditions:

(1) $q(z) \in Q_0$ and $\phi \in \Phi_n[\mathrm{h}, q]$.

(2) $q(z)$ is univalent in U and $\phi \in \Phi_n[\mathrm{h}, q_\rho]$ for some $\rho \in (0,1)$, or

(3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\in \Phi_n[\mathrm{h}_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f(z) \in A(p)$ satisfies (2.9), then $D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}) \prec \mathrm{q}(z)$ and q(z) is the best dominant.

Proof. By applying Theorem 2.3 and 2,5, we deduce that q(z) is a dominant of (2.9). Since q(z) satisfies (2.10), it is a solution of (2.9) and therefore q(z) will be dominated by all dominants of (2.9).

Hence, q(z) is the best dominantof (2.9).

In the particular case q(z) = Mz, M > 0, and in view of the Definition 1.2, the class of admissible function $\Phi_n[\Omega, q]$ denoted by $\Phi_n[\Omega, M]$ is described below.

Definition 2.7. Let Ω be a set in C, $\Re e\{m\} > 0$, $\lambda \geq 1$ and M > 0. The class of admissible functions $\Phi_n[\Omega, M]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$\phi\left(Me^{i\theta}, \frac{y+p(m-1)Me^{i\theta}}{pm}, \frac{L+[2p(m-1)y+p^2(m-1)^2Me^{i\theta}]}{p^2m^2}; z\right) \notin \Omega, \quad (2.11)$$

whenever $\theta \in R, R(Le^{i\theta}) \geq y(y-1)M, y \geq 1$ and $z \in U$.

Corollary 2.8. Let $\phi \in \Phi_n[\Omega, M]$. If $f(z) \in A(P)$ satisfy the following inclusion relationship

$$\phi\left(\begin{matrix} D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), D^{\lambda+1}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), \\ D^{\lambda+2}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}); z \end{matrix}\right) \in \Omega,$$

then

$$D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}) \prec Mz.$$

Now, in the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_n[\Omega, M]$ is denoted by $\Phi_n[M]$.

Corollary 2.9. Let $\phi \in \Phi_n[M]$. If $f(z) \in A(P)$ satisfies

$$\left|\phi\left(\begin{matrix} D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), D^{\lambda+1}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), \\ D^{\lambda+2}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}); z \end{matrix}\right)\right| < M,$$

then

$$\left|D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z})\right| < M.$$

Theorem 3.2. Let $\phi \in \Phi_n[\Omega, q]$. If $f(z) \in A(P)$, $D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}) \in Q_0$ and

$$\phi\left(\begin{matrix} D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), D^{\lambda+1}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), \\ D^{\lambda+2}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}); z \end{matrix}\right)$$ is univalent in U, then

$$\Omega \subset \phi\left(\begin{matrix} D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), D^{\lambda+1}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}), \\ D^{\lambda+2}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}); z \end{matrix}\right), \quad (3.1)$$

implies

$$\mathrm{q}(\mathrm{z}) \prec D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}). \quad (z \in U)$$

Proof. By using (2.7) and (3.1) we get

$$\Omega \subset \psi(g(z), zg'(z), z^2g''(z); z), (z \in U)$$

From (2.5), we note that the admissibility condition for $\phi \in \Phi'_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. therefore, and by Lemma 1.5 we get

$$\mathrm{q}(\mathrm{z}) \prec g(\mathrm{z}) \text{ or } \mathrm{q}(\mathrm{z}) \prec D^{\lambda}_{m,p}(\mathrm{f} * \mathrm{g})(\mathrm{z}). \quad (z \in U)$$

Theorem 3.3. Let $h(z)$ is analytic on U and $\phi \in \Phi'_n[h,q]$. If $f(z) \in A(P)$, $D^{\lambda}_{m,p}(f*g)(z) \in Q_0$ and $\phi : C^3 \times U \to C$ with

$$\phi\begin{pmatrix} D^{\lambda}_{m,p}(f*g)(z), D^{\lambda+1}_{m,p}(f*g)(z), \\ D^{\lambda+2}_{m,p}(f*g)(z)\,;z \end{pmatrix}$$ is univalent

in U, then

$$h(z) \prec \phi\begin{pmatrix} D^{\lambda}_{m,p}(f*g)(z), D^{\lambda+1}_{m,p}(f*g)(z), \\ D^{\lambda+2}_{m,p}(f*g)(z)\,;z \end{pmatrix}, \quad (3.2)$$

implies

$$q(z) \prec D^{\lambda}_{m,p}(f*g)(z).$$

Proof. By using relationship (3,2) we obtain

$$h(z) = \Omega \subset \phi\begin{pmatrix} D^{\lambda}_{m,p}(f*g)(z), \\ D^{\lambda+1}_{m,p}(f*g)(z), D^{\lambda+2}_{m,p}(f*g)(z)\,;z \end{pmatrix},$$

and from Theorem 3,2, we have

$$q(z) \prec D^{\lambda}_{m,p}(f*g)(z).$$

Collect Theorem 2.3 and 3.3, we get the following sandwich type Theorem.

Theorem 3.4. Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_n[h_2,q_2] \cap \Phi'_n[h_1,q_1]$. If $f(z) \in A(P)$, $D^{\lambda}_{m,p}(f*g)(z) \in Q_0 \cap H[0,P]$ and $\phi\left(D^{\lambda}_{m,p}(f*g)(z), D^{\lambda+1}_{m,p}(f*g)(z), D^{\lambda+2}_{m,p}(f*g)(z)\,;z\right)$ is univalent in U,

then

$$h_1(z) \prec \phi\begin{pmatrix} D^{\lambda}_{m,p}(f*g)(z), D^{\lambda+1}_{m,p}(f*g)(z), \\ D^{\lambda+2}_{m,p}(f*g)(z)\,;z \end{pmatrix} \prec h_2(z),$$

implies that

$$q_1(z) \prec D^{\lambda}_{m,p}(f*g)(z) \prec q_2(z).$$

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حول التابعية وفوق التابعية التفاضلية من الدرجة الثانية بأستخدام المؤثرالتكامل كوماتو

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المستخلص :
في هذا البحث بأستخدام خصائص مؤثر كوماتو استطعنا ان نشتق بعض خواص التابعية التفاضلية وفوق التابعية بالأعتماد على مفهوم ضرب هادمرد.