Journal of AL-Qadisiyah for computer science and mathematics Vol.10 No.1 Year 2018
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Modification of two Parameters Rayleigh into three Parameters one Through Exponentiated

Waleed meaya rodeen
Basra university
Collage of management and economic
Statistics department

Recived : 25\10\2017 **Revised : 9\11\2017** **Accepted : 19\11\2017**

Available online : 26/1/2018

DOI: 10.29304/jqcm.2018.10.1.356

Abstract
The continuous probability Rayleigh distribution is one of the important Is an important distribution that can be used To analyze signal data and statistical error data as well as time to failure, so we work on modifying two parameters Rayleigh (α, θ) into three parameters one's $(\alpha, \theta, \lambda)$ through exponentiated, the new $p.d.f$ obtained, also the formula for rth moments about origin is derived, to be used in estimating of parameters and also of reliability function. All derivation required were explained and the estimators by maximum likelihood and moments were obtained using different sets of initial values and the replicate of each experiment is $(R = 1000)$, the results are compared by using mean squared error (MSE).
Keywords: Three Parameters Rayleigh, moments Estimators, (MOM), Maximum Likelihood Estimator (MLE), reliability Function $[R_T(t)]$, Mean Squared error (MSE).

1. Introduction

The Rayleigh distribution is one of the continuous probability distribution, which is one of the family of distribution introduced by, Burr (1942), it is used in analysis of signal and also in representing statistical errors of all types. Rayleigh distribution has many applications in representing statistical model for life time data (Lawless, J.F.[1982])[7], Dey, S. and das, M.K. (2007)[4], explain Bayesian approach interval for Rayleigh distribution. The $p.d.f$ of two parameters Rayleigh (α, θ) is given by;

$$f(x;\alpha,\theta) = \begin{cases} 2\alpha\theta^2\, x\, e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha-1} & x,\alpha,\theta > 0 \\ 0 & o/w \end{cases} \quad (1)$$

Where (α) is the shape parameter and (θ) is the scale parameter.

The p.d.f in equation (1) may have different shape according to value of (α), if $(\alpha \le \frac{1}{2})$, the p.d.f in equation (1) is decreasing curve, and when $(\alpha > \frac{1}{2})$, it is skewed to right.

The cumulative distribution function corresponding to equation (1) is;

$$F(X;\alpha,\theta) = pr(X \le x) = \left(1-e^{-(\theta x)^2}\right)^{\alpha} \quad (2)$$

Also survival function is;

$$S(X;\alpha,\theta) = 1-\left(1-e^{-(\theta x)^2}\right)^{\alpha} \quad (3)$$

Also when $(\alpha\ \&\ \theta > 0)$, we can define the hazard function;

$$h(X;\alpha,\theta) = \frac{f(X;\alpha,\theta)}{S(X;\alpha,\theta)} \quad (4)$$

The $p.d.f$ in equation (1) can be transformed to a three parameters Rayleigh through using exponentiated transformation by adding another shape parameter (λ), using

$$G_X(x) = [F(x,\alpha,\theta]^{\lambda} = [\left(1-e^{-(\theta x)^2}\right)^{\alpha}]^{\lambda} = \left(1-e^{-(\theta x)^2}\right)^{\alpha\lambda} \quad (5)$$

$$g(x) = \alpha\lambda\left(1-e^{-(\theta x)^2}\right)^{\alpha\lambda-1}\left(e^{-(\theta x)^2}2x\theta^2\right)$$
$$= 2x\alpha\lambda\theta^2 e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha\lambda-1} \quad \alpha,\theta,\lambda > 0\ and\ \alpha\lambda > 1 \quad (6)$$

Equation (6) is the new $p.d.f$ of exponentiated Rayleigh which have applications in signal and errors analysis. The survival function is:

$$S(X;\alpha,\theta,\lambda) = 1-(F(X;\alpha,\theta)^{\lambda}$$

Journal of AL-Qadisiyah for computer science and mathematics Vol.10 No.1 Year 2018
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

2. Moment Estimator Method

We find the formula for the r[th] moments, about origin;

$$\mu_r' = E(x^r) = \int_0^\infty x^r\, f(X;\alpha,\theta) \tag{7}$$

After some steps we have;

$$\mu_r' = \frac{\alpha}{\theta^r}\sum_{i=0}^{\alpha-1}\frac{C_i^{\alpha-1}(-1)^{\alpha-1-i}}{(\alpha-i)^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1\right) \qquad \alpha>1 \tag{8}$$

$$\mu_r' = E(x^r) = 2\alpha\lambda\theta^2\int_0^\infty x^{r+1}\, e^{-(\theta x)^2}\left(1-e^{-(\theta x)^2}\right)^{\alpha\lambda-1} dx \tag{9}$$

Since

$$(1-t)^n = \sum_{j=0}^{n} C_j^n\,(-1)^j t^j$$

Then;

$$\left(1-e^{-(\theta x)^2}\right)^{\alpha\lambda-1} = \sum_{j=0}^{\alpha\lambda-1} C_j^{\alpha\lambda-1}\,(-1)^j\left(e^{-(\theta x)^2}\right)^j$$

$$\mu_r' = 2\alpha\lambda\theta^2\sum_{j=0}^{\alpha\lambda-1}\int_0^\infty x^{r+1}\,e^{-(\theta x)^2}C_j^{\alpha\lambda-1}(-1)^j\left(e^{-j\theta^2x^2}\right)dx \tag{10}$$

$$\mu_r' = 2\alpha\lambda\theta^2\sum_{j=0}^{\alpha\lambda-1}C_j^{\alpha\lambda-1}(-1)^j\int_0^\infty x^{r+1}\,e^{-\theta^2(i+j)x^2}dx$$

Let $Z=\theta^2(1+j)x^2 \quad\Rightarrow\quad x=\frac{\sqrt{Z}}{\theta\sqrt{1+j}} \quad\Rightarrow$

$dx = \frac{1}{2\theta\sqrt{Z}\sqrt{1+j}}\,dZ$

$$\mu_r' = k\int_0^\infty\left(\frac{\sqrt{Z}}{\theta\sqrt{1+j}}\right)^{r+1} e^{-Z}\,\frac{1}{2\theta\sqrt{Z}\sqrt{1+j}}\,dZ$$

$$= k\,\frac{1}{2\theta^{r+2}\left(\sqrt{1+j}\right)^{r+1}\sqrt{1+j}}\int_0^\infty \sqrt{Z}^{\,r}\,e^{-Z}\,dZ$$

$$\mu_r' = \frac{2\alpha\lambda\theta^2\sum_{j=0}^{\alpha\lambda-1}C_j^{\alpha\lambda-1}\,(-1)^j}{2\theta^{r+2}\left(\sqrt{1+j}\right)^{r+2}}\int_0^\infty Z^{\frac{r}{2}}e^{-Z}\,dZ$$

$$= \alpha\lambda\sum_{j=0}^{\alpha\lambda-1}C_j^{\alpha\lambda-1}\,(-1)^j\,\frac{\Gamma\left(\frac{r}{2}+1\right)}{\theta^r}\left(\frac{1}{\sqrt{1+j}}\right)^{r+2} \qquad \alpha\lambda>1 \tag{11}$$

By solving $\left(\mu_r' = \frac{\sum_{i=1}^n x_i^r}{n}\right)$ $for\ (r=1,2,3)\ and\ \alpha\lambda>1$, we can obtain moments estimates for three parameters of (E – Ray).

3. Maximum Likelihood Estimator

Let $(x_1, x_2, \dots, x_n)$ be a r. s. from $p.d.f$, $f(x;\alpha,\theta)$, then;

$$L = \prod_{i=1}^n f(x_i) = 2^n\alpha^n\lambda^n\theta^{2n}\prod_{i=1}^n x_i\,e^{-\theta^2\sum_{i=1}^n x_i^2}\prod_{i=1}^n\left(1-e^{-\theta^2x_i^2}\right)^{\alpha\lambda-1} \tag{12}$$

$$\log L = n\log 2 + n\log\alpha + 2n\log\theta + n\log\lambda + \sum_{i=1}^n\log x_i - \theta^2\sum_{i=1}^n x_i^2 + (\alpha\lambda-1)\sum_{i=1}^n\log\left(1-e^{-\theta^2x_i^2}\right)$$

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^n\log\left(1-e^{-\theta^2x_i^2}\right)=0$$

$$\hat\alpha_{MLE} = -\frac{n}{\sum_{i=1}^n\log\left(1-e^{-\theta^2x_i^2}\right)} \tag{13}$$

$$\frac{\partial\log L}{\partial\theta} = \frac{2n}{\theta} - 2\theta\sum_{i=1}^n x_i^2 + (\alpha\lambda-1)\sum_{i=1}^n\frac{(-1)\left(e^{-\theta^2x_i^2}\right)(-2\theta x_i^2)}{\left(1-e^{-\theta^2x_i^2}\right)} = 0$$

$$(\alpha\lambda-1)\sum_{i=1}^n\frac{(-1)\left(e^{-\theta^2x_i^2}\right)(-2\hat\theta x_i^2)}{\left(1-e^{-\theta^2x_i^2}\right)} = \hat\theta\sum_{i=1}^n x_i^2 - \frac{n}{\hat\theta}$$

$$\hat\theta^2\left[\sum_{i=1}^n x_i^2 - (\alpha\lambda-1)\sum_{i=1}^n\frac{(-1)\left(e^{-\theta^2x_i^2}\right)(-2\theta x_i^2)}{\left(1-e^{-\theta^2x_i^2}\right)}\right] = n$$

By using newton raphson method to get the parameter estimator of θ

4. Simulation

The estimation and comparison has been done through simulation procedures, were the values are generated from;

$$G_X(x) = \left[\left(1-e^{-(\theta x)^2}\right)^\alpha\right]^\lambda$$

$$u_i = \left[\left(1-e^{-\theta^2x^2}\right)\right]^{\alpha\lambda}$$

$$u_i^{\frac{1}{\alpha\lambda}} = \left(1-e^{-\theta^2x^2}\right)$$

$$e^{-\theta^2x^2} = 1-u_i^{\frac{1}{\alpha\lambda}}$$

$$-\theta^2x^2 = \ln\left(1-u_i^{\frac{1}{\alpha\lambda}}\right)$$

$$x^2 = -\frac{1}{\theta^2}\ln\left(1-u_i^{\frac{1}{\alpha\lambda}}\right)$$

$$x_i = -\frac{1}{\theta}\sqrt{\ln\left(1-u_i^{\frac{1}{\alpha\lambda}}\right)}$$

The initial values are;

α	λ	θ
3	0.5	0.8
4	0.8	1.5
3.5	2	2

Table (1): Estimation of ($\alpha = 3, \theta = 0.6, \lambda = 0.5$)

n	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MOM}$	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{MOM}$	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{MOM}$
30	0.6604	0.51593	0.5506	0.5560	2.4756	2.8660
	0.6403	0.52274	0.5643	0.5641	2.4814	2.9520
	0.5031	0.51499	0.5602	0.6203	2.4703	2.9830
	0.8703	1.04061	0.8613	0.6614	2.9611	2.8820
	0.8832	1.4035	0.9921	1.0021	3.0112	1.3846
	0.9994	1.0362	1.3011	1.0314	3.0611	1.4157
	1.4683	1.5266	1.2920	1.1121	3.5211	1.3992
	1.5962	1.9821	1.0122	1.2306	3.5311	1.3672
	1.6401	1.9932	1.0089	1.5216	3.8020	1.4407
	1.6612	1.9961	1.5283	1.5220	3.8840	1.4402
60	0.48776	0.5006	1.4407	1.4837	1.1254	1.5267
	0.49582	0.5832	1.4408	1.4962	1.3621	1.5382
	0.51084	0.9486	1.4425	0.9437	1.6410	1.5171
	0.8736	0.5044	0.8963	0.9882	1.0407	1.5189
	0.9921	0.9936	0.8902	0.9945	1.0475	1.2346
	0.9914	1.0052	0.8831	1.6931	1.5743	1.2204
	1.4001	1.0069	0.8801	1.4825	1.5675	1.0666
	1.4231	1.5022	0.8711	1.5119	1.0161	1.2431
	1.4303	1.5311	0.6531	1.4945	1.4241	1.3352
	1.4996	1.4958	0.6420	1.8462	1.2563	1.2453
90	0.49784	0.5042	0.5046	0.4871	2.1106	1.4837
	0.49987	0.4991	0.5049	0.4973	2.004	1.4810
	0.49417	1.0081	0.4998	0.4917	2.001	1.4707
	0.9981	1.0005	0.4999	0.9855	1.889	0.9463
	0.99046	1.0411	1.0005	0.9965	1.992	1.6472
	1.00231	1.5432	1.0024	0.9943	1.0115	1.5321
	1.51951	1.5171	1.0114	0.9931	1.5341	1.55221
	1.501009	1.51891	1.5348	1.5171	1.51718	1.4841
	1.50372	1.3264	1.51716	1.51886	1.51706	1.4742
	1.64021	1.4302	1.5189	1.5002	1.5051	1.3762

Table (2): Estimators of Reliability Function.

n	α	λ	θ	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$Best$
30	3	0.5	0.8	0.68323	0.59871	MLE
		0.8	1.5	0.76112	0.66082	MLE
		0.6	2	0.79014	0.89314	MOM
	4	0.5	0.8	0.99061	0.88541	MLE
		0.8	1.5	0.9843	0.93061	MLE
		0.6	2	0.60075	0.77431	MOM
	3.5	0.5	0.8	0.99941	0.98405	MLE
		0.8	1.5	0.78063	0.69904	MLE
		0.6	2	0.66421	0.63415	MLE
60	3	0.5	0.8	0.77435	0.90763	MOM
		0.8	1.5	0.84306	0.77531	MOM
		0.6	2	0.872310	0.99078	MLE
	4	0.5	0.8	0.843301	0.884693	MOM
		0.8	1.5	0.706492	0.77084	MLE
		0.6	2	0.77145	0.84773	MLE
	3.5	0.5	0.8	0.78715	0.77806	MLE
		0.8	1.5	0.84734	0.84707	MOM
		0.6	2	0.85321	0.9993	MOM
90	3	0.5	0.8	0.99061	0.99341	MOM
		0.8	1.5	0.84667	0.84732	MOM
		0.6	2	0.83552	0.84352	MOM
	4	0.5	0.8	0.847076	0.84352	MLE
		0.8	1.5	0.776171	0.77061	MLE
		0.6	2	0.89031	0.84315	MLE
	3.5	0.5	0.8	0.88061	0.9091	MOM
		0.8	1.5	0.84553	0.9312	MOM
		0.6	2	0.84321	0.9906	MOM

Table (3): Mean square error for $\hat{R}$.

n	α	λ	θ	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$Best$
30	3	0.5	0.8	0.0041809	0.002567	ML
		0.8	1.5	0.0026424	0.0031071	ML
		0.6	2	0.002747	0.0033751	MOM
	4	0.5	0.8	0.003617	0.00214	ML
		0.8	1.5	0.004122	0.00431	MOM
		0.6	2	0.001513	0.00221	MOM
	3.5	0.5	0.8	0.001643	0.004151	MOM
		0.8	1.5	0.003512	0.00421	MOM
		0.6	2	0.003415	0.00396	MOM
60	3	0.5	0.8	0.000436	0.000412	ML
		0.8	1.5	0.000434	0.000251	ML
		0.6	2	0.000104	0.000161	ML
	4	0.5	0.8	0.000974	0.000994	MOM
		0.8	1.5	0.000762	0.000636	ML
		0.6	2	0.000152	0.000051	ML
	3.5	0.5	0.8	0.000734	0.000626	ML
		0.8	1.5	0.000346	0.000215	ML
		0.6	2	0.000789	0.000313	ML
90	3	0.5	0.8	0.000919	0.0001021	ML
		0.8	1.5	0.000514	0.000231	ML
		0.6	2	0.000501	0.000282	ML
	4	0.5	0.8	0.000747	0.0002031	ML
		0.8	1.5	0.000671	0.000115	ML
		0.6	2	0.000621	0.000071	ML
	3.5	0.5	0.8	0.000606	0.0000241	ML
		0.8	1.5	0.000264	0.0000112	ML
		0.6	2	0.000255	0.0000151	ML

Conclusion

- Rayleigh distribution is a good probability distribution for representing signal data, also time to failure for medical physical experiments when failure happens after ($t > \alpha\ for\ example$), but in our research we work on modifying two parameters Rayleigh into three one's.
- Reliability estimators by moments and maximum likelihood indicates that ($\hat{R}_{MLE}$) is better than ($\hat{R}_{MOM}$) with percentage (75%), while ($\hat{R}_{MOM}$) with (25%).
- The sample size taken are ($n = 30,60,90$), with different sets of initial values obtained under condition ($\alpha\lambda > 1$).
- The results are compared by statistical measures mean squares errors.

References

[1]Abd – Elfatta, A.M., (2011), "Goodness of fit test for the generalized Rayleigh distribution with unknown parameters", Journal of Statistical Computation and Simulation, Vol.81, No.3, 357 – 366.
[2]Abdul-Moniem I. B. and Abdul-Hameed H.F. (2012),"On Exponentiated Lomax Distribution".Inter. Jou. Of Math. Archive -3(5), pp:2144-2150.
[3]Azeem Ali, Syed AnwerHasnain, and Munir Ahmad, (2015), MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS, Pak. J. Statist.Vol. 31(6), 697-708.
[4]Dey, S. and Das, M.K. (2007), "A note on Prediction Interval for a Rayleigh Distribution: Bayesian Approach", American Journal of Mathematical and Management Science, 27(1&20, 43 – 48.
[5]Flaih A., Elsallaukh H., Mandi E. and Milanova M. (2012),"The Exponentiated Inverted WeibullDistribution", Appl. Math. Inf. Sci., vol. 6, no. 2, pp:167-171.
[6]Kundu, D. and Raqab, M.Z., (2005), "Generalized Rayleigh Distribution: Different Methods of Estimations", Computational Statistical and Data Analysis, 49, 187 – 200.
[7]Lawless, J.F., (1982), "Statistical Models and Methods for life time data", Jhon Wiley & Sons, New York.
[8]MOHAMMAD Z. RAQAB – MOHAMED T. MADI, (2009), "Bayesian analysis for the exponentiated Rayleigh distribution", METRON - International Journal of Statistics 2009, vol. LXVII, n. 3, pp. 269-288.
[9]Raqab, M.Z., (2005), "Generalized Rayleigh Distribution Different Methods of Estimation", Computational Statistics and Data Analysis, 49,187 – 200.
[10]Raqab M.Z. and Madi M.T. (2009), "Bayesian Analysis for the Exponentiated Rayleigh Distribution", METRON-Inter. J. of stal., vol. 67, no.3, pp:269-288.
[11]Rosaiaah, K., Kantam, R.R.L. and Santosh Kumar, Ch., (2007), Reliability of Test plants for exponentiated log logistic distribution", Economic Quality Control, Vol. , 165 – 175.
[12]Reza Azimi1, FarhadYaghmaei and DarioshAzim, (2012)," Comparison of Bayesian Estimation Methods for Rayleigh Progressive Censored Data Under the Different Asymmetric Loss Function", International Journal of Applied Mathematical Research, 1 (4) (2012) 452-461.
[13]SankuDey, TanujitDey, Sudhansu S. Maiti, (2015), "Bayes Shrinkage Estimation of the Parameter of Rayleigh Distribution for Progressive Type-II Censored Data", Austrian Journal of Statistics, Volume 44, 3–15.

[14]Sanko Dey (2008), “Minimax Estimation of the parameter of the Rayleigh distribution under quadratic loss function”, Data Science Journal, Vol. 7, 23.
[15]Shankar Kumar Shrestha, Vijay Kumar, (2014), “Bayesian Analysis for the Generalized RayleighDistribution”, International Journal of Statistika and Mathematika, Volume 9, Issue 3, pp 118-131.
[16]Smail Mahdi, (2006), “Improved Parameter Estimation in RayleighModel”, Metodološkizvezki, Vol. 3, No. 1, 63-74.

وليد مية رودين
جامعة البصرة / كلية الادارة والاقتصاد / قسم الاحصاء

الملستخلص
يعتبر توزيع رايلي من التوزيعات المستمرة المهمه التي تستخدم في بيانات اوقات الفشل ، اذ تم عمل تحديث على توزيع رايلي باستخدم التحويل الاسي للحصول على توزيع بثلاث معالم وتم اشتقاق صيغة للعزم R للحصول على مقدرات العزوم وكذلك تقدير المقدرات للنموذج باستخدام دالة الامكان الاعظم وقد تم استخدام قيم ابتدائية مختلفة وبعدد تكرارات التجربة لـ(R=1000) وتم مقارنه النتائج بالاعتماد على معيار متوسط مربعات الخطاء(MSE)