Journal of AL-Qadisiyah for computer science and mathematics Vol.10 No.2 Year 2018
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

Geometric Properties for a family of p −valent Holomorphic Functions with Negative Coefficients for Operator on Hilbert Space

Abbas Kareem Wanas
Department of Mathematics
College of Computer Science and Information Technology
abbas.kareem.w@qu.edu.iq

Sema Kadhim Jebur
Department of Mathematics
College of Agriculture
semh.Alisawi@qu.edu.iq

Recived : 23\11\2017 **Revised : //** **Accepted : 8\1\2017**

Available online : 17 /2/2018

DOI: 10.29304/jqcm.2018.10.2.361

Abstract

The purpose of the present investigation is to introduce and study a certain subclass $\mathcal{A}k_p(\alpha, \beta, \delta, T)$ of p-valent holomorphic functions with negative coefficients of the operators on Hilbert space in U. Moreover, we get a number of geometric properties.

Mathematics Subject Classification: 30C45, 30C50.

Journal of AL-Qadisiyah for computer science and mathematics Vol.10 No.2 Year 2018
ISSN (Print): 2074 – 0204 ISSN (Online): 2521 – 3504

1 Introduction

Let $\mathcal{A}_p$ be the class of functions f of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1,2,\dots\}), \qquad (1.1)$$

which are holomorphic and p-valent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Let k_p denote the subclass of $\mathcal{A}_p$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad \left(a_{n+p} \geq 0,\ p \in \mathbb{N} = \{1,2,\dots\}\right). \qquad (1.2)$$

Definition 1.1: *A function $f \in k_p$ is said to be in the class $\mathcal{A}k_p(\alpha,\beta,\delta)$ if it satisfies*

$$\left|\frac{f'(z) - pz^{p-1}}{\alpha(f'(z) - \beta) + p - \beta}\right| < \delta,$$

where $0 \leq \alpha < 1,\ 0 \leq \beta < p, 0 < \delta \leq 1$ *and* $z \in U$.

Let H be a Hilbert space on the complex field. Let T be a linear operator on H. For a complex holomorphic function f on the unit disk U, we denoted $f(T)$, the operator on H defined by the usual Riesz-Dunford integral [2]

$$f(T) = \frac{1}{2\pi i}\int_c f(z)(zI - T)^{-1}\, dz\,,$$

where I is the identity operator on H, c is a positively oriented simple closed rectifiable contour lying in U and containing the spectrum $\sigma(T)$ of T in its interior domain [3]. Also $f(T)$ can be defined by the series

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n\,,$$

which converges in the norm topology [4].

Definition 1.2: *Let H be a Hilbert space and T be an operator on H such that such that $T \neq \emptyset$ and $\|T\| < 1$. Let α,β be real numbers such that $0 \leq \alpha < 1,\ 0 \leq \beta < p, 0 < \delta \leq 1$. An holomorphic function f on the unit disk is said to belong to the class $\mathcal{A}k_p(\alpha,\beta,\delta,T)$ if it satisfy the inequality*

$$\|f'(T) - pT^{p-1}\| < \delta\|\alpha(f'(T) - \beta) + p - \beta\|,$$

where $\emptyset$ denote the zero operator on H .

The operator on Hilbert space were consider recently be Xiaopei [8], Joshi [6], Chrakim et al. [1], Ghanim and Darus [5] and Selvaraj et al. [7].

2 Main Results

Theorem 2.1: *Let $f \in k_p$ be defined by (1.2). Then $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$ for all $T \neq \emptyset$ if and only if*

$$\sum_{n=1}^{\infty}(n+p)\,(1+\delta\alpha)a_{n+p} \leq \delta(p-\beta)(1+\alpha). \qquad (2.1)$$

where $0 \leq \alpha < 1,\ 0 \leq \beta < p, 0 < \delta \leq 1$.
The result is sharp for the function f given by

$$f(z) = z^p - \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)} z^{n+p}\,,\ n \geq 1. \qquad (2.2)$$

Proof: Suppose that the inequality (2.1) holds. Then, we have

$$\|f'(T) - pT^{p-1}\| - \delta\|\alpha(f'(T) - \beta) + p - \beta\| = \left\| -\sum_{n=1}^{\infty}(n+p)\, a_{n+p}T^{n+p-1} \right\| - \delta \left\| \alpha p T^{p-1} - \sum_{n=1}^{\infty} \alpha(n+p)\, a_{n+p}T^{n+p-1} + p - \beta(1+\alpha) \right\| \le \sum_{n=1}^{\infty}(n+p)\,(1+\delta\alpha)a_{n+p} - \delta(p-\beta)(1+\alpha) \le 0.$$

Hence, $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$.

To show the converse, let $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$. Then

$$\|f'(T) - pT^{p-1}\| < \delta\|\alpha(f'(T) - \beta) + p - \beta\|,$$

gives

$$\left\| -\sum_{n=1}^{\infty}(n+p)\, a_{n+p}T^{n+p-1} \right\| < \delta \left\| \alpha p T^{p-1} - \sum_{n=1}^{\infty} \alpha(n+p)\, a_{n+p}T^{n+p-1} + p - \beta(1+\alpha) \right\|$$

Setting $T = rI\ (0 < r < 1)$ in the above inequality, we get

$$\frac{\sum_{n=1}^{\infty}(n+p)\, a_{n+p}r^{n+p-1}}{\alpha p r^{p-1} - \sum_{n=1}^{\infty} \alpha(n+p)\, a_{n+p}r^{n+p-1} + p - \beta(1+\alpha)} < \delta. \qquad (2.3)$$

Upon clearing denominator in (2.3) and letting $r \longrightarrow 1$, we obtain

$$\sum_{n=1}^{\infty}(n+p)\, a_{n+p} < \delta(p-\beta)(1+\alpha) - \sum_{n=1}^{\infty} \delta\alpha(n+p)\, a_{n+p}.$$

Thus

$$\sum_{n=1}^{\infty}(n+p)\,(1+\delta\alpha)a_{n+p} \le \delta(p-\beta)(1+\alpha),$$

which completes the proof.

Corollary 2.1: *If $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$, then*

$$a_{n+p} \le \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}, \quad n \ge 1.$$

Theorem 2.2: *If $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$ and $\|T\| < 1, T \ne \emptyset$, then*

$$\|T\|^p - \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1} \le \|f(T)\| \le \|T\|^p + \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}$$

and

$$p\|T\|^{p-1} - \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^p \le \|f'(T)\| \le p\|T\|^{p-1} + \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^p.$$

The result is sharp for the function f given by

$$f(z) = z^p - \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)} z^{p+1}.$$

Proof: According to the Theorem 2.1, we get

$$\sum_{n=1}^{\infty} a_{n+p} \le \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}.$$

Hence

$$\|f(T)\| \ge \|T\|^p - \sum_{n=1}^{\infty} a_{n+p}\,\|T\|^{n+p} \ge \|T\|^p - \|T\|^{p+1}\sum_{n=1}^{\infty} a_{n+p} \ge \|T\|^p - \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}.$$

Also,

$$\|f(T)\| \le \|T\|^p + \sum_{n=1}^{\infty} a_{n+p}\,\|T\|^{n+p} \le \|T\|^p + \frac{\delta(p-\beta)(1+\alpha)}{(p+1)(1+\delta\alpha)}\|T\|^{p+1}.$$

In view of Theorem 2.1, we have

$$\sum_{n=1}^{\infty}(n+p)a_{n+p} \le \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}.$$

Thus

$$\|f'(T)\| \ge p\|T\|^{p-1} - \sum_{n=1}^{\infty}(n+p)a_{n+p}\,\|T\|^{n+p-1} \ge p\|T\|^{p-1} - \|T\|^p\sum_{n=1}^{\infty}(n+p)a_{n+p} \ge p\|T\|^{p-1} - \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^p$$

and

$$\|f'(T)\| \le p\|T\|^{p-1} + \|T\|^p\sum_{n=1}^{\infty}(n+p)a_{n+p} \le p\|T\|^{p-1} + \frac{\delta(p-\beta)(1+\alpha)}{1+\delta\alpha}\|T\|^p.$$

Therefore the proof is complete.

Theorem 2.3: *Let* $f_0(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}z^{n+p}, \quad n \ge 1.$$

Then $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty}\lambda_n f_n(z), \tag{2.4}$$

where $\lambda_n \ge 0$ *and* $\sum_{n=0}^{\infty}\lambda_n = 1$.

Proof: Assume that f can be expressed by (2.4). Then, we have

$$f(z) = \sum_{n=0}^{\infty}\lambda_n f_n(z) = z^p - \sum_{n=0}^{\infty}\frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}\lambda_n z^{n+p}.$$

Thus

$$\sum_{n=0}^{\infty}\frac{(n+p)(1+\delta\alpha)}{\delta(p-\beta)(1+\alpha)}\frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}\lambda_n = \sum_{n=0}^{\infty}\lambda_n = 1-\lambda_0 \le 1,$$

and so $f \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$.

Conversely, suppose that f given by (1.2) is in the class $\mathcal{A}k_p(\alpha,\beta,\delta,T)$. Then by Corollary 2.1, we have

$$a_{n+p} \le \frac{\delta(p-\beta)(1+\alpha)}{(n+p)(1+\delta\alpha)}.$$

Setting

$$\lambda_n = \frac{(n+p)(1+\delta\alpha)}{\delta(p-\beta)(1+\alpha)}a_n, \quad n \ge 1,$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty}\lambda_n$. Then

$$f(z) = \sum_{n=0}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem.

Theorem 2.4: *The class* $\mathcal{A}k_p(\alpha,\beta,\delta,T)$ *is a convex set .*

Proof: Let f_1 and f_2 be the arbitrary elements of $\mathcal{A}k_p(\alpha,\beta,\delta,T)$. Then for every t $(0 \le t \le 1)$, we show that $(1-t)f_1 + tf_2 \in \mathcal{A}k_p(\alpha,\beta,\delta,T)$. Thus, we have

$$(1-t)f_1 + tf_2 = z^p - \sum_{n=1}^{\infty}\left((1-t)a_{n+p} + tb_{n+p}\right)z^{n+p}.$$

Hence

$$\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)\left((1-t)a_{n+p} + tb_{n+p}\right) = (1-t)\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)a_{n+p} + t\sum_{n=1}^{\infty}(n+p)(1+\delta\alpha)b_{n+p} \le (1-t)\delta(p-\beta)(1+\alpha) + t\delta(p-\beta)(1+\alpha).$$

This completes the proof.

References

[1] Y. Chrakim, J. S. Lee and S. H. Lee, A certain subclass of analytic functions with negative coefficients for operators on Hilbert space, *Math. Japonica*, **47**(1) (1998), 155-124.

[2] N. Dunford and J. T. Schwarz, *Linear Operator*, Part I, General Theory, New York - London, Inter Science, 1958.

[3] K. Fan, Analytic functions of a proper contraction, *Math. Z.*, **160**(1978), 275-290.

[4] K. Fan, Julia's lemma for operators, *Math. Ann.*, **239**(1979), 241-245.

[5] F. Ghanim and M. Darus, On new subclass of analytic p-valent function with negative coefficients for operators on Hilbert space, *Int. Math. Forum*, **3**(2)(2008), 69-77.

[6] S. B. Joshi, On a class of analytic functions with negative coefficients for operators on Hilbert Space, *J. Appr. Theory and Appl.*, (1998), 107 – 112.

[7] C. Selvaraj, A. J. Pamela and M. Thirucheran, On a subclass of multivalent analytic functions with negative coefficients for contraction operators on Hilbert space, *Int. J. Contemp. Math. Sci.*, **4**(9)(2009), 447-456.

[8] Y. Xiapei, A subclass of analytic p-valent functions for operator on Hilbert Space, *Math. Japonica* , **40**(2)(1994), 303 – 308.

خواص هندسية لعائلة من الدوال متعددة التكافؤ ذات معاملات سالبة لمؤثر على فضاء هلبرت

عباس كريم وناس — كلية علوم الحاسوب وتكنلوجيا المعلومات — abbas.kareem.w@qu.edu.iq

سمة كاظم جبر — كلية الزراعة — semh.Alisawi@qu.edu.iq

المستخلص :

الغرض من العمل الحالي هو تقديم ودراسة صنف جزئي مؤكد $\mathcal{A}k_p(\alpha,\beta,\delta,T)$ من الدوال متعددة التكافؤ ذات معاملات سالبة لمؤثرات على فضاء هلبرت في U في . علاوة على ذلك حصلنا على عدد من الخواص الهندسية.