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Fixed Point Principles in General b −Metric Spaces and b −Menger Probabilistic spaces

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Abstract

In this work, three general principles for existence a fixed point and a common fixed point are proved in types of general metric spaces, which conclude the existence a fixed point of set−valued mapping in a general b −metric space , the existence of common fixed point of three commuting orbitally continuous $\chi-$ condensing mappings and a result of fixed point for set−valued condensing mapping defined on probabilistic bounded subset of b − Menger probabilistic metric space.

Keywords: G_b −metric spaces**,** Menger probabilistic, set−valued mappings, condensing mappings**,** fixed ponts.

Mathematics Subject Classification: 47H05, 47H09

1. Introduction

Aghajani et al. [1] introduced the notion of $\boldsymbol{G_b}$-metric space as a generalization of $\boldsymbol{b}-$ **metric** spaces. Results of $\boldsymbol{G_b}$ −metric about fixed points and its applications can be found in the research papers of Abed and Jabbar [1-3], Mustafa Khan, Arshad and Ahmad [5] and references there in.

Another generalization of metric spaces introduced by Menger [6] called probabilistic metric space. Many results on the existence of fixed points or its application in nonlinear equations in these spaces have been studied by many researchers (see e.g. [7]). the notion of a generalized probabilistic metric space or a PGM −space as a generalization of a PM −space and a G −metric space have been defined by Zhou et al. [8] .

And then, Zhuet al. [10] presented some fixed point theorems in generalized probabilistic metric spaces.

Here, there are two aims the first one is proving the existence of fixed points and common fixed points for set-valued (or single valued) condensing mappings in orbitally complete general b- metric. The second is to define a measure for probabilistic subset of a Menger general b- metric space and employ it to prove a fixed point theorem for condensing mapping.

In this paper, $\mathcal{M}$ is general $b-$ metric space and

$2^{\mathcal{M}} = \{C: \emptyset \neq C \subset \mathcal{M}\}$,

$CB(\mathcal{M}) = \{C: \emptyset \neq C \subset \mathcal{M}, C \text{ is closed and bounded}\}$,

$K(\mathcal{M}) = \{C: \emptyset \neq C \subset \mathcal{M}, C \text{ is compact}\}$,

Also, $\mathbb{R}^+$ will be denote to non-negative reals, $\mathbb{R}^*=[-\infty,\infty]$, $\boldsymbol{N}$ be positive integers, $\overline{A}$ be the closure of a set A and $\rightrightarrows$ be set-valued mapping .

2. Preliminaries

"Let $\mathcal{M}$ be a non-empty set and $\Lambda: \mathcal{M}^3 \rightarrow \mathrm{R}^+$ be a function satisfying the following for u, v, w and a in $\mathcal{M}$:

$i - \Lambda(u,v,w) = 0 \; iff \; u = v = w$

$ii - \Lambda(u,u,v) > 0 \, , u \neq v$

$iii - \Lambda(u,u,v) \leq \Lambda(u,v,w), \; u \neq v$

$iv - \Lambda(u,v,w) = \Lambda(p\{u,v,w\})$, p is permutation

$v - \Lambda(u,v,w) \leq b[\Lambda(u,a,a) + \Lambda(a,v,w)], b \geq 1.$

Then the pair $(\mathcal{M},\Lambda)$ is called general b- metric space."[1]

"A general $b-$ metric space $\mathcal{M}$ is called symmetric if $\Lambda(u,v,v) = \Lambda(v,u,u)$. For $u_0 \in \mathcal{M}$, $r > 0$ the ball with center u_0 and radius r is $B_r(u_0,r) = \{y \in \mathcal{M}: \Lambda(u_0,v,v) < r\}$ and the family $\{B_{\gamma}(u,r): u \in \mathcal{M}, r > 0\}$ is a base of a topology [12]. The diameter of a set $C \subseteq \mathcal{M}$ is $(A) = sup_{a,b,c \in C} \Lambda(a,b,c)$" . Also, the definition of generalized $b-$ metric implies that

Proposition 2.1: [1] "For all $u,v,w,a \in \mathcal{M}$, the following hold

(1) $\Lambda(u,v,w) \leq b[\Lambda(u,u,v) + \Lambda(u,u,w)]$

(2) $\Lambda(u,v,v) \leq 2b\,\Lambda(v,u,u)$

(3) $\Lambda(u,v,w) \leq b[\Lambda(u,a,w) + \Lambda(a,v,w)]$, $b \geq 1$"

From this Proposition, we have

(5) $\Lambda(u,v,w) \leq b\,\Lambda(u,a,a) + b^2\,\Lambda(v,a,a) + b^2\Lambda(w,a,a)$

"A sequence $\{u_n\} \subseteq \mathcal{M}$ is [1] or [2]

(1) Cauchy sequence if ,$\forall \, \varepsilon > 0$ $\exists n_0 \in \boldsymbol{N}$ such that $\forall m,n,i \geq n_0$, $\Lambda(u_n,u_m,u_i) < \varepsilon$

(2) Convergent to a point $u \in \mathcal{M}$if $\forall \epsilon > 0, \exists n_0 \in \boldsymbol{N}$ such that $\forall$ $n,m \geq n_0$, $\Lambda(u_n,u_m,u) < \varepsilon$.

A space $\mathcal{M}$ is called complete if every Cauchy sequence is convergent in $\mathcal{M}$." Throughout this paper $(\mathcal{M},\Lambda)$ denotes general b $-$metric space.

Proposition 2.2: [1] " Let $\{u_n\}$ be a sequence in $\mathcal{M}$, then $\{u_n\}$ is Cauchy iff $\forall \; \varepsilon > 0$, $\exists \; n_0 \in \mathbf{N}$ such that $\Lambda(u_n,u_m,u_m) < \varepsilon, \forall \; m,n \geq n_0$."

Proposition 2.3: [4] "$\{u_n\}$ is convergent to u ⇔ $G(u, u_n, u) \to 0$ ⇔ $G(u_n, u, u) \to 0$ ⇔ $G(u_n, u_m, u) \to 0$ as $n, m \to \infty$. The sequence $\{u_n\}$ is Cauchy ⇔ G(u_n, u_m, u_m)→0 as n, m→∞."

Definition 2.4: Let ($\mathcal{M}$, Λ) and ($\mathcal{M}'$, Λ') be general b −metric spaces, and $T : \mathcal{M} \to \mathcal{M}'$ be a function. Then T is continuous at a point $a \in \mathcal{M}$iff for ∀ $\varepsilon > 0$, $\exists \delta > 0$ such that $u, v \in \mathcal{M}$and $G(a, u, v) < \delta$ implies $\Lambda'(T(a), T(u), T(v)) < \varepsilon$. A function T is continuous on $\mathcal{M}$ iff it is continuous at $\forall a \in \mathcal{M}$.

Definition 2.5: Let $T: \mathcal{M} \to \mathcal{M}$ then

(1) The orbit of a point $u \in \mathcal{M}$ is the set $O(u) = \{u, Tu, T^2u, \ldots..\}$, and let $C \subseteq \mathcal{M}$the orbit of a set C by T is $O_{T(C)} = \{ T^n(u) : n = 0, 1, 2, \ldots, u \in C \}$.

(2) An orbit of a point z is said to be bounded if ∃ $K > 0$ such that $\Lambda(u, v, w) \leq K$ $\forall u, v, w \in O(z)$, K is called a bound of $O(z)$.

(3) $\mathcal{M}$is said to be $T-$ orbitally bounded if $\delta(O(z)) < \infty$ $\forall z \in \mathcal{M}$.

(4) $\mathcal{M}$is said to be $T-$ orbitally complete if every Cauchy sequence in $O(z)$ converges to a point in $\mathcal{M}$.

(5) T is a called orbitally continuous if for any $\{u_n\} \subseteq O(u)$ and $u_n \to u$ implies $Tu_n \to Tu$, $\forall u \in \mathcal{M}$.

(6) The point $u \in \mathcal{M}$ is called a fixed point of the set-valued mapping $T : \mathcal{M} \to 2^{\mathcal{M}}$ if $u \in Tu$ and u is fixed point of a single mapping $T : \mathcal{M} \to \mathcal{M}$ if $u = Tu$.

The orbit of u by two mappings S, T is, $O(u)_{ST} =: \{u, Su, TSu, STSu, \ldots\}$, when T, S are commuting then $O(u) = \{T^m S^n u : m, n = 0, 1, \ldots\}$.

Analogous to the general Hausdorff distance in [11] we define the following

Defintion 2.6 The function $\boldsymbol{\Lambda}: [CB(\mathcal{M})]^3 \to \mathbb{R}^+$ is called general $b-$ Hausdorff distance if

$$\boldsymbol{\Lambda}(A, B, C) = max\{sup_{x \in A}\Lambda(x, B, C), sup_{x \in B}\Lambda(x, C, A), sup_{x \in C}\Lambda(x, A, B)\},$$

where,

$$\Lambda(x, B, C) = \Lambda(x, B) + \Lambda(B, C) + \Lambda(x, C),$$

$$\Lambda(x, B) = \inf\{\Lambda(x, y), y \in B\},$$

$$\Lambda(A, B) = inf\{\Lambda(a, b), a \in A, b \in B\}.$$

Directly, we obtain the following

Lemma 2.7: If $A, B \in CB(X)$ and $a \in A$, then $\forall \varepsilon > 0$, $\exists b \in B \ni$ $\Lambda(a, b, b) \leq \boldsymbol{\Lambda}(A, B, B) + \varepsilon$. And, if B is compact then $\Lambda(a, b, b) \leq \boldsymbol{\Lambda}(A, B, B)$.

Let C be a bounded subset of $\mathcal{M}$, the measure of non−compactness of C is

$$\chi(C) = inf\{r > 0: C \subset \bigcup_{i=1}^{n} C_i \text{ and } \delta(C_i) \leq r\}.$$

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Clearly, χ satisfies the following:

i- $\chi(\emptyset) = 0$

ii- $\chi(C) = 0 \Leftrightarrow$ C is relatively compact

iii- $0 \le \chi(C) \le diam(C)$

iv- $C \subseteq D \Rightarrow \chi(C) \le \chi(D)$

v- $\chi(C + D) \le \chi(C) + \chi(D)$

vi- $\chi(C) = \chi(\overline{C})$

vii- $\chi(\cup C_i) = max\{\chi(C_i)\}$

A set-valued mapping $T: \mathcal{M} \rightrightarrows 2^{\mathcal{M}}$ is called χ- condensing if for any bounded set $C \subset \mathcal{M}$, $T(C)$ is bounded and $\chi(T(C)) < \chi(C)$, $\chi(C) > 0$.

The space of all probability distribution functions is
$\Delta^+ = \{s: \mathbb{R}^* \to [0,1]: s$ is $left -$ continuous, non decreasing $on\ \mathbb{R}$, $s(0) = 0$, $s(+\infty) = 1\}$ and
D+ = $\{s \in \Delta+ : \ell^- s(+\infty) = 1\}$.
Here,
$\ell^- s(a_0) = \ell^- s(a_0) = \lim_{a \to a_0^-} s(a)$.
The space Δ^+ is partially ordered by ordering
$s \le r$ iff $(a) \le r(a), \forall a \in \mathbb{R}$.
The maximal element for $\Delta+$ is the probability distribution function

$$h(a) = \begin{cases} 0, & if\ a \le 0 \\ 1, & if\ a > 0 \end{cases} \quad \ldots \quad (1)$$

Definition 2.8: A mapping $\Delta : [0,1]^2 \to [0,1]$ is a continuous $t -$norm if Δ satisfies the following
(i) Δ is commutative and associative;
(ii) Δ is continuous;
(iii) $\Delta(a,1) = a, \forall a \in [0,1]$;
(iv) $\Delta(a,b) \le \Delta(c,d)$, whenever $a \le c$ and $c \le d$, $a,b,c,d \in [0,1]$.

Definition 2.9: A Menger probabilistic $b -$ metric space (briefly, $b -$ Menger space) is a triple $(\mathcal{M}, \Lambda, \Delta)$, where, $\emptyset \ne \mathcal{M}$ Δ is a continuous $t -$norm , and $\Lambda: \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to D^+$ such that, if $\Lambda_{x,y,z}$ denotes the value of Λ at the triple (x,y,z) , the following conditions hold for all $x,y,z,a \in \mathcal{M}$ and $\forall\, t,s \ge 0$

(1) $\Lambda_{x,y,z}(t) = 1$, iff $x = y = z$;
(2) $\Lambda_{x,y,z}(t) < 1$ iff $x \ne y$;
(3) $\Lambda_{x,y,z}(t) = \Lambda_{y,z,x}(t) = \Lambda_{z,x,y}(t) = \ldots$,
(4) $\Lambda_{x,y,z}(t + s) \ge \Delta(\Lambda_{x,a,a}(t), \Lambda_{a,y,z}(s))$, .

Definition 2.10: A $b -$ Menger space is called symmetric if $\Lambda_{x,y,y}(t) = \Lambda_{y,x,x}(t), \forall x,y \in \mathcal{M}$.

Definition 2.11: Let $(\mathcal{M}, \Lambda, \Delta)$ be a $b -$ Menger space.

i- A sequence $\{x_n\}$ in $\mathcal{M}$ is said to be

-Convergent to $x \in M$ if $\forall \varepsilon > 0$, $\lambda > 0$, $\exists N \in \boldsymbol{N}$ such that $\Lambda_{x,x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $m, n \ge N$.
-Cauchy sequence if, $\forall \varepsilon > 0$ and $\lambda > 0$, $\exists N \in \boldsymbol{N}$ such that $\Lambda_{x_n,x_m,x_l}(\varepsilon) > 1 - \lambda$ whenever $n,m,l \ge N$.

ii- $\mathcal{M}$ is complete if every Cauchy sequence in $\mathcal{M}$ converges to a point in $\mathcal{M}$.

iii- The strong $\lambda -$ neighborhood of x is
$N_x(\lambda) = \{q \in \mathcal{M} : \Lambda_{x,q,q}(t)(\lambda) > 1 - \lambda\}$
and neighborhood system for $\mathcal{M}$ is the union $\cup_{x \in V} N_x$ where $N_x = \{N_x(\lambda) : \lambda > 0\}$.

iv- The (ϵ, λ) – topology in $\mathcal{M}$ is introduced by the family of neighborhoods given by

$U_v(\epsilon, \lambda) = \{ u; T_{u,v,v}(\epsilon) > 1 - \lambda \}$.

If $t-$ norm Δ is continuous then $\mathcal{M}$ is a metrizable topology space, with respect to $(\epsilon,\lambda)-$ topology.

v- The probabilistic diameter of a subset A of $\mathcal{M}$ is

$D_C(t) = sup_{s<t}\ inf_{p,q,q\in A} T_{p,q,q}(s), t \in \mathbb{R}^+$,

and the set C is probabilistic bounded if and only if $sup_{t\in\mathbb{R}+} D_C(t) = 1$.

Analogously with Measure of non-compact set, we give

Definition 2.12: Let $\mathcal{M}$ be a $b-$ Menger space and $C \subset \mathcal{M}$ be a probabilistic bounded, the function $\chi_C : \mathbb{R} \to [0,1]$ is $\chi_C = \sup\{\epsilon > 0$, there is a finite family $\{C_j\}_{j\in J}$ in $\mathcal{M}$ such that $C = \bigcup_{j\in J} C_j$ and $D_{C_j}(u) \geq \epsilon, \forall j\in J\}$.

The χ_C function has the following properties:

i- $\chi_A(t) \geq D_A(t)$, $\forall t \in \mathbb{R}^+$

ii- $\emptyset \neq A \subset B \subset \mathcal{M} \Rightarrow \chi_A(t) \geq \chi_B(t)$, $\forall t \in \mathbb{R}^+$

iii- $\chi_{A\cup B}(t) = \{\chi_A(t), \chi_B(t)\}$, $\forall t \in \mathbb{R}^+ \chi_A(t) = \chi_{\bar{A}}(t), t \in \mathbb{R}^+$

iv- $\chi_A = h \Leftrightarrow A$ is precompact , where $h(x)$ as in (1)

Definition 2.13: Let $\subset \mathcal{M}$, K is probabilistic bounded , $T : K \to 2^{\mathcal{M}}$ and $T(K)$ is probabilistic bounded subset of $\mathcal{M} \ni \forall B \subset K$ and

$\chi_{T(B)}(t) \leq \chi_B(t), \forall t > 0$

implies that B is pre-compact then T is called a condensing mapping on K w.r.t. χ.

3. Fixed Points in general $b-$ metric spaces

Let Φ denoted the class of all function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ which satisfying the following conditions:

(1) φ is continuous ,
(2) φ is non- decreasing,
(3) $\varphi(t) < t. \forall t > 0$,and
(4) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, $\forall t \in \mathbb{R}^+$.

Thus $\varphi^n(0) = 0$ for each n and $\lim_{n\to\infty} \varphi^n(t) = 0\ \forall t > 0$.

Directly, we have the following Lemma

Lemma 3.1: If $\{u_n\}$ is a bounded sequence in $\mathcal{M}$ with constant bound K satisfying

$\Lambda(u_n, u_{n+1}, u_m) \leq \varphi^n(k), \forall m > n \in N$,

where $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\sum_{n=1}^{\infty} \varphi(t) < \infty, \forall t \in R^+$ then $\{u_n\}$ is Cauchy.

Theorem 3.2: Let $\mathcal{M}$ be a general $b-$ metric space and $T: \mathcal{M} \rightrightarrows K(\mathcal{M})$ be a set-valued mapping. if $\mathcal{M}$ is $T-$ orbitally complete and

$\boldsymbol{\Lambda}(Tx, Ty, z) \leq \varphi(\Lambda(x,y,z))$ ... (2) $\forall x, y, z \in \mathcal{M}$ with $x \notin T(x), y \notin T(y)$

for all $x, y, z \in \mathcal{M}$ with $x \notin T(x), y \notin T(y)$, where $\varphi \in \Phi$. Then T has a fixed point.

Proof: Let $x_0 \in \mathcal{M}$, define $\{x_n\}$ by $x_{n+1} \in Tx_n$, $n \geq 0$

The proof is divided into three steps: we must prove that

- $\Lambda(x_n, x_{n+1}, x_m) \leq \varphi^n(L)$ ... (3)

for any $m > n$, $L = \lim_{m\to\infty} \sum_{i=0}^{m} 2b^{m-i} \varphi^i(q)$, $q = max\{\Lambda(x_0, x_1, x_0), \Lambda(x_0, x_1, x_1)\}$.

- $\{x_n\}$ in (2) is bounded .
- $\lim_{n\to\infty} x_n = u \in Tu$

For first step, from (2)

$\Lambda(x_n,x_{n+1},x_m) \leq \boldsymbol{\Lambda}(Tx_{n-1},Tx_n , x_m)$
$\leq \varphi ((x_{n-1},x_n,x_m)), \ldots$ (4)
$n = 1,2, \ldots$

By induction,

$\Lambda (x_n,x_{n+1},x_m) \leq \varphi(\Lambda(x_{n-1}, x_n, x_m)) \leq \varphi^2(\Lambda(x_{n-2}, x_{n-1}, x_m))\ldots \leq \varphi^n(\Lambda(x_0, x_1, x_m))$ … (5)

By using Proposition (2.1) , definition of general $b-$ metric and (5) we get

$\Lambda(x_0, x_1, x_m) \leq b\, \Lambda (x_0, x_{m-1}, x_{m-1}) + b^2 \Lambda (x_1, x_{m-1}, x_{m-1}) + b^2\Lambda \left(x_m, x_{m-1,} x_{m-1}\right)$
$\leq b\Lambda(x_{0,} x_{1,} x_{m-1}) + b^2\varphi^{m-1} (\Lambda(x_0, x_1, x_0)) + b^2 \varphi^{m-1}(\Lambda(x_{0,} x_{1,} x_1))$

Since $q = max\{(x_{0,} x_{1,} x_0), \Lambda(x_{0,} x_{1,} x_1)\}$, then

$\Lambda (x_0, x_1, x_m) \leq \sum_{i=0}^{m} 2b^{m-i}\, \varphi^i$ (q)
$\leq L < \infty$

Substituting into (5) yield (3). For second step, for any integers $s \geq m \geq n$, there exists p and r such that

$\Lambda(x_n, x_m ,x_s) = \Lambda(x_n , x_{n+p}, x_{n+r})$

By similar argument, we have $\Lambda (x_n, x_m ,x_s) < \infty$. This showing the second step. Moreover,

Lemma (3.1) implies that $\{x_n\}$ is Cauchy, hence convergent to $u \in \mathcal{M}$.

For third step

$\Lambda(x_{n+1}, Tu, x_n) \leq \boldsymbol{\Lambda}(Tx_n, Tu ,x_n)$
$\leq \varphi (\Lambda(x_n, u ,x_n))$
$\Rightarrow (u,Tu ,u) = 0$, as $n \to \infty$, and hence $u \in Tu$.

As a special case of the above theorem, we give

Corollary 3.3: Let $\mathcal{M},\Lambda$ and T be in Theorem (3.2) such that

$\boldsymbol{\Lambda} (Tu,Tv,w) \leq \lambda\, \Lambda (u,v ,w)$, $u,v ,w \in \mathcal{M}$ with $u \notin T(u)$, $u \notin T(u)$ … (6)

where $0 \leq \lambda<1$. Then T has a fixed point.

For $\chi-$ condensing, we need the following lemma:

Lemma 3.4: Let $\mathcal{M},\Lambda$ and $\mathcal{M} \rightrightarrows 2^{\mathcal{M}}$ be $\chi-$condensing mapping. If $\mathcal{M}$ an $T-$ orbitally bounded and complete Then $\overline{O(u)}$ is compact, $\forall u \in \mathcal{M}$.

Proof: Let $u \in \mathcal{M}$ and $M = \{u_n\}$, where $u_n = T^n u$. Then

$M = \{u\} \cup \{ Tu ,T^2u ,\ldots\}$
$= \{u\} \cup T(M)$

If M is not pre-compact, then $\chi (M) = \chi (T(M)) < \chi (M)$, which is a contradiction.

Therefore, $\overline{M} = \overline{O(u)}$ is compact, since $\overline{M}$ is complete.

Theorem 3.5: Let $T : \mathcal{M} \rightrightarrows CB(\mathcal{M})$ be an orbitally continuous $\chi-$ condensing mapping on a $T-$ orbitally bounded complete general $b-$ metric space $\mathcal{M}$. If

$\boldsymbol{\Lambda}(Tx,Ty ,z)< \Lambda(x,y ,z), \ldots$ (7)

$\forall x,y ,z \in \mathcal{M}$, x $\notin$ T(x) ,y $\notin$ T(y)

then T has a fixed point .

Proof: Suppose that x_0 in $\mathcal{M}$, then by Lemma(3.4) $M = \overline{O(x_0)}$ is compact. Since T is continuous on M then $\boldsymbol{\Lambda} (Tx,Ty ,z)$ and $\Lambda(x,y ,z)$ are bounded.

Define the well-defined

$S : M^3 \to R^+$ by $S(x,y,z) = \frac{\Lambda(Tx,Ty,z)}{\Lambda(x,y,z)}$ $\forall x,y ,z \in M$

By the continuity of T, S is continuous. The compact of product sets is compact implies that M^3 is compact. So, S attains its maximum at $(u,v,w) \in M^3$. Call the value C from (3.1) , $0 < C < 1$. By definition of S we get

$$\frac{\Lambda(Tx,Ty,z)}{\Lambda(x,y,z)} = S(x,y,z) \leq S(u,v,w) = C$$

for all $x,y,z \in \mathcal{M}$ with $x \notin T(x), y \notin T(y)$, Now, the result follows from Corollary (3.3). This completes the proof.

Define

$\delta(x,y,z) = \delta(O(x) \cup O(y) \cup O(z))$

$\bar{\delta}(x,y,z) = \delta(\overline{O(x)} \cup \overline{O(y)} \cup \overline{O(z)})$

Theorem 3.6: : Let $T_i : \mathcal{M} \to \mathcal{M}$ are commuting orbitally continuous $\chi-$ condensing mappings , $i = 1,2,3$, such that

$\Lambda(T_1x, T_2y, T_3z) < \bar{\delta}(x,y,z)$, $\forall x,y,z \in \mathcal{M}$ and $T_1x \neq T_2y \neq T_3z$ (8)

If $\mathcal{M}$ is orbitally bounded and complete. Then $\exists! u \in \mathcal{M}, T_i(u) = u, \forall i$.

Proof: Let $O_3(x) = \{ T_1^k \, T_2^m \, T_3^n : k,m,n = 0,1,2 \ldots \}$ be the orbit of x by T_1, T_2, T_3 .

Since, $\chi(O_{T_1}(x)) = max\{\chi(x), \chi(O_{T_1}(T_1 x))\} = \chi(T_1(O_{T_1}(x)))$ and T_1 is condensing $\Rightarrow O_{T_1}(x)$ pre-compact. Similarly for $O_3(x)$;

$\chi(O_3(x)) = max\{\chi(O_{T_1}(x)), \chi(O_{T_2}(O_{T_1}(x))), \chi(O_{T_3}(O_{T_2}(O_{T_1}(x))))\}$.

Therefore, the condition of condensing$\Rightarrow$ $O_3(x)$ is totally bounded, $\Rightarrow$ pre-compact.

Now, if $M_0 = \overline{O_3(x)}$, so, M_0 is compact . Clearly $T_i(M_0) \subset M_0$, $i = 1,2,3$. Now, let

$M_i = \cap_{n=1}^{\infty} T_i^n(M_{i-1})$, $i = 1,2,3$

Thus M_i is T_i-invariant, $i = 1,2,3$. The finite intersection property assures that M_i is non-empty compact subsets, $i = 1,2,3$.

Suppose $u \in M_1$, there exist $x_n \in T_1^{n-1}(M_0)$ such that $T_1(x_n) = u$, $n = 1,2,\ldots$

Thus a subsequence, say also (x_n) converges to $v \in M_0$.

Since $\{x_{n+1}, x_{n+2}, \ldots\} \subset T_1^n(M_0)$ and $T_1^n(M_0) \subset M_0 \Rightarrow v \in T_1^n(M_0)$, $n = 1,2,\ldots$

We have $v \in M_1$ and $T_1(v) = u$ $\Rightarrow T_1(M_1) = M_1$.

The properties of M_2 and $T_1 \Rightarrow T_1(M_2) = M_2$, $T_1(M_3) = M_3$.

Similarly, $T_i(M_2) = M_2$, $T_i(M_3) = M_3$,
i =1, 2,3.

Now, We claim that M_3 singleton, and $M_3 = \{x\}$, then x is the singleton fixed point of T_1, T_2, T_3 .

If not, $\delta(M_3) > 0$, the compactness of $M_3 \Rightarrow \exists a,b,c \in M_3, a \neq b \neq c \ni \delta(M_3) = \Lambda(a,b,c)$. This implies that

$a \in T_1(a_1), b \in T_2(b_1), c \in T_3(c_1)$, for $a_1, b_1, c_1 \in M_3$, hence, by (8), we get

$\overline{O_{T_1}(a_1)} \cup \overline{O_{T_2}(b_1)} \cup \overline{O_{T_3}(c_1)} \subset \overline{M_3} = M_3$,

$0 < \delta(M_3) = \Lambda(a,b,c) < \delta(\overline{O_{T_1}(a_1)} \cup \overline{O_{T_2}(b_1)} \cup \overline{O_{T_3}(c_1)}) \leq \delta(M_3)$

which is a contradiction. So x is unique.

Note that, it is possible to modify Theorem (3.6) for finite commuting continuous condensing mappings. Also, the composition of two compact (moreover, $\chi -$ condensing) mappings is compact ($\chi -$ condensing), implies that

Corollary 3.7: Let $\mathcal{M}$ be as theorem (3.6) and $T : \mathcal{M} \to \mathcal{M}$ be an orbitally continuous compact mapping such that $\Lambda\,(T^r u,\, T^s v,\, T^t w) < \delta(u\,,v,w)$ … (9)

$\forall u\,,v,w \in \mathcal{M}$ with $\neq Tu$, $v \neq Tv\,,w \neq Tw$ and r , s , and t are fixed positive integers . Then T has a unique fixed point in $\mathcal{M}$.

Proof: Fix $T_1 = T^r$, $T_2 = T^s$, $T_3 = T^t$, and then apply Theorem (3.6).

Corollary 3.8: Let $\mathcal{M}$ and T be as Corollary (3.7) such that

$$\Lambda(Tu\,,Tv,Tw) < max\begin{Bmatrix} \Lambda(u\,,v,w), \Lambda(u\,,Tv\,,w), \\ \Lambda(v\,,Tv\,,w), \Lambda(u\,,Tv\,,w), \\ \Lambda(v,Tu\,,w) \end{Bmatrix}$$

… (10)

$\forall u\,,v,w \in \mathcal{M}$ with $u \neq Tu\,,v \neq Tv, or\; w \neq Tw$. Then T has a unique fixed point p in $\mathcal{M}$. Moreover T is continuous at p.

Proof: The inequality (10) implies that $\Lambda(Tu\,,Tv,Tw) < \delta(u\,,v,w)$and the existence and uniqueness of a fixed point p follow from corollary (3.7). For continuity, let $\{w_n\} \subset \mathcal{M}$ with $w_n \neq$ p f or each n and $\lim_{n\to\infty} w_n$ = p. From (10)

$\Lambda(p\,,p,Tw_n) = \Lambda\,(Tp\,,Tp\,,Tw_n)$

$< \Lambda\,(p,Tp\,,w_n)$ …(11)

Taking the limit as $n \to \infty$ implies that T is continuous at p.

The following example shows that the condensing conditions in (8) and (9) are essential.

Example 3.9: Let $\mathcal{M}$ =$\boldsymbol{N}$, $\Lambda(m.n.k) =$

$$\begin{cases} 0\;, & m = n = k \\ \frac{r+1}{n+1}, & n < m,k,r \text{ is any positive real number} \end{cases}$$

Then($\mathcal{M}$, Λ) is complete general $b-$metric with $b = 1$. Consider $T_1(n) = T_2(n) = T_3(n) = n + 1, \forall n$ which have no fixed point in $\mathcal{M}$. Easily, one checks that T_1, T_2, T_3 satisfy conditions (8), (9) except the condensing property, since. $\delta(\mathcal{M})$= $\delta(\mathcal{M}\backslash\{1\}) = r$.

Theorem 3.10: Let $T : \mathcal{M} \to \mathcal{M}$ be a $\chi -$ condensing orbitally continuous mapping and $\mathcal{M}$be a complete bounded general $b -$ metric space. Let $\in \mathcal{M}$. If (9) holds on $\overline{O(a)}$, then T has a unique fixed point $p \in \overline{O(a)}$, and $\lim_{n\to\infty} T_n x = p, \forall\, x \in \overline{O(a)}$.

Proof: Lemma (3.4) and hypotheses $\Rightarrow T$ is compact. Now apply corollary (3.7).

Corollary 3.11: Let $\mathcal{M}$ as in Theorem (3.9) and $T : \mathcal{M} \to \mathcal{M}$ be a continuous χ −condensing mapping satisfying (10) for all $x, y, z \in \mathcal{M}$ with $x \neq Tx$, $y \neq Ty$, or $z \neq Tz$. Then T has a unique fixed point $p \in \mathcal{M}$.

Theorem 3.12: Let $T : \mathcal{M} \to \mathcal{M}$ be a mapping on a general b −metric space $\mathcal{M}$. Suppose that $\exists a \in \mathcal{M} \ni \overline{O(a)}$ is bounded and complete. Suppose that T is continuous and χ − condensing on $\overline{O(a)}$ and satisfies (7) ,$\forall x, y, z \in \overline{O(x)}$, and $x \neq Tx$, $y \neq Ty$, $z \neq Tz$. Then T has a fixed point in $\overline{O(a)}$.

Proof: If $\exists\, n \in \mathbf{N} \ni T^n(a) = T^{n+1}(a) \Rightarrow T$ has a fixed point in $\overline{O(a)}$ (since Lemma (3.4) implies that $\overline{O(x)}$ is compact). Assume that $T^n(a) \neq T^{n+1}(a)$, $\forall n$.Let u be an accumulation point of $\overline{O(a)}$ and $u \neq Tu$,.Then T satisfies condition (7) $\forall x, y, z \in \overline{O(a)}$. Therefore, by Corollary (3.7) , T has a unique fixed point $p \in \overline{O(a)}$. This contradicts the assumption that $u \neq Tu$. Hence $u = T\,u$, for some accumulation point $u \in \overline{O(a)}$.

4. Fixed Points in $b-$ Menger Spaces

Consider

$$\omega = \left\{ \begin{matrix} \omega : R^+ \to R^+ : \omega \text{ is continuous} \\ \omega(0) = 0, \\ \omega(a+b) \geq \omega(a) + \omega(b), \\ a, b \in \mathbb{R}^+ \end{matrix} \right\},$$

If Δ_f is an Archimedean t – norm with the additive generator f and Δ is $t-$ norm $\ni \Delta \geq \Delta_f$ then by

$\Lambda_{\omega_1,\omega_2,\omega_2}(p, q, q) = sup\{u; u \geq 0,$ $\omega_1(u) \leq f \circ \Lambda_{p,q,q}(\omega_2(u)) p, q \in \mathcal{M}$; $\omega_1, \omega_2 \in \omega\}$

a metric on $b-$ Menger space $(\mathcal{M}, \Lambda, \Delta)$ is defined and $\Lambda_{\omega_1,\omega_2,\omega_2}$ induces the (ϵ, λ) - topology.

Theorem 4.1 : Let $(\mathcal{M}, \Lambda, \Delta)$ be a $b-$ Menger space $T: \mathcal{M} \rightrightarrows CB(\mathcal{M})$ be a closed mapping and $\emptyset \neq K \subset \mathcal{M}$ be a probabilistic bounded such that $T(K)$ is probabilistic bounded. If $\mathcal{M}$ is symmetric and:

a) there exist $\omega_1, \omega_2 \in \omega$ and $f: [0,1] \to [0, b]$ is a decreasing function, $b > 0 \ni$ $inf_{x \in K}\ inf_{y \in Tx}\ sup\{u;\ u \geq 0, \omega_1(u) \leq f \circ \Lambda_{x,y,y}(\omega_2(u))\} = 0$.

b) T is $\chi-$ condensing on K.

Then T has a fixed point.

Proof: The condition (a) follows that $\forall n \in N, \exists x_n \in K$ and $y_n \in Tx_n$,

(1) $sup\{u;\ u > 0,\ \omega_1(u) \leq f \circ \Lambda_{x_n, y_n, y_n}(\omega_2(u))\} < 2^{-n}$.

and then,

(2) $\omega_1(2^{-n}) > f \circ \Lambda_{x_n, y_n, y_n}(\omega_2(2^{-n}))$

We shall prove that (2) implies that $\forall\epsilon > 0$, $\lim_{n\to\infty} \Lambda_{x_n, y_n, y_n}(\epsilon) = 1$ $\Rightarrow \forall\epsilon > 0$ and $\lambda \in (0,1)$,$\exists n_0(\epsilon,\lambda) \in N$ so that $\Lambda_{x_n, y_n, y_n}(\epsilon) > 1-\lambda$, $\forall n \geq n_0(\epsilon,\lambda)$. Since ω_1 is continuous and $\omega_1(0) = 0$, $\exists n_0(b) \in N \Rightarrow \omega_1(2^{-n}) < b$, $\forall n \geq n_0(b) \Rightarrow$ for $n \geq n_0(b)$, from (2) it follows that, $f^{-1}[\omega_1(2^{-n})] < \Lambda_{x_n, y_n, y_n}(\omega_2(2^{-n}))$
Let $n_1(\epsilon,\lambda) \in N$ such that $\omega_1(2^{-n}) < f(1-\lambda)$, $\omega_2(2^{-n}) < \epsilon$, $n \geq n_1(\epsilon,\lambda)$.
then

$\Lambda_{x_n, y_n, y_n}(\epsilon) \geq \Lambda_{x_n, y_n, y_n}(\omega_2(2^{-n})) > f^{-1}[\omega_1(2^{-n})] > 1-\lambda$

For every $n \geq n_2(\epsilon,\lambda,b) = max\{n_0(b), n_1(\epsilon,\lambda)\}$, which means that $\lim_{n\to\infty} \Lambda_{x_n, y_n, y_n}(\epsilon) = 1$.

From (b), we obtain that $\{x_n ; n \in N\}$ is compact $\Rightarrow$there exists $\{x_{n_k}\}_{k\in N}$ a convergent sub-sequence. If $z = \lim_{k\to\infty} x_{n_k} \Rightarrow \lim_{k\to\infty} y_{n_k} = z$. Since $y_{n_k} \in T_{x_{n_k}} \Rightarrow z \in Tz$, by closeness of T.

For $u > 0$, we need to prove that $\chi_{\{x_n ; n\in N\}}(u) = \chi_{\{y_n ; n\in N\}}(u))$.
Let $\epsilon \in (0, u)$ and $\chi_{\{y_n ; n\in N\}}(u-\epsilon) > 0$. It is enough to prove that

$\chi_{\{y_n ; n\in N\}}(u-\epsilon) \leq \chi_{\{x_n ; n\in N\}}(u)$.

Let $r < \chi_{\{y_n ; n\in N\}}(u-\epsilon) \Rightarrow$there exists $A_1, A_2, \dots, A_n \subset \mathcal{M}$ such that $\{y_n ; n \in \mathbf{N}\} = \cup_{j=1}^{n} A_j$, $D_{A_j}(u-\epsilon) \geq r, \forall j \in \{1,2,\dots,n\}$.
Thus, $inf_{x,y\in A_j} \Lambda_{x,y,y}(u-\epsilon) > r$ and so $\Lambda_{x,y,y}(u-\epsilon) > r$, $\forall x,y \in A_j$.
Let $p \in (0,r)$ and $q \in (0,1)$ such that $1 \geq u, w > 1-q \Rightarrow \Delta(u, \Delta(r,w)) > r-p$.

Since $\Delta(1, \Delta(r,1)) = 1$and the mapping $(u,w) \to \Delta(u, \Delta(r,w))$ is continuous such a number q exists. For $j \in \{1,2,\dots,n\}$, $B_j = \{ z ; \Lambda_{z,y,y}(\frac{\epsilon}{4}) > 1-q$, for some $y \in A_j \}$.
If $n_1(\epsilon,q) \in \mathbf{N}$ is such that $\Lambda_{x_n, y_n, y_n}(\frac{\epsilon}{4}) > 1-q$, $\forall n \geq n_1(\epsilon,q)$
then, $\{x_n ; n \geq n_1(\epsilon,q)\} \subseteq \cup_{j=1}^{n} B_j$.
We shall prove that

$sup_{s<u}\ inf_{x,y\in B_j} \Lambda_{x,y,y}(s) \geq r-p$.

If $x \in B_j$ and $y \in B_j$, then there exists $x^* \in T_j$ and $y^* \in A$ so that $\Lambda_{x,x^*,x^*}(\frac{\epsilon}{4}) > 1-q$, $\Lambda_{y,y^*,y^*}(\frac{\epsilon}{4}) > 1-q$.
Since $\Lambda_{x^*,x^*,y^*}(u-\epsilon) \geq r$ we have that

$\Lambda_{x,y,y}(u-\frac{\epsilon}{2}) \geq \Delta(\Lambda_{x,x^\sim,x^\sim}(\frac{\epsilon}{4}), \Delta(\Lambda_{x^*,y^*,y^*}(u-\epsilon), \Lambda_{x^*,y,y}(\frac{\epsilon}{4})))$
$\geq \Delta(\Lambda_{x,x^*,x^*}(\frac{\epsilon}{4}), \Delta(r, \Lambda_{x^*,y,y}(\frac{\epsilon}{4}))) > r-P$

which implies

$sup_{s<u}\ inf_{x,y\in B_j} \Lambda_{x,y,y}(s) \geq r-P$.

and so

$\chi_{\{x_n ; n\geq n_1(\epsilon,q)\}}(u) \geq r-p$.

Since

$\chi_{\{x_n ; n\in N\}}(u) = \chi_{\{x_n ; n\geq n_1(\epsilon,q)\}}$ (u),

we obtain that $\chi_{\{x_n ; n\in N\}}(u \geq$ r).
then $\chi_{\{y_n ; n\in N\}}(u \leq \chi_{\{x_n ; n\in N\}}(u)$, $\forall u > 0$. Similarly, $\chi_{\{x_n ; n\in N\}}(u) \leq \chi_{\{y_n ; n\in N\}}(u)$.
So, we proved that

$\chi_{\{y_n ; n\in N\}}(u) = \chi_{\{x_n ; n\in N\}}(u)$, $\forall u > 0$.

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مبادئ للنقطة الصامدة في فضاءات - المترية المعممة وفضاءات - منكر المحتملة

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المستخلص

في هذا العمل تم البرهنة على ثلاثة مبادئ عامة حول ايجاد النقطة الصامدة النقطة الصامدة المشتركة في انماط من الفضاءات - المترية المعممة حيث تضمنت وجود نقطة صامدة لتطبيقات متعددة القيم في تعميم لفضاء -المتري و وجود نقطة صامدة مشتركة لثلاثة تطبيقات المكثفة المستمرة مساريا والمتبادلة وكذلك نتيجة للنقطة الصامدة للتطبيقات المكثفة المتعددة القيم المعرفة على مجموعة مقيدة محتملة جزئية من - فضاء منكر المحتملة.