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On the third natural representation module *M* (n-3,3) of the permutation groups

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Abstract:

The main purpose of this work is to propose the third natural representation M (n-3,3)of the symmetric groups over a field **F** and prove that M (n-3,3)is split iff p does not divide $\frac{n(n-1)(n-2)}{6}$.

Keywords: symmetric group, group algebra FS_n , FS_n −module, Spechet module, exact sequence.

Mathematics Classification :20C30.

1. Introduction

In 1935, W .Specht introduced tableau correspondence polynomials ,known Specht polynomials, that proved how a given polynomial can be written as a linear combination of other polynomials.(see[Kerber:2004]).This was the results of Specht study on representation theory of symmetric groups, after he faced the problem when the symmetric group acts, in natural way, as a tableaux. However, the result of permutation a standard tableau can be a nonstandard tableau and this nonstandard tableau can be written as a linear combination of Specht polynomials. On the other hand, the representation with partition $\mu = (n-1,1)$ for a positive integer n, was first studied by H.K.Farahat in 1962 [Farahat:1962]. This type of representation is called the natural representation. Seven years later, M.H.Peel introduced in [Peel:1969] and [Peel:1971] the second representation of the symmetric groups and renamed Farahat natural representation by the first natural representation .In Peel's representation, the partition was then $\mu = (n-2,2)$ for a positive integer n. He also represented the r^{th}-Hook representation where the partition $\mu = (n-r, 1^r)$, for any $r \geq 1$.

For the author's knowledge, no one has studied the 3rd-natural representation so far . Therefore, this work represent of the symmetric groups over a field **F** and $x_1, x_2, \dots, x_n$ defined to be linearly independent commuting variables over **F**.

2. Preminaries

Definition 1: Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set, then the symmetric group on X is the group whose elements "permutations" can be viewed as a bijective function from $\mathbf{F}[x_1, x_2, \dots, x_n]$ onto $\mathbf{F}[x_1, x_2, \dots, x_n]$.The symmetric group on X is denoted by $\mathbf{S_X}$ or $\mathbf{S_n}$. Then $\mathbf{FS_n}$ is called the group algebra of the symmetric group $\mathbf{S_n}$ with respect to addition of functions, composition of functions and product of functions by scalars [Joyce:2008].

Definition 2: Let n be a natural number then the sequence $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ is called a partition of n if $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0$ and $\mu_1 + \mu_2 + \cdots + \mu_l = n$, the set $D_\mu = \{(i,j) | i = 1,2, \dots, l; 1 \leq j \leq \mu_i\}$ is called $\mu-$diagram and any bijective function $t : D_\mu \to \{x_1, x_2, \dots, x_n\}$ is called a μ–tableau. A μ–tableau may be thought as an array consisting of l rows and μ_1 columns of distinct variables $t((i,j))$ where the variables appear in the first μ_i positions of the i^{th} row and each variable $t((i,j))$ appears in the i^{th} row and the j^{th} column $((i,j)$-position) of the array.$t((i,j))$ will be denoted by $t(i,j)$ for each $(i,j) \in D_\mu$. The set of all μ-tableaux will be denoted by T_μ. i.e $T_\mu = \{t | t \; is \; a \; \mu - tableau\}$. Then the function $h: T_\mu \to F[x_1, x_2, \dots, x_n]$ which is defined by $h(t) = \prod_{i=1}^{l} \prod_{j=1}^{\mu_i} (t(i,j))^{i-1}$, $\forall\, t \in T_\mu$ is called the row position monomial function of T_μ, and for each μ–tableau t, $h(t)$ is called the row position monomial of t .So $M(\mu)$ is the cyclic FS_n −module generated by $h(t)$ over FS_n.[Ellers:2007]

3.The Third Natural Representation of S_n

In the beginning, we determine some denotations which we need them in this paper.

1. Let $\sigma_1(n) = \sum_{i=1}^{n} x_i$.

2. Let $\sigma_2(n) = \sum_{1 \leq i < j \leq n} x_i x_j$.

3. Let $\sigma_3(n) = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k$.

4. Let $C_l(n) = x_l\,(\sigma_2(n) - \sum_{\substack{j=1 \\ j \neq l}}^{n} x_l\, x_j)$; $l = 1,2, \dots, n$. Then $\sum_{i=1}^{n} c_i\,(n) = \sigma_3(n)$ and $dim_F(F\sigma_1(n))$ $=dim_F(F\sigma_2(n))$= $dim_F(F\sigma_3(n))$ = 1. $F\sigma_1(n), F\sigma_2(n)$ and $F\sigma_3(n)$ are all FS_n-modules, since $\tau\sigma_k(n) = \sigma_k(n)\ \forall\, k = 1,2,3$.

5. Let $u_{ij}(n) = C_i(n) - C_j(n)$; $i,j = 1,2, \dots, n$.

We denote $\bar{V}$ to be the FS_n-modules generated by $C_1(n)$ over FS_n and $\bar{V}_0$ to be the FS_n-submodule of $\bar{V}$ generated by $u_{12}(n)$ over S_n .

Definition3.1: The FS_n-module $M(n-r,r)$ defined by

$$M(n-r,r) = FS_n x_1 x_2 \dots x_r \quad ; \quad n \geq r$$

is called r^th^- natural representation module of S_n over F.

Lemma3.2: The set $B(n-3,3) = \{x_i x_j x_l : 1 \leq i < j < l \leq n\}$is a F-basis of $M(n-3,3)$ and $dim_F M(n-3,3) = \binom{n}{3}; n \geq 3$.

Proof: Clear

Theorem3.3: The set $B_0(n-3,3) = \{x_i x_j x_l - x_1 x_2 x_3 : 1 \leq i < j < l \leq n, (i,j,l) \neq (1,2,3)\}$ is a F-basis of $M_0(n-3,3)$and $dim_F M_0(n-3,3) = \binom{n}{3} - 1$; $n \geq 3$.

Proof: Since $M_0(n-3,3) = \{ \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l :$ $\sum_{1 \leq i < j < l \leq n} k_{ijl} = 0$ and $k_{ijl} \in F\}$, we get that $B_0(n-3,3) \subset M_0(n-3,3)$.To prove $B_0(n-3,3)$ generates $M_0(n-3,3)$ over F .

Let $x \in M_0(n-3,3)$.$\Rightarrow x = \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l$; $\sum_{1 \leq i < j < l \leq n} k_{ijl} = 0$

$$\Rightarrow x = \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l - 0.\, x_1 x_2 x_3$$

$$\Rightarrow x = \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l - \Big(\sum_{1 \leq i < j < l \leq n} k_{ijl}\Big) x_1 x_2 x_3$$

$$\Rightarrow x = \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l - \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_1 x_2 x_3$$

$\Rightarrow x = \sum_{1 \leq i < j < l \leq n} k_{ijl}\, (x_i x_j x_l - x_1 x_2 x_3)$ with the term 123 excluded from the summation since $k_{ijl}(x_1 x_2 x_3 - x_1 x_2 x_3) = 0$.Hence $B_0(n-3,3)$ generates $M_0(n-3,3)$ over F .Moreover $B_0(n-3,3)$ is linearly independent since if

$$\sum_{1 \leq i < j < l \leq n} k_{ijl}\, (x_i x_j x_l - x_1 x_2 x_3) = 0 \Rightarrow$$

$$\sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l - \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_1 x_2 x_3 = 0$$

$\Rightarrow \sum_{1 \leq i < j < l \leq n} k_{ijl}\, x_i x_j x_l = 0$,where k_{123} = - $\sum_{1 \leq i < j < l \leq n} k_{ijl}$ with $\sum_{1 \leq i < j < l \leq n} k_{ijl} = 0$ and $(i,j,l) \neq (1,2,3)$. By lemma (3.2) we have $B(n-3,3)$ is linearly independent. Thus we get $k_{ijl} = 0\ \forall i,j,l ; 1 \leq i < j < l \leq n$. Hence $B_0(n-3,3)$ is a F-basis of $M_0(n-3,3)$and $dim_F M_0(n-3,3) = \binom{n}{3} - 1$; $n \geq 3$.■

Theorem3.4: The set $B = \{C_i(n) | i = 1,2,\ldots,n\}$ is a F-basis for $\bar{V}(n) = FS_nC_1(n)$.

Proof: Let $\tau_i = (x_1x_i) \in S_n; 1 < i \leq n$. Then $\tau_i(C_1(n)) = C_i(n); i = 1,2,\ldots,n$.

Thus $C_i(n) \in \bar{V}(n); i = 1,2,\ldots,n$ .Hence B $\subset \bar{V}(n)$.

Now if $w \in \bar{V}(n) .\Longrightarrow w = \sum_{i=1}^{(n-1)!}\sum_{j=1}^{n} k_{ij}\tau_{ij}C_1(n)$ where $\tau_{ij} \in S_n, k_{ij} \in F$ and $\tau_{ij}(x_1) = x_j$, which implies that $\tau_{ij}(C_1(n)) = C_j(n); j = 1,2,\ldots,n$.

$\Longrightarrow w = \sum_{i=1}^{(n-1)!}\sum_{j=1}^{n} k_{ij}\tau_{ij}C_1(n)$

$=\sum_{j=1}^{n}(\sum_{i=1}^{(n-1)!} k_{ij})C_j(n) = \sum_{j=1}^{n} d_jC_j(n)$

,where $d_j = \sum_{i=1}^{(n-1)!} k_{ij}$ Hence B generates $\bar{V}(n) = FS_nC_1(n)$ over F.

If $\sum_{i=1}^{n} k_iC_i(n) = 0. \Longrightarrow k_1C_1(n) + k_2C_2(n) + \cdots + k_nC_n(n) = 0$.

$\Longrightarrow k_1 + k_2 + \cdots + k_n = 0$ since $C_l(n) = \sum_{1\leq i<j<l\leq n} x_ix_jx_l$.Thus B is a linearly independent. Therefore B is a basis of $\bar{V}(n)$ and $dim_F\bar{V}(n) = n$.

■

Theorem3.5: $\bar{V}(n) = FS_nC_1(n)$ and $M(n-1,1)$ are isomorphic over FS_n.

Proof: Let $\varphi : M(n-1,1) \rightarrow \bar{V}(n)$ be defined as follows:

$\varphi(x_i) = C_i(n)$; $i = 1,2,\ldots,n$. Then for each $\tau = (x_ix_j) \in S_n$ such that $\tau(x_i) = x_j$ we get that $\varphi(\tau x_i) = \varphi(x_j) = C_j(n) = \tau C_i(n) = \tau\varphi(x_i)$ .Hence φ is a FS_n-homomorphism .Also $y = \sum_{i=1}^{n} k_iC_i(n)$ for any $y \in \bar{V}$.Thus for all $y \in \bar{V}$, $\exists\ w = \sum_{i=1}^{n} k_ix_i \in M(n-1,1)$ such that $\varphi(w) = \varphi(\sum_{i=1}^{n} k_ix_i)$ $= \sum_{i=1}^{n} \varphi(k_ix_i) = \sum_{i=1}^{n} k_i\varphi(x_i) = \sum_{i=1}^{n} k_iC_i(n) = y$. Hence φ is an epimorphism.

Thus $dim_F\, ker\varphi = dim_F M(n-1,1) - dim_F\bar{V} = n - n = 0 .\Longrightarrow ker\varphi = 0$. Then φ is a monomorphism .

Thus φ is a FS_n – isomorphism. Hence $M(n-1,1)$ and $\bar{V}$ are isomorphic over FS_n.

Theorem3.6: If p does not divides n , then $\bar{V}(n) = \bar{V}_0(n) \oplus F\sigma_3(n)$.

Proof : From Theorem (3.5) we have a FS_n– isomorphism $\varphi : M(n-1,1) \rightarrow \bar{V}(n)$ such that $\varphi(x_i) = C_i(n)$; $i = 1,2,\ldots,n$.

And since $M_0(n-1,1) = FS_n(x_2 - x_1) \subset M(n-1,1)$, then $\psi = \varphi|_{M_0(n-1,1)}$ is a FS_n – isomorphism .Thus $\bar{V}_0(n)$ and $M_0(n-1,1)$ are isomorphic over FS_n which is irreducible submodule over FS_n when p does not divides n and $\sigma_3(n) \notin \bar{V}_0(n)$ when p does not divide n since the sum of the coefficients of the $C_i(n)$ in $\sigma_3(n)$ is n. Hence $\bar{V}_0(n) \cap F\sigma_3(n) = 0$, $F\sigma_3(n) \subset \bar{V}(n)$ and $\bar{V}_0(n) \subset \bar{V}(n)$ .But $dim_F\bar{V}_0(n) + dim_F F\sigma_3(n) = n - 1 + 1 = n = dim_F\bar{V}(n)$.

Hence $\bar{V}_0(n) \oplus F\sigma_3(n) = \bar{V}(n)$ when p does not divides n .

Proposition 3.7 : If p does not divides n ,then $\bar{V}$ has the following two composition series

$0 \subset \bar{V}_0(n) \subset \bar{V}(n)$ and $0 \subset F\sigma_3(n) \subset \bar{V}(n)$.

Proof : Since p does not divides n , then by Theorem (3.6) we have

$\bar{V} = \bar{V}_0(n) \oplus F\sigma_3$, and $\bar{V}_0(n)$ is irreducible submodule when p does not divide n .Hence $\frac{\bar{V}}{F\sigma_3(n)} = \frac{\bar{V}_0(n)\oplus F\sigma_3(n)}{F\sigma_3(n)} \simeq \bar{V}_0(n)$.Thus $\frac{\bar{V}}{F\sigma_3(n)}$ is irreducible module when p does not divide n. Since $dim_F F\sigma_3(n)$ =1. Then $F\sigma_3(n)$ is irreducible submodule over FS_n . But $\frac{\bar{V}}{\bar{V}_0(n)} = \frac{\bar{V}_0(n)\oplus F\sigma_3(n)}{\bar{V}_0(n)} \simeq F\sigma_3(n)$. Therefore $\frac{\bar{V}}{\bar{V}_0(n)}$ is irreducible module over FS_n . Thus we get the following two composition series

$0 \subset \bar{V}_0(n) \subset \bar{V}$ and $0 \subset F\sigma_3(n) \subset \bar{V}$.

Theorem 3.8: The following sequence

$$O \rightarrow M_0(n-3,3) \xrightarrow{i} M(n-3,3) \xrightarrow{f} F \rightarrow O \quad \ldots(1)$$

over a field F is split iff p does not divide $\frac{n(n-1)(n-2)}{6}$.

Proof: If p does not divide $\frac{n(n-1)(n-2)}{6}$. For any $k \in F$ we have $f(\sum_{1\leq i<j<l\leq n} k_{ijl}\, x_ix_jx_l) = \sum_{1\leq i<j<l\leq n} k_{ijl} = k$ .Hence f is on to. Moreover

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$kerf = \{ \sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l :$

$f \left(\sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l \right) = 0 \} =$

$\{ \sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l : \sum_{1\le i<j<l\le n} k_{ijl} = 0 \} =$

$M_0(n-3,3)= Im\, i$ .Hence the sequence (1) is an exact sequence.

So we can defined a function $h: F \to M(n-3,3)$ by $h(k) = \frac{6k\sigma_3(n)}{n(n-1)(n-2)}$ which is a FS_n – homomorphism since

$\sum_{\tau\in S_n} r\tau h(k) = \sum_{\tau\in S_n} r\tau \left(\frac{6k\sigma_3(n)}{n(n-1)(n-2)}\right) = \sum_{\tau\in S_n} \frac{6rk\tau\sigma_3(n)}{n(n-1)(n-2)} = \sum_{\tau\in S_n} \frac{6rk\sigma_3(n)}{n(n-1)(n-2)}$

$=\sum_{\tau\in S_n} rh(k) = h(\sum_{\tau\in S_n} rk) = h(\sum_{\tau\in S_n} r\tau(k))$ .And since

$fh(k) = f\left(\frac{6k\sigma_3(n)}{n(n-1)(n-2)}\right) = \frac{6k}{n(n-1)(n-2)} f(\sigma_3(n)) = \frac{6k}{n(n-1)(n-2)} f\left(\sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l \right) = \frac{6k}{n(n-1)(n-2)} \cdot \frac{n(n-1)(n-2)}{6} = k$ .Hence $fh = I$ on F. Thus the sequence(1) is split.

Now assume the sequence (1) is split.

Then there exist a FS_n –homomorphism $f_1: F \to M(n-3,3)$ s.t. $ff_1 = I$ on F.

Let $f_1(1) = \sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l$. Then $\tau f_1(1) = f_1(\tau(1)) = f_1(1)$,where $\tau = (x_r x_s) \in S_n$,$1 \le r < s \le n$.Thus $f_1(1) - \tau f_1(1) = 0$.

$\Rightarrow 0 = \sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l - \sum_{1\le i<j<l\le n} k_{ijl}\, \tau(x_i x_j x_l)$

$= \sum_{1\le i<j<l\le n} k_{ijl}\, (x_i x_j x_l - \tau(x_i x_j x_l))$

$= \sum_{\substack{r+1<j<l\le n \\ j,l\ne s}} k_{sjl}\, (x_r x_j x_l - x_s x_j x_l) + \sum_{i=1}^{r-1} \sum_{\substack{l=r+1 \\ l\ne s}}^{n} k_{irl}\, (x_i x_r x_l - x_i x_s x_l) + \sum_{1\le i<j<r} k_{ijr}\, (x_i x_j x_r - x_i x_j x_s) =$

$\sum_{\substack{j=r+1 \\ j<l\le n}}^{n-1} (k_{rjl} - k_{sjl}) x_r x_j x_l + \sum_{i=1}^{r-1} \sum_{\substack{l=r+1 \\ l\ne s}}^{n} (k_{irl} - k_{isl})\, x_i x_r x_l + \sum_{1\le i<j<r} (k_{ijr} - k_{isl})\, x_i x_j x_r$.

So by equaling the coefficient ,we get

$k_{rjl} = k_{sjl}\ \forall\, r < j < l \le n$,

$k_{irl} = k_{isl}\ \forall\, 1 \le i < r$ and $r < l \le n , l \ne s$

$k_{ijr} = k_{ijs}\ \forall\, 1 \le i < j < r$.

Hence for each r , s such that $1 \le r < s \le n$ we get

$k_{ijl} = k ; 1 \le i < j < l \le n$ .Then $f_1(1) = \sum_{1\le i<j<l\le n} k_{ijl}\, x_i x_j x_l = \sum_{1\le i<j<l\le n} k\, x_i x_j x_l = k\sigma_3(n)$.

$\because ff_1 = I$

$\therefore 1 = ff_1(1) = f(k\sigma_3(n)) = kf(\sigma_3(n)) = kf\left(\sum_{1\le i<j<k\le n} x_i x_j x_k \right) = k\frac{n(n-1)(n-2)}{6}$ .Hence p does not divide $\frac{n(n-1)(n-2)}{6}$.

Corollary3.9:M(n-3,3) is a direct sums of $M_0(n-3,3)$ and $F\sigma_3(n)$ when p does not divide $\frac{n(n-1)(n-2)}{6}$.

Proof: Since p does not divide $\frac{n(n-1)(n-2)}{6}$,then the sequence (1) is split .Thus M(n-3,3) decomposable into $kerf = M_0(n-3,3)$ and $Imf = F\sigma_3(n)$ since for each $k\sigma_3(n) \in F\sigma_3(n)$ we have $f\left(\frac{kn(n-1)(n-2)}{6}\right) = \frac{kn(n-1)(n-2)}{6}\left(\frac{6\sigma_3(n)}{n(n-1)(n-2)}\right) = k\sigma_3(n)$.

Hence $M(n-3,3) = M_0(n-3,3) \oplus F\sigma_3(n)$.

Theorem 3.10: The following sequence

$$0 \to ker\bar{d}_2 \xrightarrow{i} M_0(n-3,3) \xrightarrow{\bar{d}_2} M_0(n-2,2) \to 0 \qquad ....(2)$$

is split iff p does not divide neither (n-2) nor (n-3).

Proof: Since $\bar{d}_2\left(\frac{1}{2}(x_1 x_i x_k - x_1 x_2 x_i + x_2 x_i x_k - x_1 x_2 x_k)\right) = \frac{1}{2}(x_1 x_i + x_1 x_k + x_i x_k - x_1 x_2 - x_1 x_i - x_2 x_i + x_2 x_i + x_2 x_k + x_i x_k - x_1 x_2 - x_1 x_k - x_2 x_k) = \frac{1}{2}(2(x_i x_k - x_1 x_2)) = x_i x_k - x_1 x_2$ .Which is the generated of $M_0(n-3,3)$ .Hence $\bar{d}_2$ is on to map. Moreover $Imi = ker\bar{d}_2$ .Thus the sequence (2) is exact.

If p does not divide neither (n-2)nor(n-3).

Let $\phi: M_0(n-2,2) \to M_0(n-3,3)$ be defined as follows:

$\phi(x_i x_j - x_1 x_2) = \frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} (x_1 x_i x_j - x_1 x_2 x_i + x_2 x_i x_j - x_1 x_2 x_j)$

Then for any $\in S_n , s.t. \tau(x_1) = x_1 , \tau(x_2) = x_2$.

$\phi\left(\tau(x_i x_j - x_1 x_2)\right) = \phi(\tau(x_i)\tau(x_j) - \tau(x_1)\tau(x_2)) = \frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} (x_1 x_{i_1} x_{j_1} - x_1 x_2 x_{i_1} + x_2 x_{i_1} x_{j_1} - x_1 x_2 x_{j_1})$Where $\tau(x_i) = x_{i_1}$ and $\tau(x_j) = x_{j_1}$.Then

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$\phi\left(\tau(x_ix_j - x_1x_2)\right) = \frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} \tau\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j) = \frac{1}{(n-2)(n-3)}\tau \sum_{2\langle i\langle j\le n} (x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j) = \tau\phi(x_ix_j - x_1x_2)$ .Hence ϕ is a FS_n –homomorphism .Moreover $\bar{d}_2\phi(x_ix_j - x_1x_2) = \bar{d}_2\left(\frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} (x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)\right) = \frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} \bar{d}_2\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j) = \frac{1}{(n-2)(n-3)} \sum_{2\langle i\langle j\le n} 2(x_ix_j - x_1x_2) = \frac{1}{(n-2)(n-3)} \frac{(n-2)(n-3)}{2}\left(2(x_ix_j - x_1x_2)\right) = x_ix_j - x_1x_2$

So $\bar{d}_2\phi = I$ on $M_0(n-2,2)$. Hence the sequence (2) is split if p does not divide neither (n-2) nor (n-3).Thus $M_0(n-3,3) = \ker \bar{d}_2 \oplus L_0$, where $L_0 = \phi(M_0(n-2,2))$.

Now assume if the sequence (2) is split.

Let $\phi: M_0(n-2,2) \to M_0(n-3,3)$ be a FS_n –homomorphism such that $\bar{d}_2\phi = I$.Thus we can define ϕ as follows $\phi(x_{i_1}x_{j_1} - x_1x_2) = \sum_{2\langle i\langle j\le n} k_{ij}\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)$.$\Rightarrow \bar{d}_2\phi(x_{i_1}x_{j_1} - x_1x_2) = \bar{d}_2\left(\sum_{2\langle i\langle j\le n} k_{ij}\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)\right) = \sum_{2\langle i\langle j\le n} \bar{d}_2(k_{ij}\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)) = \sum_{2\langle i\langle j\le n} k_{ij}\,(2(x_ix_j - x_1x_2)) = 2\left(\sum_{2\langle i\langle j\le n} k_{ij}\,(x_ix_j - x_1x_2)\right) = x_{i_1}x_{j_1} - x_1x_2$.

$$\Rightarrow 2\sum_{2\langle i\langle j\le n} k_{ij} = \begin{cases} 0 & \text{if } (i,j) \ne (i_1,j_1) \\ 1 & \text{if } (i,j) = (i_1,j_1) \end{cases}$$

Moreover if $\tau = (x_rx_s) \in S_n$; $2 < r < s \le n$ such that$\tau(x_{i_1}x_{j_1}) = x_{i_1}x_{j_1}$.Then $\phi\left(\tau(x_{i_1}x_{j_1} - x_1x_2)\right) = \phi(x_{i_1}x_{j_1} - x_1x_2) = \tau\phi(x_{i_1}x_{j_1} - x_1x_2) \Rightarrow \phi(x_{i_1}x_{j_1} - x_1x_2) - \tau\phi(x_{i_1}x_{j_1} - x_1x_2) = 0 \Rightarrow \sum_{2\langle i\langle j\le n} \left(k_{ij}\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)\right) - \tau\left(\sum_{2\langle i\langle j\le n} \left(k_{ij}\,(x_1x_ix_j - x_1x_2x_i + x_2x_ix_j - x_1x_2x_j)\right)\right) = 0.$

$\Rightarrow \sum_{\substack{j=r+1\\ j\ne s}}^{n} (k_{rj} - k_{sj})\, x_1x_rx_j - \sum_{\substack{j=r+1\\ j\ne s}}^{n} (k_{rj} - k_{sj})\, x_1x_2x_r + \sum_{\substack{j=r+1\\ j\ne s}}^{n} (k_{rj} - k_{sj})\, x_2x_rx_j + \sum_{2<i<r} (k_{ir} - k_{is})\, x_1x_ix_r + \sum_{2<i<r} (k_{ir} - k_{is})\, x_1x_ix_r + \sum_{2<i<r} (k_{ir} - k_{is})\, x_2x_ix_r - \sum_{2<i<r} (k_{ir} - k_{is})\, x_1x_2x_r = 0$

$\Rightarrow$ By equaling the above equitation we get $k_{rj} = k_{sj}$; $r < j \le n \,\&\, j \ne s$,and $k_{ir} = k_{is}$; $2 < i < r$.

$\Rightarrow k_{rj} = k_{sj} = k_{ir} = k_{is} = k$.Thus since $2\sum_{2\langle i\langle j\le n} k_{ij} = 0 \Rightarrow 2\sum_{2\langle i\langle j\le n} k = 0 \Rightarrow 2\binom{n-2}{2}k = (n-2)(n-3)k = 0$.Then $k = 0$ *or* $p|(n-2)$ *or* $p|(n-3)$.

$\because\ 2\sum_{2\langle i\langle j\le n} k_{ij} = 1$when$(i,j) = (i_1,j_1) \Rightarrow 2\frac{(n-2)(n-3)}{2}k_1 = 1$.

$\Rightarrow p \nmid (n-2)\,, p \nmid (n-3)$ *and* $k_1 \ne 0$.Hence we get $p \nmid (n-2)\,, p \nmid (n-3)\,, k = 0$ *and* $k_1 \ne 0$ .Thus if the sequence is split then p does not divide neither (n-2) nor (n-3).

Proposition3.11: $S(n-3,3)$ is a proper submodule of $ker\bar{d}_2$overFS_n.

Proof: Since $S(n-3,3) = FS_n(x_2 - x_1)(x_4 - x_3)(x_6 - x_5)$. *i.e.* $(x_2 - x_1)(x_4 - x_3)(x_6 - x_5)$ is the generator of $S(n-3,3)$ over FS_n, and $\bar{d}_2((x_2 - x_1)(x_4 - x_3)(x_6 - x_5)) = 0$. Hence $S(n-3,3) \subseteq ker\bar{d}_2$. Since $\bar{d}_2$ is an epimorphism. Hence

$$dim_F ker\bar{d}_2 = dim_F M_0(n-3,3) - dim_F M_0(n-2,2) = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} = \frac{n(n-1)(n-5)}{6}$$

While $dim_F S(n-3,3) = \frac{n(n-1)}{6}$.Thus $dim_F S(n-3,3) < dim_F ker\bar{d}_2$.

Hence $S(n-3,3)$ is a proper submodule of $ker\bar{d}_2$ over FS_n.

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Proposition3.12: If $p \neq 2,3$ and p divides (n+1), then we get the following series:

1) $0 \subset \bar{V}_0 \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

2) $0 \subset \bar{V}_0 \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V}_0 \oplus ker\bar{d}_2 \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

3) $0 \subset \bar{V}_0 \subset \bar{V} \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

4) $0 \subset F\sigma_3 \subset \bar{V} \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

5) $0 \subset F\sigma_3 \subset F\sigma_3 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

6) $0 \subset S(n-3,3) \subset F\sigma_3 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

7) $0 \subset S(n-3,3) \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V}_0 \oplus ker\bar{d}_2 \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

8) $0 \subset S(n-3,3) \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

Proof:

If $p \neq 2,3$ and $p\ |(n+1)$.

Then p does not divide n nor $(n-1)$ nor $(n-2)$.Since $\sigma_3(n) = \sum_{1 \le i < j < k \le n} x_i x_j x_k$ and the sum of the coefficients is $\frac{n(n-1)(n-2)}{6}$, then $\sigma_3(n) \notin M_0(n-3,3)$. $i.e$ $F\sigma_3 \cap M_0(n-3,3) = 0$.which implies that $M(n-3,3) = M_0(n-3,3) \oplus F\sigma_3$.Moreover we have $S(n-3,3) \subset ker\bar{d}_2$ and $F\sigma_3(n) \not\subset ker\bar{d}_2$,thus $F\sigma_3 \cap S(n-3,3) = 0$.

$Since\ p \nmid n$, then by Theorem (3.6) we have $\bar{V}_0(n)$ is an irreducible submodule overFS_n , and $\bar{V} = \bar{V}_0(n) \oplus F\sigma_3$,then $\bar{V} \cap ker\bar{d}_2 = 0$. Thus $\bar{V} \cap S(n-3,3) = 0$which implies that $F\sigma_3 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3)$ and $\bar{V}_0(n) \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3)$ .Therefore we get the following series:

1) $0 \subset \bar{V}_0 \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

2) $0 \subset \bar{V}_0 \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V}_0 \oplus ker\bar{d}_2 \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

3) $0 \subset \bar{V}_0 \subset \bar{V} \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

4) $0 \subset F\sigma_3 \subset \bar{V} \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\sigma_3$.

5) $0 \subset F\sigma_3 \subset F\sigma_3 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

6) $0 \subset S(n-3,3) \subset F\sigma_3 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

7) $0 \subset S(n-3,3) \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V}_0 \oplus ker\bar{d}_2 \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

8) $0 \subset S(n-3,3) \subset \bar{V}_0 \oplus S(n-3,3) \subset \bar{V} \oplus S(n-3,3) \subset \bar{V} \oplus ker\bar{d}_2 \subset M_0(n-3,3) \oplus F\,\sigma_3$.

Theorem 3.13: The following sequence of a FS_n- modules is short exact sequence.

$$0 \to ker\bar{\bar{d}}_2 \xrightarrow{i} G \xrightarrow{\bar{\bar{d}}_2} S(n-2,2) \to 0 \qquad \text{.........(3)}$$

where $G = FS_n(x_1x_3x_6 - x_1x_4x_6 + x_2x_4x_6 - x_2x_3x_6)$

Proof: From the definition of G we get that $G \subset M_0(n-3,3)$ and by Theorem(3.10) we have $\bar{d}_2 : M_0(n-3,3) \to M_0(n-2,2)$ is on to map. Since $S(n-2,2) = FS_n(x_2 - x_1)(x_4 - x_3) \subset M_0(n-2,2)$.

Then $(x_2 - x_1)(x_4 - x_3) = x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3$ and $\bar{d}_2\big((x_1x_3x_6 - x_1x_4x_6 + x_2x_4x_6 - x_2x_3x_6) + (x_1x_3x_5 - x_1x_4x_5 + x_2x_4x_5 - x_2x_3x_5)\big) = 2(x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3) = 2(x_2 - x_1)(x_4 - x_3)$.

$Thus\ \bar{\bar{d}}_2 = \bar{d}_2|_G : G \to S(n-2,2)$ is on to map. Moreover the inclusion map i is one-to-one map and $ker\ \bar{\bar{d}}_2 = Imi$. Hence the sequence (3) is short exact sequence.

Corollary 3.14: The short exact sequence (3) is split when p does not divide (n-4).

Proof: Assume $p \nmid (n-4)$. Let $\varphi: S(n-2,2) \to G$ be define as follows: $\varphi\big((x_r - x_s)(x_t - x_l)\big) = \frac{1}{n-4} \sum_{\substack{k=1 \\ k \neq r,s,t,l}}^{n} (x_rx_tx_k - x_rx_lx_k - x_sx_tx_k + x_sx_lx_k)$ .Then for any $\tau \in S_n$ we get $\varphi\big(\tau(x_r - x_s)(x_t - x_l)\big) = \varphi\big((\tau x_r - \tau x_s)(\tau x_t - \tau x_l)\big) = \varphi\Big(\big(x_{r_1} - x_{s_1}\big)\big(x_{t_1} - x_{l_1}\big)\Big) = \frac{1}{n-4} \sum_{\substack{k_1=1 \\ k_1 \neq r_1, s_1, t_1, l_1}}^{n} (x_{r_1}x_{t_1}x_{k_1} - x_{r_1}x_{l_1}x_{k_1} - x_{s_1}x_{t_1}x_{k_1} + x_{s_1}x_{l_1}x_{k_1}) = \frac{1}{n-4}$

$\sum_{\substack{k_1=1\\ k_1\neq r_1,s_1,t_1,l_1}}^{n} \tau(x_r x_t x_k - x_r x_l x_k - x_s x_t x_k + x_s x_l x_k) = \frac{1}{n-4}\tau(\sum_{\substack{k=1\\ k\neq r,s,t,l}}^{n}(x_r x_t x_k - x_r x_l x_k - x_s x_t x_k + x_s x_l x_k)) = \tau\varphi\big((x_r - x_s)(x_t - x_l)\big)$. Hence φ is a FS_n-homomorphism. Moreover we have

$\bar{\bar{d}}_2\varphi\big((x_r - x_s)(x_t - x_l)\big) = \bar{\bar{d}}_2(\frac{1}{n-4}\sum_{\substack{k=1\\ k\neq r,s,t,l}}^{n}(x_r x_t x_k - x_r x_l x_k - x_s x_t x_k + x_s x_l x_k))$

$=\frac{1}{n-4}\sum_{\substack{k=1\\ k\neq r,s,t,l}}^{n}\bar{\bar{d}}_2(x_r x_t x_k - x_r x_l x_k - x_s x_t x_k + x_s x_l x_k) = \frac{1}{n-4}\big((n-4)(x_r - x_s)(x_t - x_l)\big) = (x_r - x_s)(x_t - x_l)$ Thus $\bar{\bar{d}}_2\varphi = I \ on\ S(n-2,2)$. Hence the sequence(3) is split when $p \nmid (n-4)$. Moreover $G = ker\ \bar{\bar{d}}_2 \oplus \bar{G}$; $\bar{G} = \varphi(S(n-2,2))$.

Proposition 3.15: $S(n-3,3)$ is a proper $KS_n -$ submodule of G.

Proof: Since $S(n-3,3) = FS_n(x_2 - x_1)(x_4 - x_3)(x_6 - x_5)$ and $G = FS_n(x_1x_3x_6 - x_1x_4x_6 + x_2x_4x_6 - x_2x_3x_6)$ then

y=$(x_2 - x_1)(x_4 - x_3)(x_6 - x_5) = (x_1x_3x_6 - x_1x_4x_6 + x_2x_4x_6 - x_2x_3x_6) + (x_1x_4x_5 - x_1x_3x_5 + x_2x_3\,x_5 - x_2x_4x_5) \in F$. Thus $S(n-3,3) \subset G$ .Moreover since $\bar{\bar{d}}_2 = \bar{d}_2|_G$, then we get $ker\ \bar{\bar{d}}_2 \subset ker\bar{d}_2$ and since $kerd_2 = ker\bar{d}_2$. Hence $ker\ \bar{\bar{d}}_2 \subset kerd_2$ and by definition of $\bar{\bar{d}}_2$ we get $\bar{\bar{d}}_2(y) = 0$ which implies that $(n-3,3) \subset ker\ \bar{\bar{d}}_2 \subset G$. Hence S(n-3,3) is a proper $FS_n -$ submodule of G .

Theorem 3.16: If $p \neq 2,3$ and $p|(n-3)$ then we have the following series:

1)0$\subset F\sigma_3 \subset F\sigma_3 \oplus S(n-3,3) \subset F\sigma_3 \oplus ker\ \bar{\bar{d}}_2 \subset F\sigma_3 \oplus G \subset F\sigma_3 \oplus M_0(n-3,3)$.

2)0$\subset S(n-3,3) \subset F\sigma_3 \oplus S(n-3,3) \subset F\sigma_3 \oplus ker\ \bar{\bar{d}}_2 \subset F\sigma_3 \oplus G \subset F\sigma_3 \oplus M_0(n-3,3)$.

Proof: Since $p|(n-3)$,then $p \nmid (n-4)$ and by Corollary (3.19) we get $G = ker\ \bar{\bar{d}}_2 \oplus \bar{G}$; $\bar{G} = \varphi\big(S(n-2,2)\big) \cong S(n-2,2)$ and by Proposition(3.20) we have $S(n-3,3) \subset ker\ \bar{\bar{d}}_2 \subset G$. $Since\ \sigma_3(n) = \sum_{1\le i<j<k\le n} x_i x_j x_k$ and the sum of coefficients of $\sigma_3(n)$ is $\frac{n(n-1)(n-2)}{6}$ then $\sigma_3(n) \notin M_0(n-3,3)$ and $\sigma_3(n) \notin G$ which implies that $\sigma_3(n) \notin ker\ \bar{\bar{d}}_2.\ i.e.$ $F\sigma_3 \cap G = 0$ and $F\sigma_3 \cap ker\ \bar{\bar{d}}_2 = 0$ .Hence $F\sigma_3 \oplus ker\ \bar{\bar{d}}_2 \subset F\sigma_3 \oplus G$. Moreover we have $F\sigma_3 \oplus S(n-3,3) \subset F\sigma_3 \oplus ker\ \bar{\bar{d}}_2$ .Thus we get the following series:

1)0$\subset F\sigma_3 \subset F\sigma_3 \oplus S(n-3,3) \subset F\sigma_3 \oplus ker\ \bar{\bar{d}}_2 \subset F\sigma_3 \oplus G \subset F\sigma_3 \oplus M_0(n-3,3)$.

2)0$\subset S(n-3,3) \subset F\sigma_3 \oplus S(n-3,3) \subset F\sigma_3 \oplus ker\ \bar{\bar{d}}_2 \subset F\sigma_3 \oplus G \subset F\sigma_3 \oplus M_0(n-3,3)$.

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في التمثيل الطبيعي الثالث $M(n-3,3)$ للزمر النتاظرية

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المستخلص :

ان الهدف من هذا العمل هو دراسة التمثيل الطبيعي الثالث للزمرالنتاظرية$M(n-3,3)$ ضمن حقلF وبرهان بان$M(n-3,3)$ يمكن ان تجزئ اذا وفقط اذا كان p لا تقسم $n(n-1)(n-2)/6$