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On Differential Sandwich Theorems of Meromorphic Univalent Functions

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Abstract

By using of linear operator, we obtain some Subordinations and superordinations results for certain normalized meromorphic univalent analytic functions in the in the punctured open unit disk U^*. Also we derive some sandwich theorems .

Keywords :Analytic Function, Differential Subordination, Hadamard Product, Meromorphic Univalent Function.

Mathematics Subject Classification :30C45

1. Introduction

Let $\mathcal{H}$ be the Linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let
$\mathcal{H}[a,n] = \{f \in \mathcal{H}: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}$.
For two functions f and g analytic in U .We say that the function g is subordinate to f in U and write $g(z) \prec f(z)$, if there exists a Schwarz function ω, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $g(z) = f(\omega(z)), (z \in U)$.

If the function $f(z)$ is if the function f is univalent in U, then we have

$g(z) \prec f(z) \Leftrightarrow g(0) = f(0)$ and $g(U) \subset f(U)$,

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \ , \qquad (1.1)$$

which are analytic and meromorphic univalent function in the punctured open unit disk $U^* = \{z: z \in \mathbb{C}$ and $0 < |z| < 1\}$.

Let $p\, , h \in \mathcal{H}$, and $\emptyset(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$.

If p and $\emptyset(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second- order superordination

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z), \ (z \in U), \qquad (1.2)$$

then p is called a solution of the differential superordination (1.2) , (if f subordinate to g, then g is superordinate to f).

An analytic function q is called a subordinate of the differential superordination if $q \prec p$ for all p satisfying (1.2). A univalent subordinate $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinates q of (1.2)is said to be the best subordinate. Recently Miller and Mocnu [3] obtained sufficient conditions on the functions $h\, , p$ and $\emptyset$ for which the following implication holds :

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z), (z \in U). \qquad (1.2)$$

If $f \in W$ is given by (1.1) and $g \in W$ given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).$$

Using the results, Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination preserving integral operator [1]. Ali et al. [5], have used the results of Bulboacă [4] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [6] obtained a sufficient conditions for starlikeness of f in terms of the quantity

$$\frac{f''(z)f(z)}{(f'(z))^2}.$$

Recently, Shanmugam et al. [7,8] and Goyal et al. [9] also obtained sandwich results for certain classes of analytic functions.

Ali et al. [10] introduced and investigated the linear operator

$I_1(n,\lambda): W \to W$

which is defined as follows:

$$I_1(n,\lambda)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k,$$

$$(z \in U^*, \lambda > 1). \qquad (1.4)$$

The general Hurwitz- lerch zeta function

$$\Phi(z,s,r) = \sum_{k=0}^{\infty} \frac{z^k}{(r+k)^s} \ , r \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$$

when $0 < |z| < 1$.

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Definition 1.1. Let $f \in W, z \in U^*, r \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$ and $\lambda > 1$, we define the operator $\mathcal{J}_{s,r,1}(n,\lambda)f(z): W \longrightarrow W$, where

$$\mathcal{J}_{s,r,1}(n,\lambda)f(z) = \frac{\Phi(z,s,r)}{zr^{-s}} * I_1(n,\lambda)f(z)$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{r}{1+k+r}\right)^s\left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k \qquad (1.5)$$

We note from (1.5) that, we have

$$\lambda\, \mathcal{J}_{s,r,1}(n,\lambda)f(z) = z\left(\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)' - (\lambda-1)\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z), \qquad (1.6)$$

$\mathcal{J}_{0,r,1}(n,\lambda)f(z) = I_1(n,\lambda)f(z)$

and $\mathcal{J}_{0,r,1}(0,\lambda)f(z) = f(z)$.

The main object of this idea is to find sufficient conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q_2(z),$$

and

$$q_1(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given uninvent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.

Definition 2.1. [2]: Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $\overline{U} = U \cup \{z \in \partial U\}$, and

$$\mathrm{E}(q) = \left\{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\right\} \qquad (1.7)$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Lemma 2.1. [5] Let $q(z)$ be convex univalent function in U, let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $p(z)$ is analytic in U and

$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$,

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2. [1]

Let q be univalent in U and let $\emptyset$ and θ be analytic in the domain D containing $q(U)$ with $\emptyset(w) \neq 0$, when $w \in q(U)$. Set

$Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$,

suppose that

1 – Q is starlike univalent in U,

2 – $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\emptyset(p(z)) + zp'(z)\emptyset(p(z)) \prec \emptyset(q(z)) + zq'(z)\emptyset(q(z)),$$

then $p \prec q$, and q is the best dominant.

Lemma 2.3. [3] Let q(z) be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

1 – $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,

2 – $zq'(z)\phi(q(z))$ is starlike univalent in $z \in U$.

If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \qquad (1.8)$$

then $q \prec p$, and q is the best subordinant

Lemma 2.4. [3]: Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\mathrm{Re}\{\beta\} > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in U, then

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$q(z)+\beta zq'(z) \prec p(z)+\beta zp'(z)$,

which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Subordination Results

Theorem 3.1. Let $q(z)$ be convex univalent in U with $q(0)=1, \eta, \delta \in \mathbb{C}\backslash\{0\}$. Suppose that

$$\mathrm{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \mathrm{Re}\left(\frac{\delta}{\eta}\right)\right\}. \qquad (3.1)$$

If $f \in W$ is satisfies the Subordination

$$G(z) \prec q(z)+\frac{\eta}{\delta} zq'(z), \qquad (3.2)$$

where

$$\mathrm{G}(z)=\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times \left(1+\eta \times \left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+(\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)+(\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right)\right), \qquad (3.3)$$

then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q(z), \qquad (3.4)$$

and $q(z)$ is the best dominant.

Proof. Define a function $g(z)$ by $g(z)=$

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta}, \qquad (3.5)$$

then the function $g(z)$ is analytic in U and q(0)=1,therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.6) in the resulting equation,

$$\mathrm{G}(z)=\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times \left(1+\eta \times \left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+(\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)+(\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right)\right)$$

$$=g(z)+\frac{\eta}{\delta} zg'(z).$$

Thus the subordination (3.2) is equivalent to

$$g(z)+\frac{\eta}{\delta} zg'(z) \prec q(z)+\frac{\eta}{\delta} zq'(z).$$

An application of Lemma (2.1) with $\beta=\frac{\eta}{\delta}$ and $\alpha=1$, we obtain (3.4).

Taking $q(z)=\frac{1+Az}{1+Bz}(-1 \le B < A \le 1)$, in Theorem (3.1), we obtain the following Corollary.

Corollary3.2. Let $\eta, \delta \in \mathbb{C}\backslash\{0\}$ and ($-1 \le B < A \le 1$). Suppose that

$$\mathrm{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following Subordination condition :

$$G(z) \prec \frac{1+Az}{1+Bz}+\frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ given by (3.3), then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is best dominant .

Taking $A=1$ and $B=-1$ in Corollary (3.2), we get following result.

Corollary 3.3. Let $\eta, \delta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following Subordination

$$G(z) \prec \frac{1+z}{1-z}+\frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where $G(z)$ given by (3.3), then

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z)+\beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is best dominant .

Theorem 3.4. Let $q(z)$ be convex univalent in unit disk U with $q(0)=1$, let $\varsigma\, \eta, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \mu, \tau \in \mathbb{C}, f \in W$ and suppose that f and q satisfy the following conditions

$$\mathrm{Re}\left\{\frac{\mu}{\varsigma}q(z)+\frac{2\tau\alpha}{\varsigma}q^2(z)+1+z\frac{q''(z)}{q'(z)}-z\frac{q'(z)}{q(z)}\right\} > 0, \qquad (3.6)$$

and

$$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0. \qquad (3.7)$$

If

$$r(z) \prec t + \mu q(z) + \tau\alpha q^2(z) + \varsigma \frac{zq'(z)}{q(z)}, \qquad (3.8)$$

where

$$r(z) = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \times ((\mu + \tau\alpha\left(\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}\right) + t + \varsigma\delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)} - 1\right), \qquad (3.9)$$

then

$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q(z)$, and $q(z)$ is best dominant.

Proof . Define analytic function $g(z)$ by

$$g(z) = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}. \qquad (3.10)$$

Then the function $g(z)$ is analytic in U and $g(0) = 1$, differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zg'(z)}{g(z)} = \delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)} - 1\right). \qquad (3.11)$$

By setting $\theta(w) = t + \mu w + \tau\alpha w^2$ and $\emptyset(w) = \frac{\varsigma}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\emptyset(w)$is analytic in $\mathbb{C}\backslash\{0\}$ and that $\emptyset(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$.

Also, if we let

$Q(z) = zq'(z)\emptyset(z) = \varsigma\frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z)$

$$= t + \mu q(z) + \tau\alpha q^2(z) + \varsigma\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U ,we have

$$h'(z) = \mu q'(z) + 2\tau\alpha q(z)q'(z) + \varsigma\frac{q'(z)}{q(z)} + \varsigma z\frac{q''(z)}{q(z)} - \varsigma z\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\mu}{\varsigma}q(z) + \frac{2\tau\alpha}{\varsigma}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\mu}{\varsigma}q(z) + \frac{2\tau\alpha}{\varsigma}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\mu g(z) + \tau\alpha g^2(z) + \frac{zg'(z)}{g(z)} = \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \times \left(\mu + \tau\alpha\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}\right) + t + \varsigma\delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)} - 1\right).$$

By using (3.8), we have

$$\mu g(z) + \tau\alpha g^2(z) + \varsigma\frac{zg'(z)}{g(z)} \prec \mu q(z) + \tau\alpha q^2(z) + s\frac{zq'(z)}{q(z)}$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $g(z) \prec q(z)$ and the function $q(z)$ is the best dominant.

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Theorem (3.4),the condition (3.6) becoms

$$\operatorname{Re}\left(\frac{\mu}{\varsigma}\frac{1+Az}{1+Bz} + \frac{2\tau\alpha}{\varsigma}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0 (s \in \mathbb{C}\backslash\{0\}), \qquad (3.12)$$

hence, we have the following Corollary.

Corollary 3.5. Let $(-1 \leq B < A \leq 1), s, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \tau, \mu \in \mathbb{C}$. Assume that (3.12) holds.

If $f \in W$ and

$$r(z) \prec t + \mu\frac{1+Az}{1+Bz} + \tau\alpha\left(\frac{1+Az}{1+Bz}\right)^2 + \varsigma\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9), then

$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec \frac{1+Az}{1+Bz}$,and $\frac{1+Az}{1+Bz}$ is best dominant.

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < \rho \leq 1)$, in Theorem (3.4),the condition(3.6) becoms

$$\operatorname{Re}\left\{\frac{\mu}{\varsigma}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\tau\alpha}{\varsigma}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{2z^2}{1-z^2}\right\} > 0, \quad (\varsigma \in \mathbb{C}\backslash\{0\}), \qquad (3.13)$$

hence, we have the following Corollary.

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Corollary 3.6. Let $0 < \rho \le 1, \varsigma, \delta \in \mathbb{C}\backslash\{0\}, \alpha, t, \tau, \mu \in \mathbb{C}$. Assume that (3.13) holds.

If $f \in W$ and

$$r(z) \prec t + \mu\left(\frac{1+z}{1-z}\right)^{\rho} + \tau\alpha\left(\frac{1+z}{1-z}\right)^{2\rho} + \varsigma\frac{2\rho z}{1-z^2},$$

where $r(z)$ is defined in (3.9), then $\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$,and $\left(\frac{1+z}{1-z}\right)^{\rho}$ is best dominant.

4. Superordination Results

Theorem 4.1. Let $q(z)$ be convex univalent in U with $q(0) = 1, , \delta \in \mathbb{C}\backslash\{0\}$,$\mathrm{Re}\{\eta\} > 0$, if $f \in W$, such that

$$\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1} \neq 0$$

and

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q. \qquad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z) + \frac{\eta}{\delta} zq'(z) \prec \mathrm{G}(z), \qquad (4.2)$$

holds, then

$$q(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \quad (4.3)$$

and $q(z)$ is the best subordinant .

Proof . Define a function $g(z)$ by

$$g(z) = \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta}. \qquad (4.4)$$

Differentiating (4.4) with respect to z logarithmically ,we get

$$\frac{zg'(z)}{g(z)} = \delta\left(\frac{(1-\beta)z(\mathcal{J}_{s,r,1}(n,\lambda)f(z))' + \beta z(\mathcal{J}_{s,r,1}(n+1,\lambda)f(z))'}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right). \quad (4.5)$$

A simple computation and using (1.6), from (4.5) ,we get

$$\mathrm{G}(z) = \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \times ((1+\eta \left(\frac{(\beta\lambda-\lambda+1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + (\lambda-1-2\lambda\beta+2\beta)\mathcal{J}_{s,r,1}(n+1,\lambda)f(z) + (\beta\lambda-\beta)\mathcal{J}_{s,r,1}(n+2,\lambda)f(z)}{(1-\beta)\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}\right)),$$

$$= g(z) + \frac{\eta}{\delta} zg'(z),$$

now, by using Lemma(2.4), we get the desired result .

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (4.1), we get the following Corollary.

Corollary 4.2. Let $\mathrm{Re}\{\eta\} > 0, \delta \in \mathbb{C}\backslash\{0\}$ and $-1 \le B < A \le 1$, such that

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $G(z)$ given by (3.3) is univalent in U and $f \in W$ satisfies the following superordination condition :

$$\frac{1+Az}{1+Bz} + \frac{\eta}{\delta}\frac{(A-B)z}{(1+Bz)^2} \prec G(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem 4.3. Let $q(z)$ be convex univalent in unit disk U, let $\varsigma, \delta \in \mathbb{C}\backslash\{0\}\alpha, t, \mu, \tau \in \mathbb{C}, , q(z) \neq 0$, and $f \in W$. Suppose that $\mathrm{Re}\left\{\frac{q(z)}{s}(2\tau\alpha q(z) + \mu)\right\}q'(z) > 0,$

and satisfies the next conditions

$$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q, \qquad (4.6)$$

and

$$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0,$$

If the function $r(z)$ is given by (3.9) is univalent in U ,

$$\mathrm{t} + \mu q(z) + \tau\alpha q^2(z) + \varsigma\frac{zq'(z)}{q(z)} \prec r(z), \qquad (4.7)$$

implies

$q(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta}$, and $q(z)$ is the best subordinant.

Proof . Let the function $g(z)$ defined on U by (3.14) . Then a computation show that

$$\frac{zg'(z)}{g(z)} = \delta(\lambda-1)\left(\frac{\mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\mathcal{J}_{s,r,1}(n,\lambda)f(z)} - 1\right), \qquad (4.8)$$

By setting $\theta(w) = t + \mu w + \tau\alpha w^2$ and $\emptyset(w) = \frac{\varsigma}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\emptyset(w)$is analytic in $\mathbb{C}\backslash\{0\}$ and that $\emptyset(w) \neq 0$ $(w \in \mathbb{C}\backslash\{0\})$. Also, we get

$$Q(z) = zq'(z)\emptyset(q(z)) = \varsigma\frac{zq'(z)}{q(z)},$$

it observed that $Q(z)$ is starlike univalent in U.

Since $q(z)$ is convex , it follows that

$$\text{Re}\left(\frac{z\theta'(q(z))}{\emptyset(q(z))}\right) = \text{Re}\left\{\frac{q(z)}{\varsigma}(2\tau\alpha q(z)+\mu)\right\}q'(z) > 0.$$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$$\theta(q(z) + zq'(z)\emptyset(q(z))) = \theta(g(z) + zg'(z)\emptyset(g(z))),$$

thus, by applying Lemma (2.3), the proof is completed .

5 . Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem .

Theorem 5.1. Let q_1 and $q_2$be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1) . Suppose that $\text{Re}\{\eta\} > 0$, $\eta, \delta \in \mathbb{C}\backslash\{0\}$.

If $f \in W$, such that

$$\left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q,$$

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \frac{\eta}{\delta} z\, q_1'(z) \prec \mathrm{G}(z) \prec q_2(z) + \frac{\eta}{\delta} z\, q_2'(z), \qquad (5.1)$$

then

$$q_1(z) \prec \left(\frac{(1-\beta)\, z\mathcal{J}_{s,r,1}(n,\lambda)f(z) + \beta z\, \mathcal{J}_{s,r,1}(n+1,\lambda)f(z)}{\beta+1}\right)^{\delta} \prec q_2(z),$$

where $q_1$and q_2 are, respectively ,the subordinant and the best dominant of (5.1).

Combining Theorem (3.4) with Theorem (4.3), we obtain the following sandwich theorem .

Theorem 5.2. Let q_i be two convex univalent functions in U,such that $q_i(0) = 1$, $q_i(0) \neq 0 (i = 1,2)$ .Suppose that $q_1$and q_2 satisfies (4.8) and (3.8), respectively.

If $f \in W$ and suppose that f satisfies the next conditions :

$$\left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q,$$

and

$$z\mathcal{J}_{s,r,1}(n,\lambda)f(z) \neq 0,$$

and $r(z)$ is univalent in U, then

$$\mathrm{t} + \mu\, q_1(z) + \tau\alpha\, q_1^2(z) + \varsigma\frac{zq_1'(z)}{q_1(z)} \prec r(z) \prec \mathrm{t} + \mu\, q_1(z) + \tau\alpha\, q_1^2(z) + \varsigma\frac{zq_1'(z)}{q_1(z)}, \qquad (5.2)$$

implies

$$q_1(z) \prec \left(z\mathcal{J}_{s,r,1}(n,\lambda)f(z)\right)^{\delta} \prec q_2(z),$$

and q_1 and $q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

References

[1] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, (2005).

[2] S. S. Miller and P. T. Mocanu, Differential subordinations: Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics, 225, Marcel Dekker, New York and Basel, (2000).

[3] S. S. Miller and P. T. Mocanu, Subordinations of differential superordinations, Complex Variables , 48(10),815‒826, (2003).

[4] T. Bulboacă, Classes of first order differential superordinations, Demonstration Math. , 35(2), 287– 292, (2002).

[5] R. M. Ali, V. Ravichandran, K. G. Subramanian, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci., 15, 87– 94, (2004).

[6] N. Tuneski, on certain sufficient conditions for starlikeness, Internat. J. Math. Sci. 23(8), 521– 527, (2000).

[7] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, Differential Sandwich theorems for subclasses of analytic functions, Aust. J. Math. Anal. Appl., 3, Article 8, 1– 11, (2006).

[8] T. N. Shanmugam, S. Shivasubramanian and H. Silverman, On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci., Article ID 29684, 1 – 13, (2006).

[9] S. P. Goyal, P. Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions, Int. J. Math. Math. Sci., Article ID 561638, 1– 12, (2008).

[10] R. M. Ali, V. Ravichandran and N. Seenivasagan ,On Subordination and superordination of the multiplier transformation for meromorphic functions ,Bull. Malays. Math. Sci. Soc., 33,311 – 324, (2010).

حول مبرهنات الساندوج التفاضلية لدوال احادية التكافؤ الميرومورفيه

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المستخلص :
باستخدام المؤثر الخطي، حصلنا على بعض النتائج للتبعية التفاضلية والتبعية التفاضلية العليا للدوال التحليلية الاحادية التكافؤ الاكيدة في قرص الوحدة المثقوب U^*.ايضا اشتقينا بعض مبرهنات الساندوج.