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Coefficient Estimates for Subclasses of Bi-Univalent Functions

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Abstract

In the present paper, we introduce two new subclasses of the class $\sum$ consisting of analytic and bi-univalent functions in the open unit disk U. Also, we obtain the estimates on the Taylor-Maclurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. We obtain new special cases for our results.

Keywords : Analytic function , Univalent function , Bi-univalent function , Coefficient estimates .

Mathematics Subject Classification : 30C45.

1. Introduction

Let $\mathcal{H}$ be the class of the functions of the form :

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j , \qquad (z \in U), \qquad (1.1)$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let S denoted the class of all functions in $\mathcal{H}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$ in U [1]. It is well known that every univalent function f has inverse f^{-1} satisfying:

$$f^{-1}(f(z)) = z \ \ (z \in U),$$

and

$$f(f^{-1}(w)) = w \ \ (|w| < r_0(f);\ r_0(f) \geq \tfrac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \qquad (1.2)$$

A function $f \in \mathcal{H}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U given by (1.1). For a brief history and interesting example in the class Σ (see [2]). However , the familiar Koebe function is not bi-univalent. The class Σ of bi-univalent functions was first inverstigated by Lewin [3] and it was shown that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$. Later , Netanyahu [5], showed that if $f \in \Sigma$, then $\max|a_2| = \frac{4}{3}$.

Recently, Srivastava et al. [6], Frasin and Aouf [7], BansaL and Sokol [8] and Srivastava and BansaL [2] are also introduced and investigated the various subclasses of bi- univalent functions and obtained bounds for the initial coefficients $|a_2|$ and $|a_3|$.

The coefficient estimate problem involving the bound of $|a_n|$ $(n \in N\backslash\{1,2\}; N = \{1,2,3,...\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

The object of this work is to find estimates on the Taylor –Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclasses $S_\Sigma(\tau,\gamma,\delta;\alpha)$ and $S_\Sigma(\tau,\gamma,\delta;\beta)$ of the functions class Σ. Several related classes are also considered and connections to earlier known results are made.

In order to prove in our main results, we require the following lemma.

Lemma 1.1. [1] If $h \in p$ the $|c_k| \leq 2$ for each k, where p is the family of all functions h analytic in U for which $\mathrm{Re}(h(z)) > 0$

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \quad \text{for } z \in U$$

2. Coefficient Bounds for Function class $S_\Sigma(\tau,\gamma,\delta;\alpha)$

To prove our main results , we need to introduce the following definition .

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $S_\Sigma(\tau,\gamma,\delta;\alpha)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 < \alpha \leq)$ if the following conditions are satisfied :

$$f \in \Sigma, \left|\arg\left(1 + \frac{1}{\tau}\left[\frac{z(f'(z) + \gamma z f''(z))}{\gamma z(f'(z) + \delta z f''(z)) + (1-\gamma)(\delta z f'(z) + (1-\delta) f(z))} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (z \in U) \qquad (2.1)$$

and

$$g \in \Sigma, \left|\arg\left(1 + \frac{1}{\tau}\left[\frac{w(g'(w) + \gamma w g''(w))}{w\gamma(g'(w) + \delta g''(w)) + (1-\gamma)(\delta w g'(w) + (1-\delta) g(w))} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (w \in U) \qquad (2.2)$$

where the function $g(w)$ is given by (1.2).

Theorem 2.2. Let $f(z)$ given by (1.1) be in the class $S_\Sigma(\tau,\gamma,\delta;\alpha)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 < \alpha \leq 1)$. Then

$$|a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\tau\alpha((2-2\delta+4\gamma-4\delta\gamma) - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)) + (1-\alpha)(1-\delta+\gamma-\delta\gamma)^2|}} . \qquad (2.3)$$

and

$$|a_3| \leq \frac{2|\tau|\alpha}{|(2-2\delta+4\gamma-4\delta\gamma)|} + \frac{4|\tau|^2\alpha^2}{(1-\delta+\gamma-\delta\gamma)^2}. \qquad (2.4)$$

Proof: Let $f(z) \in S_{\Sigma}(\tau,\gamma,\delta;\alpha)$.Then

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))}-1\right] = [r(z)]^{\alpha} \qquad (2.5)$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w\, g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))}-1\right] = [h(w)]^{\alpha}. \qquad (2.6)$$

Where $r(z)$ and $h(w)$ are in p and have the following series representations :

$$r(z) = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \cdots \qquad (2.7)$$

and

$$h(w) = 1 + h_1 w + h_2 w^2 + h_3 w^3 + \cdots. \qquad (2.8)$$

Since

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))}-1\right] =$$
$$1+\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 z + \frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_3$$
$$-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)\, a_2^2 z^2 + \cdots, \qquad (2.9)$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w\, g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))}-1\right] = 1-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 w + \frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)$$
$$(2a_2^2-a_3)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2 w^2 + \cdots. \qquad (2.10)$$

Now , equating the coefficients in (2.5) and (2.6), we get

$$\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = \alpha r_1\,, \qquad (2.11)$$

$$\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_3-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2$$
$$-\delta^2\gamma^2)a_2^2) = \alpha r_2 + r_1^2\frac{\alpha(\alpha-1)}{2}, \qquad (2.12)$$

$$-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = \alpha h_1\,, \qquad (2.13)$$

and

$$\frac{1}{\tau}((2-2\delta+4\gamma-4\delta\gamma)(2a_2^2-a_3)-(1+2\gamma+\gamma^2-2\delta^2\gamma$$
$$-\delta^2-\delta^2\gamma^2)a_2^2 = \alpha h_2 + h_1^2\frac{\alpha(\alpha-1)}{2}\,. \qquad (2.14)$$

From (2.11) and (2.13) , we find

$$r_1 = -h_1 \qquad (2.15)$$

and

$$\frac{2}{\tau^2}(1-\delta+\gamma-\delta\gamma)^2 a_2^2 = \alpha^2(r_1^2+h_1^2). \qquad (2.16)$$

Also, from (2.12), (2.14) and (2.16), we find that

$$\frac{2}{\tau}((2-2\delta+4\gamma-4\delta\gamma)a_2^2 - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2) = \alpha(r_2+h_2) + \frac{\alpha(\alpha-1)}{2}(r_1^2+h_1^2) = \alpha(r_2+h_2) + \frac{(\alpha-1)}{\alpha\tau^2}(1-\delta+\gamma-\delta\gamma)^2\, a_2^2\,. \qquad (2.17)$$

Therefore ,we obtain

$$a_2^2 = \frac{\alpha^2\tau^2(r_2+h_2)}{2\alpha\tau((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))+(1-\alpha)(1-\delta+\gamma-\delta\gamma)^2}.$$

Applying Lemma (1.1) for the coefficients r_2 and h_2, we readily get

$$|a_2| \leq \frac{2\alpha|\tau|}{\sqrt{|2\tau\alpha((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))+(1-\alpha)(1-\delta+\gamma-\delta\gamma)^2|}}.$$

The last inequality gives the desired estimate on $|a_2|$ given in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.12) and (2.14), we get

$$\frac{1}{\tau}(2-2\delta+4\gamma-4\delta\gamma)(2a_3-2a_2^2) = (\alpha r_2 + r_1^2\frac{\alpha(\alpha-1)}{2})$$
$$-(\alpha h_2 + h_1^2\frac{\alpha(\alpha-1)}{2}). \qquad (2.18)$$

It follows from (2.15) , (2.16) and (2.18), that

$$a_3 = \frac{\alpha\tau(r_2-h_2)}{2(2-2\delta+4\gamma-4\delta\gamma)} + \frac{\alpha^2\tau^2(r_1^2-h_1^2)}{2(1-\delta+\gamma-\delta\gamma)^2}\,.$$

Applying Lemma (1.1) once again for the coefficients r_1 , r_2, h_1 and h_2 , we immediately

$$|a_3| \leq |\frac{2\alpha|\tau|}{(2-2\delta+4\gamma-4\delta\gamma)} + \frac{4|\tau|^2\alpha^2}{(1-\delta+\gamma-\delta\gamma)^2}.$$

This complete the proof of Theorem (2.2) .

3. Coefficient Bounds for Function class $S_\Sigma\ (\tau,\gamma,\delta;\beta)$

To prove our main results , we need to introduce the following definition .

Definition 3. 1. A function $f(z)$ given by (1.1) is said to be in the class $S_\Sigma\ (\tau,\gamma,\delta;\beta)(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 \leq \beta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \mathrm{Re}\left(1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))} - 1\right]\right) > \beta \quad (z \in U) \tag{3.1}$$

and

$$g \in \Sigma, \mathrm{Re}\left(1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))} - 1\right]\right) > \beta \quad (w \in U), \tag{3.2}$$

where the function $g(w)$ is given by (1.2).

Theorem 3. 2. Let $f(z)$ given by (1.1) be in the class $S_\Sigma\ (\tau,\gamma,\delta;\beta)\ (\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \gamma \leq 1, 0 \leq \delta < 1, 0 \leq \beta < 1)$. Then

$$|a_2| = \sqrt{\frac{2|\tau|(1-\beta)}{|(1+2\gamma-\gamma^2-2\delta-4\delta\gamma+2\delta^2\gamma+\delta^2+\delta^2\gamma^2)|}} \tag{3.3}$$

and

$$|a_3| \leq \frac{|\tau|(1-\beta)}{|1-\delta+2\gamma-2\delta\gamma|} + \frac{4|\tau|^2\,(1-\beta)^2}{|(1-\delta+\gamma-\delta\gamma)^2|}. \tag{3.4}$$

Proof : Let $f(z) \in S_\Sigma\ (\tau,\gamma,\delta;\beta)$. Then

$$1+\frac{1}{\tau}\left[\frac{z(f'(z)+\gamma z f''(z))}{\gamma z(f'(z)+\delta z f''(z))+(1-\gamma)(\delta z f'(z)+(1-\delta)f(z))} - 1\right] = \beta + (1-\beta)r(z) \tag{3.5}$$

and

$$1+\frac{1}{\tau}\left[\frac{w(g'(w)+\gamma w g''(w))}{w\gamma(g'(w)+\delta g''(w))+(1-\gamma)(\delta w g'(w)+(1-\delta)\, g(w))} - 1\right] = \beta + (1-\beta)h(w)\,, \tag{3.6}$$

where $g(w) = f^{-1}(w), r(z)$ and $h(w)$ have form (2.7)and (2.8), respectively.

Now, equating the coefficients in (3.5) and (3.6) , we get

$$\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = (1-\beta)r_1\,, \tag{3.7}$$

$$\frac{1}{\tau}\left((2-2\delta+4\gamma-4\delta\gamma)a_3 - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2\right) = (1-\beta)r_2\,, \tag{3.8}$$

$$-\frac{1}{\tau}(1-\delta+\gamma-\delta\gamma)a_2 = (1-\beta)h_1\,, \tag{3.9}$$

and

$$\frac{1}{\tau}\left((2-2\delta+4\gamma-4\delta\gamma)(2a_2^2-a_3) - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2)a_2^2\right) = (1-\beta)h_2\,. \tag{3.10}$$

From (3.7) and (3.9), we obtain

$$r_1 = -h_1 \tag{3.11}$$

and

$$\frac{2}{\tau^2}(1-\delta+\gamma-\delta\gamma)^2 a_2^2 = (1-\beta)^2(r_1^2+h_1^2). \tag{3.12}$$

Also , from (3.8) and (3.10) , we have

$$\frac{2}{\tau}((2-2\delta+4\gamma-4\delta\gamma) - (1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))a_2^2 = (1-\beta)\,(r_2+h_2)\,. \tag{3.13}$$

Therefore , we get

$$a_2^2 = \frac{\tau(1-\beta)(r_2+h_2)}{2((2-2\delta+4\gamma-4\delta\gamma)-(1+2\gamma+\gamma^2-2\delta^2\gamma-\delta^2-\delta^2\gamma^2))}. \tag{3.14}$$

Applying Lemma (1.1) for coefficients r_2 and h_2 , we obtain

$$|a_2| \leq \sqrt{\frac{2|\tau|(1-\beta)}{|(1+2\gamma-\gamma^2-2\delta-4\delta\gamma+2\delta^2\gamma+\delta^2+\delta^2\gamma^2)|}}\,. \tag{3.15}$$

This gives the bound on $|a_2|$ as asserted in (3.3).

Next in order to find the bound on $|a_3|$, by subtracting (3.8) and (3.10) , we thus get

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$$\frac{1}{\tau}(2-2\delta+4\gamma-4\delta\gamma)(2a_3-2a_2^2)=(1-\beta)\,(r_2-h_2) \tag{3.16}$$

or, equivalently,

$$a_3=\frac{\tau(1-\beta)(r_2-h_2)}{4(1-\delta+2\gamma-2\delta\gamma)}+a_2^2 \tag{3.17}$$

It follows from (3.12) and (3.17) , that

$$a_3=\frac{\tau(1-\beta)(r_2-h_2)}{4(1-\delta+2\gamma-2\delta\gamma)}+\frac{\tau^2(1-\beta)^2(r_1^2+h_1^2)}{2(1-\delta+\gamma-\delta\gamma)^2}.$$

Applying Lemma (1.1) once again for the coefficients r_1 , r_2 , h_1 and h_2 , we obtain

$$|a_3|\leq\frac{|\tau|(1-\beta)}{|(1-\delta+2\gamma-2\delta\gamma)|}+\frac{4|\tau|^2(1-\beta)^2}{|(1-\delta+\gamma-\delta\gamma)^2|}.$$

This completes the prove of Theorem (3.2) .

4. Corollaries and Consequence

This section is devoted to the presentation of some special cases of the main results .

These results are given in the form of corollaries :

If we set τ=1 and δ=0 in Theorems (2.2) and (3.2) , then ,we get following results due to Keerthi and Raja [9] :

Corollary 4.1. Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma\ (\gamma;\alpha)(0\leq\gamma\leq 1, 0<\alpha\leq 1)$. Then

$$|a_2|\leq\frac{2\alpha}{\sqrt{|4\alpha(1+2\gamma)+(1-3\alpha)(\gamma+1)^2|}}$$

and

$$|a_3|\leq\frac{4\alpha^2}{(1+\gamma)^2}+\frac{\alpha}{1+2\gamma}.$$

Corollary 4.2. Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma\ (\gamma\,;\beta)\ (0\leq\gamma\leq 1, 0\leq\beta<1)$. Then

$$|a_2|\leq\sqrt{\frac{2(1-\beta)}{|1+2\gamma-\gamma^2|}}$$

and

$$|a_3|\leq\frac{4(1-\beta)^2}{(1+\gamma)^2}+\frac{1-\beta}{1+2\gamma}.$$

The classes $\mathcal{B}_\Sigma\ (\gamma;\alpha)$ and $\mathcal{B}_\Sigma\ (\gamma\,;\beta)$ are respectively defined as follows:

Definition 4.3. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma\ (\gamma;\alpha)\ \ (0\leq\gamma\leq 1, 0<\alpha\leq 1)$ if the following conditions are satisfied:

$$f\in\Sigma,\left|\arg\left(\frac{z(f'(z)+\gamma zf''(z))}{(1-\gamma)f(z)+\gamma zf'(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(z\in U)$$

and

$$g\in\Sigma,\left|\arg\left(\frac{w(g'(w)+\gamma wg''(w))}{(1-\gamma)g(w)+\gamma wg'(w)}\right)\right|<\frac{\alpha\pi}{2},\quad(w\in U)$$

where the function $g(w)$ is given by (1.2) .

Definition 4.4. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\beta;\gamma)(0\leq\gamma\leq 1, 0\leq\beta<1)$if the following conditions are satisfied:

$$f\in\Sigma\ ,\mathrm{Re}\left(\frac{z(zf'(z)+\gamma zf''(z))}{(1-\gamma)f(z)+\gamma zf'(z)}\right)>\beta\quad(z\in U)$$

and

$$g\in\Sigma,\mathrm{Re}\left(\frac{w(g'(w)+\gamma wg''(w))}{(1-\gamma)g(w)+\gamma zg'(w)}\right)>\beta\,,\quad(w\in U)$$

where the function $g(w)$ is given by (1.2).

If we set $\tau=1$ and $\gamma=0$ in Theorems (2.2) and (3.2), then the classes $S_\Sigma\ (\tau,\gamma,\delta;\alpha)$ and $S_\Sigma\ (\tau,\gamma,\delta;\beta)$ reduce to the classes $\mathcal{G}_\Sigma\ (\delta;\alpha)$ and $\mathcal{G}_\Sigma\ (\delta;\beta)$ investigated by Murugusundaramoorthy et al. [10] ,which are defined as follows :

Definition 4.5. *A* function $f(z)$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma\ (\delta;\alpha)(0<\alpha\leq 1, 0\leq\delta<1)$ if the following conditions are satisfied:

$$f\in\Sigma,\left|\arg\left(\frac{zf'(z)}{(1-\delta)f(z)+\delta f'(z)}\right)\right|<\frac{\alpha\pi}{2}\quad(z\in U)$$

and

$$g\in\Sigma,\left|\arg\left(\frac{wg'(w)}{(1-\delta)g(w)+\delta g'(w)}\right)\right|<\frac{\alpha\pi}{2},\quad(w\in U)$$

where the function $g(w)$ is given by (1.2).

Definition 4.6. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma, \mathrm{Re}\left(\frac{zf'(z)}{(1-\delta)f(z)+\delta f'(z)}\right) > \beta \qquad (z \in U)$$

where and

$$g \in \Sigma, \mathrm{Re}\left(\frac{wg'(w)}{(1-\delta)g(w)+\delta g'(w)}\right) > \beta, \quad (w \in U)$$

the function $g(w)$ is is given by (1.2).

In this case .Theorems (2.2) and (3.2) reduce to the following:

Corollary 4.7. Let $f(z)$ given by (1.1)be in the class $\mathcal{G}_\Sigma$ $(\delta;\alpha)(0 < \alpha \le 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{2\alpha}{(1-\delta)\sqrt{(\alpha+1)}}$$

and

$$|a_3| \le \frac{4\alpha^2}{(1-\delta)^2} + \frac{\alpha}{1-\delta}.$$

Corollary 4.8. Let $f(z)$ given by (1.1) be in the class $\mathcal{G}_\Sigma$ $(\delta;\alpha)(0 \le \beta < 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{\sqrt{2(1-\beta)}}{(1-\delta)}$$

and

$$|a_2| \le \frac{4(1-\beta)^2}{(1-\delta)^2} + \frac{1-\beta}{(1-\delta)}.$$

Letting $\tau = 1$ and $\gamma = 1$ in Theorems (2.2) and (3.2) gives the following corollaries:

Corollary 4.9. Let $f(z)$ given by (1.1) be in the class $\mathcal{D}_\Sigma$ $(\delta;\alpha)(0 < \alpha \le 1, 0 \le \delta < 1)$.Then

$$|a_2| \le \frac{2\alpha}{\sqrt{|2\alpha(6-6\delta)-(4-4\delta^2))+(1-\alpha)(2-2\delta)^2|}}$$

and

$$|a_3| \le \frac{\alpha}{3(1-\delta)} + \frac{\alpha^2}{(1-\delta)^2}.$$

Corollary 4.10. Let $f(z)$ given by (1.1) be in the class $\mathcal{D}_\Sigma$ $(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ .Then

$$|a_2| \le \sqrt{\frac{(1-\beta)}{|2-3\delta+2\delta^2|}}$$

and

$$|a_3| \le \frac{(1-\beta)}{|3-3\delta|} + \frac{(1-\beta)^2}{(1-\delta)^2}.$$

The classes $\mathcal{D}_\Sigma$ $(\delta;\alpha)$and $\mathcal{D}_\Sigma$ $(\tau,\delta;\beta)$ are given explicitly in the next definitions .

Definition 4.11. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{D}_\Sigma$ $(\delta;\alpha)$ $(0 < \alpha \le 1, 0 \le \delta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \left|\arg\left(\frac{f'(z)+zf''(z)}{f'(z)+\delta zf''(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U)$$

and

$$g \in \Sigma, \left|\arg\left(\frac{g'(w)+w\,g''(w)}{g'(w)+\delta wg''(w)}\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in U)$$

where the function $g(w)$ is given by (1.2).

Definition 4.12. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{D}_\Sigma$ $(\delta;\beta)(0 \le \beta < 1, 0 \le \delta < 1)$ if the following conditions are satisfied :

$$f \in \Sigma, \mathrm{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\delta zf''(z)}\right) > \beta \qquad (z \in U)$$

and

$$g \in \Sigma, \mathrm{Re}\left(\frac{g'(w)+w\,g''(w)}{g'(w)+\delta g''\,g(w)}\right) > \beta, \quad (w \in U)$$

where the function $g(w)$ is given by (1.2).

References

[1] P. Duren ,Univalent Functions , Grandlehrender Mathematischen Wissenscaften,259, Springer, New York ,(1983).

[2] H. M. Srivastava and D. Bansal , Coefficient estimates for a subclass of analytic and bi- univalent functions, J .Egypt. Math. Soc.,1-4, (2014).

[3] M. Lewin ,On a coefficient problem for bi-univalent functions ,Proc. Am. Math. Soc.,18, 63-68, (1967).

[4] D.A. Brannan ,J. Clunie Aspects of contemporary complex Analysis ,Academic press ,New York London,(1980).

[5] E. Netanyahau, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal., 32(2),100-112, (1969).

[6] H. M. Srivastava ,A. K. Mishra and P. Gochhayat ,Certain subclasses of analytic and bi-univalent functions ,appl. Math. Lett., 23,1188-1192, (2010).

[7] B.A. Frasin , M. K. Aouf ,New subclasses of bi-univalent functions ,Appl. Math. Lett., 24 (9),1569-1573, (2011).

[8] D. BansaL ,J. Sokol , Coefficient bound for a new class of analytic and bi-univalent functions, J. Fract . Clac. Appl., 5(1), 122-128,(2014).

[9] B. S. Keerthi, B. Raja, Coefficient inequality for certain new subclasses of analytic bi- univalent functions, Theoretical Mathematics and Applications, 31(1),1-10, (2013).

[10] G. Murugusundaramoorthy , N. Magesh , V. Prameela , Coefficient bounds for certain subclasses of bi-univalent function, Abstr. Appl. Anal., 2013(Article ID 573017),1-3),(2013).

مخمنات المعامل لاصناف جزئيه من الدوال الثنائية التكافؤ

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المستخلص :

في هذ البحث قدمنا صفيين جزئيين جديدين من الصنف Σ متكون من الدوال ثنائية التكافؤ التحليلية في قرص الوحدة المفتوح U و حصلنا على مخمنات حول معاملات تايلر – ماكلورين $|a_2|$ و $|a_3|$ للدوال في هذه الاصناف الجزئية. حصلنا ايضا على حالات خاصة جديده لنتائجنا