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Certain Types of Complex Lie Group Action

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Abstract

The main aim in this paper is to look for a novel action with new properties on $Complex\ \ Lie\ Group$ from the $Lemma\ of\ Schure$, the Literature are concerned with studying the action of $Lie\ algebra$ of two representations , one is usual and the other is the dual, while our interest in this work is focused on some actions on complex Lie group[10] . Let G be a matrix complex Lie group , and π is representation of G. In this study we will present and analytic the concepts of action of complex Lie group on $Hom-space$. We recall the definition of tensor product of two representations of Lie group and construct the definition of action of Lie group on $Hom-space$, then by using the equivalent relation $Hom(w_2, w_1) \cong w_2^* \otimes w_1$ between Hom and $Tensor\ product$, we get a new action : $Action - complex\ Lie\ Group\ on\ tensor\ product$. The two actions are forming smooth representation of G [8], [9]. This we have new action which called $triple\ action\ of\ Complex\ Lie\ Group\ G$ denoted by $TAC-complex\ Lie\ group$ which acting on $Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*))$.
This TAC is smooth representation of G. The theoretical Justifications are developed and prove supported by some concluding remarks and illustrations.

Key words : Hom – Space , Tensor Product , Action of Lie Group , Complex Lie Group .

Mathematics subject classification: 64S40

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1- Introduction

A complex Lie group is a finite dimensional analytic manifold G together with a group structure on G, Such that the multiplication $GxG \to G$ and attaching of an inverse $g \to g^{-1} : G \to G$ is analytic map [4],[6].

A matrix Lie group is any subgroup G of $GL(n, \mathbb{C})$ with the following property [7] . If A_m converges to some Matrix A, then $A \in G$ or A is not invertible [5]. $The\ Schur's\ lemma$ introduced the concepts of Lie algebra on the space of Linear maps from W_2 $into$ W_1 ,which denoted by $Hom(W_2, W_1)$ [1],[3]. Also introduced the concepts of action on $Hom-space$ of two representations of $Lie\ algebr$ [1].

Also the main work here is to give a representation of complex Lie group by intertwine these actions (representations) and to give representation by intertwine duel of these actions (representations) and Then generalizing them.

2- The TcoA of complex Lie Groups on Hom - Space

In [2- P327], $the\ Schur's\ lemma$ introduced the concepts of action of $Lie\ algebra$ on $Hom\ space$ of Two representations of Lie algebra.

Lemma (2.1) [2]:

Suppose that $\pi_1\ and\ \pi_2$ are two representations of $Lie\ algebra\ g$ action on finite dimensional space $W_1\ and\ W_2$ respectively . Define an Co-action of g on $Hom_k(W_2, W_1)$, $\pi: g \to gl(Hom_k(W_2, W_1))$ for all $x \in g$, $F \in Hom_k(W_2, W_1), \pi_1(x)F - F\pi_2(x)$ $\pi(x)(v) = \pi_1(x)F(v)\ and\ Hom\ (W_2, W_1) = W_2^* \otimes W_1$ as equivalence of representations.

Lemma (2.2):

Put $Hom(Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$ the $K-vector-space$ of all Linear maps $(Hom(W_5 \oplus W_4), W_3^*)\ onto\ (Hom(W_2 \oplus W_1, W^*)$.

Define $\pi: G \to GL(Hom(Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$,

by $\pi(a) = \pi^*(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2\pi_3^*(a) \circ F_3 \circ (\pi_5(a) \oplus \pi_4(a))^{-1}$,

for all $a \in G$, $F_1 \in Hom(W_2^* \oplus W_1, W)$

$F_2 \in Hom(W_5 \oplus W_4, W_3^*)$

$F_3 \in Hom(Hom(W_5 \oplus W_4, W_3^*), Hom(W_2 \oplus W_1, W^*))$.

$$\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a) \oplus \pi_1(a))^{-1} \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1})_{(v)}$$

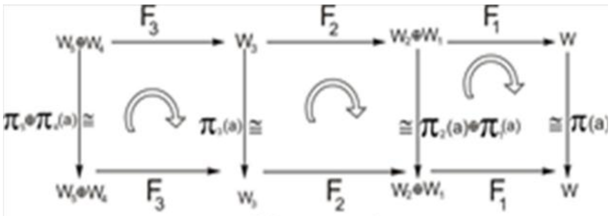

Diagram 1.

For all $a \in G, v \in (W_5 \oplus W_4)$

$$\pi(a)_v = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a)^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ (\pi_5(a)^{-1} \oplus \pi_4(a)^{-1})_{(v)}$$

For all $a \in G, v \in (W_5 \oplus W_4)$.

Where the arrow that makes the diagram 1 commutative π is $homomorphism$ of groups $G\ into\ GL((Hom(W_5 \oplus W_4), W_3^*), (Hom(W_2 \oplus W_1, W^*)))$.

Let $\pi_i: G \to GL(W_i)$, and $\pi_i^*: G \to GL(W_i^*)$, $for\ i = 1,2,3,4,5$.

The TAS of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$ is given by a representation π such that

$\pi(a) = \pi(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a))^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ \pi_5(a)^{-1} \oplus \pi_4(a)^{-1})$,

For all $a \in G$.

Then the TAS of complex Lie group G on $Hom_k(Hom((W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W^*))$ is also given by representation π^* such that

$$\pi^*(a) = \pi^*(a) \circ F_1 \circ (\pi_2^*(a)^{-1} \oplus \pi_1^*(a))^{-1}) \circ F_2 \circ (\pi_3^*(a) \circ F_3 \circ \pi_5^*(a)^{-1} \oplus \pi_4^*(a)^{-1}).$$

Proof of Lemma (2.2):

Let $TCoA$ of complex Lie group G on $Hom(Hom(W_5 \oplus W_4), W_3^*), Hom(W_2 \oplus W_1, W))$ is

induced by the representation

$\pi : \mathrm{G} \to \mathrm{GL}(\mathrm{Hom}(Hom\ (W_5 \oplus W_{4,} W_3^*), Hom(W_2 \oplus W_1, W^*))$

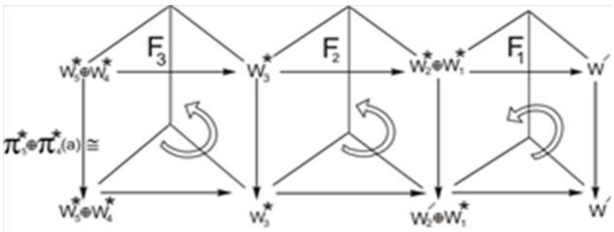

Diagram 2.

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such that $\pi(a) = \acute{\pi}(a) \circ F_1 \circ (\pi_2(a)^{-1} \oplus \pi_1(a))^{-1}) \circ F_2 \circ (\pi_3(a) \circ F_3 \circ \pi_5(a)^{-1} \oplus \pi_4(a)^{-1})$
For all $a \in G$. Thus π^* is a representation from G to $Hom - space$

This arrow makes the diagram 2 commutative .

Remark (2.4):
Since $\mathrm{Hom}\big(Hom\ (W_5 \oplus W_{4,} W_3^*), Hom\big(W_2 \oplus W_1, W^{*/}\big)\big) \cong ((W_5 \oplus W_{4,} W_3^*)^* \otimes ((W_2 \oplus W_1)^* \otimes W^{*/}) \cong ((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/})$
So we construct an action of G on the $product$,
Let $\pi(G) \to GL(((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/}))$, then π forms a representation of G acting on $vector\ space$ $(((W_5 \oplus W_4) \otimes W_3) \otimes ((W_2^* \oplus W_1^*) \otimes W^{*/}))$.

$$\begin{tikzcd}
(W_5^* \times W_4^*) \times W_3 \times (W_2^* \times W_1^*) \times W^{/} \arrow[r, "canianical"above, "map"below] \arrow[d, "f"', "Linear\ map"] & (W_5^* \oplus W_4^*) \otimes W_3 \otimes (W_2 \oplus W^{/}) \arrow[dl, "g"', "bilinear\ map"] \\
K \times (W_5 \times W_4) \times W_3 \times K \times (W_2 \times W_1) \times W^{/} &
\end{tikzcd}$$

Diagram 3.

$$f\left((W_5^* \times W_4^*) \times W_3 \times (W_2^* \times W_1^*) \times W^{/}\right) = w^{/} \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

$$g\left((W_5^* \oplus W_4^*) \otimes W_3 \otimes (W_2^* \oplus W_1^*) \otimes W^{/}\right) = w^{/} \circ (w_2^* \oplus w_1^*) \circ w_3 \circ (w_5^* \times w_4^*)$$

For all $w_5 \in W_5 , w_5 \in W_5 , w_4 \in W_4 , w_3 \in W_3 , w_2 \in W_2 , w_1 \in W_1 , w^{/} \in W^{/}$.

$$\begin{tikzcd}
G \arrow[r, "\Delta"] & G \times G \arrow[r, "\pi_3\ \pi_2\ \pi_1"] \arrow[dr, "\pi"'] & GL(W_5 \oplus W_4) \times GL(W_3) \times GL(W_2 \oplus W_1) \times GL(W^{/}) \arrow[d] \\
& & GL(W_5 \oplus W_4) \otimes GL(W_3) \otimes GL(W_2 \oplus W_1) \otimes GL(W^{/})
\end{tikzcd}$$

***Diagram* 4.**

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & G \times G \arrow[r] & GL(W_5 \oplus W_4) \times GL(W_3) \times GL(W_2 \oplus W_1) GL(W^{/}) \arrow[dl] \\
& GL((W_5 \oplus W_4) \times W_3 \times (W_2 \oplus W_1) \times W^{/}) \arrow[d] & \\
& GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W^{/}) &
\end{tikzcd}$$

***Diagram* 5.**

That π is a representation of G acting on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W^{/})$ where $\pi_i, i = 1,2,3,4,5$ are five representations of G acting on W_i , $i = 1,2,3,4,5$ respectively ,thus :
$\pi(ab) = \pi^{/}\ (ab) \circ W^{/} \otimes (W_2 \oplus W_1) \otimes W_3 \otimes (W_5 \oplus W_4)$
$\pi^{/}\ (ab) \otimes \pi_1^{/}(ab) \circ (W^{/})_{(v)} \circ \pi_2(ab)^{-1}\ (W_2 \oplus W_1)_{(v)} \circ \pi_3(ab)\ W_3(v) \circ\ \pi_4(ab)^{-1}\ (W_5 \oplus W_4)_{(v)} = \pi_4(ab)^{-1} \oplus \pi_4(ab)) \otimes W_3(ab)^{-1} \otimes \pi_2(ab) \otimes \pi_2(ab) \otimes \pi_1(ab)^{-1}\ \otimes \pi^{/}\ (ab)$
$\pi(a) \circ \pi(b) = \pi(b)(\pi(a)\ ) = \pi(b)(\ \pi^{/}(a) \circ (W^{/})_{(v)} \circ \pi_1(a)(W_2 \oplus W_1)_{(v)} \circ \pi_2(a) W_3(v) \circ \pi_3(a)(W_5 \oplus W_4)_{(v)} = \pi^{/}(b)(\ \pi^{/}(a) \circ (W^{/})_{(v)} \circ \pi_1(b)\pi_1(a)(W_2 \oplus W_1)_{(v)} \circ \pi_2(b)\pi_2(a) W_3(v) \circ \pi_3(b)\pi_3(a)(W_5 \oplus W_4)_{(v)} = \pi^{/}(ab) \circ W^{/} \otimes (W_2 \oplus W_1) \otimes W_3 \otimes (W_5 \oplus W_4)$.

$$\pi(ab) = \pi(a) \circ \pi(b)$$

π is a group homomorphism of G on $GL((W_5 \oplus W_4) \otimes W_3 \otimes (W_2 \oplus W_1) \otimes W^{/})$.

$$\begin{tikzcd}
W_5 \oplus W_4 \arrow[r] \arrow[d, "\pi_3(a)"', "\cong"] & W_3 \arrow[r] \arrow[d, "\pi_2(a)"', "\cong"] & W_2 \oplus W_1 \arrow[r] \arrow[d, "\pi_1(a)"', "\cong"] & W^{/} \arrow[d, "\cong"', "\pi^{/}(a)"] \\
W_5 \oplus W_4 \arrow[r] \arrow[d, "\pi_3(b)"', "\cong"] & W_3 \arrow[r] \arrow[d, "\pi_2(b)"', "\cong"] & W_2 \oplus W_1 \arrow[r] \arrow[d, "\pi_1(b)"', "\cong"] & W^{/} \arrow[d, "\cong"', "\pi^{/}(b)"] \\
W_5 \oplus W_4 \arrow[r] & W_3 \arrow[r] & W_2 \oplus W_1 \arrow[r] & W^{/}
\end{tikzcd}$$

***Diagram* 6.**

3- The $TCoA$ of Complex Lie Groups on Tensor Product

We have been introduced the triple Co-action of complex Lie groups by the tensor product of the five representations, which are $TCoA$-complex Lie groups on tensor product

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($W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ and constructed this definition. Depending on what has been mentioned above, Π is called Triple Co-Action of complex Lie groups on tensor product denoted by ($TCoA$-complex Lie groups).

Example (3.1):

Let $\Pi_1:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_1(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times2}$, for all $a \in \mathbb{R}$, $\Pi_2:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_2(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}_{2\times2}$, for all $a \in \mathbb{R}$, $\Pi_3:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_3(a) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}_{2\times2}$, for all $a \in \mathbb{R}$, $\Pi_4:\mathbb{R} \longrightarrow GL(2,\mathbb{C})$ such that $\Pi_4(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times2}$, for all $a \in \mathbb{R}$ and $\Pi_5:\mathbb{R} \longrightarrow GL(3,\mathbb{C})$ such that $\Pi_5(a) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{3\times3}$, for all $a \in \mathbb{R}$.

The representation Π of $GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is:

$\Pi:G \longrightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))) \cong GL(M(8\times3),\mathbb{C})$, such that

$\Pi(a) = (((\Pi_1(a) \otimes \Pi_3^*(a)^{-1} \oplus \Pi_2(a) \otimes \Pi_3^*(a)^{-1}) \otimes \Pi_4(a)) \otimes \Pi_5^*(a)^{-1})$, where Π^* is dual representation

$$= \left(\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}\right] \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

$$= \left(\left[\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & -2 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}_{4\times4} \oplus \begin{pmatrix} 1 & 2 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}_{4\times4}\right] \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times2}\right) \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times3}$$

$$= \left[\begin{pmatrix} 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{2\times2}\right] \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times3}$$

$$= \begin{pmatrix} 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 \\ 1 & 2 & -2 & -4 & 1 & 2 & -2 & -4 \\ 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 4 & 0 & 0 & 2 & 4 \end{pmatrix}_{8\times8} \otimes \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}_{3\times3}$$

$$= \left(\begin{array}{cccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 1 & 2 & -2 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 2 & 4 \\
-1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 \\
-\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 \\
0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \\
0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 \\
-1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & -4 & -4 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & -2 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & -2 & -4 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & -1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 2 & -1 & -1 & 2 & 2 & 2 & 2 & -4 & -4 & 2 & 2 & -4 & -4 & 1 & 1 & -2 & -2 & 1 & 1 & -2 & -2 \\
-\frac{1}{2} & -1 & 1 & 2 & -\frac{1}{2} & -1 & 1 & 2 & 1 & 2 & -2 & -4 & 1 & 2 & -2 & 0 & -3 & 3 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{2} & -\frac{3}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & -3 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & -3 & 3 & 0 & \frac{3}{2} & -\frac{3}{2} & 0 & -\frac{3}{2} & \frac{3}{2}
\end{array}\right)_{24\times24}$$

Proposition (3.2):

Let $\Pi_i:G \longrightarrow GL(W_i)$, $\Pi_i^*:G \longrightarrow GL(W_i^*)$ for $i = 1,2,3,4,5$ and the $TCoA$-complex Lie groups of G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1))))$ is given by a representation Π such that

$\Pi(a) = [(\Pi_1(a) \circ W_1 \circ W_3^* \circ \Pi_3^*(a^{-1})) \oplus (\Pi_2(a) \circ W_2 \circ W_3^* \circ \Pi_3^*(a)^{-1}) \circ \Pi_4(a) \circ W_4 \circ W_5^* \circ \Pi_5^*(a)^{-1}$,

for all $a \in G$.

Then the $TCoA$-complex Lie group of G on (W_5^* ⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) is also given by a representation Π*, such that:

$$\Pi^*(a)=\Pi_5^*(a)^{-1}\circ W_5\circ W_3^*\circ(\Pi_4^*(a)\circ(\Pi_3^*(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a)\oplus\Pi_3^*(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a)))$$

,
for all $a \in$ G.

Proof: Let $TCoA$-complex Lie group G on (W_5^* ⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) is induced by the representation Π:G $\longrightarrow$ GL((W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) such that

$\Pi(a)$ = $(((\Pi_1(a)\circ W_1\circ W_3^*\circ\Pi_3(a^{-1}))\oplus(\Pi_2(a)\circ W_2\circ W_3^*\circ\Pi_3(a)^{-1})\circ\Pi_4(a)\circ W_4\circ W_5^*\circ\Pi_5(a)^{-1})$,

for all $a\in$G, $W_3'\times W_1'\in W_3\times W_1$, $\Pi_3 \in (W_3,(W_2\times W_1))$, $\Pi_4\in W_5\times W_4$.

To show that Π*:G $\longrightarrow$ GL(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1))))* is a representation, such that

$\Pi^*(a)$ = $(((\Pi_5^*(a)^{-1}\circ W_5\circ W_4^*\circ(\Pi_4^*(a)\circ(\Pi_3^*(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a)\oplus\Pi_3^*(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a))$

is a representation for all $a\in$G and $\Pi_4^*\in(W_5^*\otimes W_4)^*$, $\Pi_3^*\in(W_3,(W_2\otimes W_1))^*$, $\Pi_2^*\times\Pi_1^*\in(W_2\otimes W_1)^*$ since

$$\Pi^*(a)=\Pi_5(a)^{-1}\circ W_5\circ W_4^*\circ(\Pi_4(a)\circ((\Pi_3(a)^{-1}\circ W_3\circ W_2^*\circ\Pi_2^*(a))\oplus(\Pi_3(a)^{-1}\circ W_3\circ W_1^*\circ\Pi_1^*(a)))$$

For all $a \in$ G, $\Pi_4^*:W_4^*\longrightarrow W_5^*$ and $\Pi^*(ab)=(\Pi(ab))^*=(\Pi(b)\circ\Pi(a))^*=\Pi^*(a)\circ\Pi^*(b)$. Thus Π* is a representation from G (Π* is a group *homoorphism of G*)

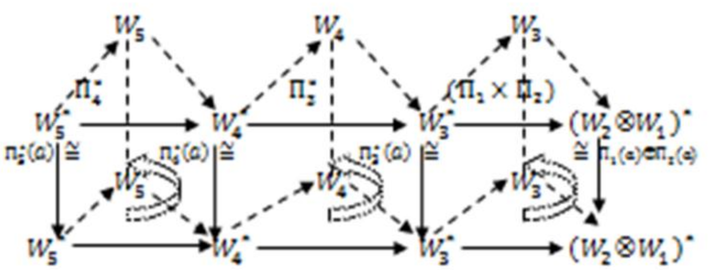

Diagram 7.

This arrow makes the diagram 7 commutative.

Proposition (3.3):

Let Wi for i = 1, 2, 3, 4, 5 are finite vector spaces, W_i^* is the dual of vectors W_i, for i = 1, 2, 3, 4, 5 then the following assertions are equivalent:

(1) [(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1))))]*.

(2) (((W_3^*⊗W_2)*⊕ (W_3^*⊗W_1)*)⊗W_4^*)*⊗ W_5^{**}).

(3) (W_2^*⊗W_3)⊕(W_1^*⊗W_3)⊗ W_4^*)⊗W_5).

(4) ((($W_2^*\otimes W_1^*$)⊗W_3)⊗ W_4^*)⊗W_5).

(5) (($W_2^*\otimes W_1^*$)⊗(W_3^*,K)⊗ W_4^*)⊗W_5).

(6) (($W_2^*\otimes W_1^*$)⊗W_3) ⊗ W_4^*)⊗(W_5^{**},K)).

(7) ((W_1⊗W_2)*⊗ W_4^*)⊗(W_3,K)⊗ W_4^*))⊗W_5).

(8) [(W_5⊗(W_4⊗(W_3⊗W_2)⊕(W_3⊗W_1))]***

$$=\begin{cases}(W_5\otimes(W_4\otimes(W_3\otimes W_2)\oplus(W_3\otimes W_1)) & \text{if } n \text{ is an even number}\\ (W_3\otimes W_2)^*\oplus(W_3\otimes W_1)^*\otimes W_4^*)\otimes W_5 & \text{if } n \text{ is an odd number}\end{cases}$$

Proof:

(1) ≅ (2) To show [(W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1))]* ≅ (((W_3^*⊗W_2)*⊕(W_3^*⊗W_1)*)⊗ W_4^*)*⊗ W_5^{**}).

Let $\Pi_4\in(W_5^*\otimes(W_4)$, $\Pi_3\in(W_4\otimes W_3^*)$, $\Pi_2\times\Pi_1\in(W_3^*\otimes(W_2\otimes W_1))$, $\Pi_4^*\in(W_4^*\otimes W_5)$,

$\Pi_3^*\in(W_3,W_4^*),\Pi_1^*\times\Pi_2^*=(\Pi_2\times\Pi_1)^*\in((W_1^*\otimes W_2^*)\otimes W_3)$

and there exists an intertwining map

ψ: (W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕(W_3^*⊗W_1)))) $\longrightarrow$ (((W_3^*⊗W_2)*⊕(W_3^*⊗W_1)*)⊗ W_4^*)*⊗ W_5^{**},

such that

ψ(Π*(a))(v) = Π*(a)ψ(v), for all v ∈ $W_1^*\times W_2^*$

and ψ is an invertible map.

(1) ≅ (3) To show (W_5^*⊗(W_4⊗((W_3^*⊗W_2)⊕

(2) (W_3^*⊗W_1)))* ≅

(($W_2^*\otimes W_3^*$)⊕($W_1^*\otimes W_3^*$)⊗ W_4^*)⊗W_5), since (W_3^*,W_2)* ≅ (W_2^*, W_3^{**}), (W_3^*,W_1)* ≅ (W_1^*, W_3^{**}), $W_5^{**}\cong W_5$ and $W_3^{**}\cong W_3$.

By the same methods, we have the other parts. ■

Example (3.4):

Let Π_i, $i = 1,2,3,4$, $\Pi_i : S^1 \longrightarrow SO(2) \subset GL(2,\mathbb{C})$ and $\Pi_5 : S^1 \longrightarrow O(3) \subset GL(3,\mathbb{C})$, where $G = S^1$, $(n = 2,\ m = 3)$ and Wi, $i = 1,2,3,4$ are the $\mathbb{C}$-vector spaces of dimensional 2 and W_5 is the $\mathbb{C}$-vector space of dimensional 3 such that,

$$\Pi_1(e^{i\theta}) = \Pi_2(e^{i\theta}) = \Pi_3(e^{i\theta}) = \Pi_4(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

, $0 \le \theta \le 2\pi$, $i^2 = -1$, $\Pi_5(e^{i\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, $0 \le \theta \le 2\pi$. The $TCoA$-complex Lie group G on $(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ is a representation:

$\Pi :\ S^1 \longrightarrow GL(W_5^* \otimes (W_4 \otimes ((W_3^* \otimes W_2) \oplus (W_3^* \otimes W_1)))$ such that

$$\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\sin\theta\cos\theta & \cos^2\theta & \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin^2\theta & \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}_{4\times4} \Bigg]$$

$$= \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta & 0 & -\sin\theta\cos\theta & \sin^2\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta & 0 & -\sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & \sin\theta\cos\theta & -\sin^2\theta & 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin^2\theta & \sin\theta\cos\theta & 0 & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}_{6\times6} \otimes$$

$$\begin{pmatrix} 2\cos^2\theta & 2\sin\theta\cos\theta & -2\sin\theta\cos\theta & -2\sin^2\theta \\ -2\sin\theta\cos\theta & 2\cos^2\theta & 2\sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 2\sin^2\theta & 2\cos^2\theta & 2\sin\theta\cos\theta \\ -2\sin^2\theta & 2\sin\theta\cos\theta & -2\sin\theta\cos\theta & 2\cos^2\theta \end{pmatrix}_{4\times4}$$

$$\Pi^*(a) = (\Pi_5(a))^{-1} \circ W_5 \circ W_4^* \circ (\Pi_4^*(a) \circ ((\Pi_3(a))^{-1} \circ W_3 \circ W_2^* \circ \Pi_2^*(a)) \oplus (\Pi_3(a))^{-1} \circ W_3 \circ W_1^* \circ \Pi_1^*(a)))$$

$$\Pi^*(e^{i\theta}) = \Pi_5(e^{-i\theta}) \otimes (\Pi_4^*(e^{i\theta}) \otimes ((\Pi_3(e^{-i\theta}) \otimes \Pi_2^*(e^{i\theta})) \oplus (\Pi_3(e^{-i\theta}) \otimes \Pi_1^*(e^{i\theta}))$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \otimes \left(\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) \oplus \right.$$

$$\left. \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) \right)$$

$$= \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta & 0 & -\sin\theta\cos\theta & \sin^2\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta & 0 & -\sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & \sin\theta\cos\theta & -\sin^2\theta & 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin^2\theta & \sin\theta\cos\theta & 0 & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}_{6\times6} \otimes$$

$$\left[\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\sin\theta\cos\theta & \cos^2\theta & \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin^2\theta & \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}_{4\times4} \oplus \right.$$

$$\begin{bmatrix}
2\cos^3\theta & 0 & 0 & -2\cos^2\theta\sin\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta \\
0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 \\
0 & 2\sin\theta\cos^2\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta \\
0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta \\
-2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 \\
0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta \\
0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta \\
-2\sin\theta\cos^2\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta \\
2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & -2\sin^3\theta & 0 & 0 \\
0 & -2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta \\
0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta \\
2\sin^2\theta\cos\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta \\
0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta \\
2\sin^2\theta\cos\theta & 0 & 0 & 2\sin^3\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
-2\sin^3\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 \\
0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^2\theta\cos\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta \\
0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta
\end{bmatrix}$$

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$$
\left.\begin{matrix}
-2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & 2\sin^3\theta & 0 & 0 \\
0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta \\
0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta \\
-2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^3\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
2\sin^2\theta\cos\theta & 0 & 0 & -2\sin^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^4\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta \\
0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^4\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta \\
2\sin^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 \\
0 & 2\sin^3\theta\cos\theta & -2\sin^4\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta \\
0 & 2\sin^4\theta & 2\sin^3\theta\cos\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta \\
2\cos^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & -2\sin^2\theta\cos\theta & 0 & 0 \\
0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^2\theta & 2\sin^2\theta\cos^2\theta & 0 & -2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta \\
0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta \\
2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\sin\theta\cos^3\theta & 0 & 0 \\
0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta \\
0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^3\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & 2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin^2\theta\cos^2\theta \\
-2\sin\theta\cos^2\theta & 0 & 0 & 2\sin^2\theta\cos\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 \\
0 & -2c\sin\theta os^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin^2\theta\cos^2\theta & -2\sin^3\theta\cos\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta \\
0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^3\theta\cos\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta \\
-2\sin^2\theta\cos\theta & 0 & 0 & -2\sin\theta\cos^2\theta & 0 & 0 & 2\sin\theta\cos^2\theta & 0 & 0 & 2\cos^3\theta & 0 & 0 \\
0 & -2\sin^2\theta\cos^2\theta & 2\sin^3\theta\cos\theta & 0 & -2\sin\theta\cos^3\theta & 2\sin^2\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & -2\sin^2\theta\cos^2\theta & 0 & 2\cos^4\theta & -2\sin\theta\cos^3\theta \\
0 & -2\sin^3\theta\cos\theta & -2\sin^2\theta\cos^2\theta & 0 & -2\sin^2\theta\cos^2\theta & -2\sin\theta\cos^3\theta & 0 & 2\sin^2\theta\cos^2\theta & 2\sin\theta\cos^2\theta & 0 & 2\sin\theta\cos^3\theta & 2\cos^4\theta
\end{matrix}\right]_{24\times 24}
$$

Proposition (3. 5):

Let Π_i, $i = 1,2,3,4,5$ be representations of G acting on K-finite dimensional vector spaces W_i, $i = 1,2,3,4,5$ respectively, then the $TCoA$-reductive Lie group of G on $\underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is equivalent to the representation $\Pi_5^*\otimes(\Pi_4\otimes((\Pi_3^*\otimes\Pi_2)\oplus(\Pi_3^*\otimes\Pi_1))))$ of G on $GL(W_5^*\otimes(W_4\otimes((W_3^*,W_2)\oplus(W_3^*,W_1))))$.

Proof: To show that:

$\psi:(W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is bilinear map, defined by $\psi(W_5^*,w_1) = F$ for all $W_5^* \in W_5$ and $w_1 \in W_1$, where $F: W_5 \longrightarrow W_1$ is a linear map defined by $F(v) = w_5^*(v)w_1$, for all w_5^*, $w_5^{*'} \in W_5^*$, $v \in W_5$, $\alpha, \beta \in K$, $w_1 \in W_1$

$\psi(\alpha w_5^* + \beta w_5^{*'},w_1) = (\alpha w_5^* + \beta w_5^{*'}(v))w_1$

$= \alpha w_5^*(v)w_1 + \beta w_5^{*'}(v)w_1$

$= \alpha\psi(w_5^*,w_1) + \beta\psi(w_5^{*'},w_1)$

Other for all w_1, $w_1' \in W_1$ and $w_5^* \in W_5^*$

$\psi(,\alpha w_1 + \beta w_1') = (w_5^*(v)(\alpha w_1 + \beta w_1')$

$= w_5^*(v)(\alpha w_1) + w_5^{*'}(v)(\beta w_1')$

$= \alpha W_5^*(v)w_1 + \beta w_5^{*'}(v) w_1'$

$\psi(W_5^*,\alpha w_1 + \beta w_1') = \alpha\psi(W_5^*,w_1) + \beta\psi(W_5^*, w_1')$.

So $\psi: W_5^*\times(W_4\times((W_3^*\times W_2)\oplus(W_3^*\times W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$ is a bilinear map, thus by using the tensor product and universal property of this tensor product, we get a unique linear map ϕ.

$$
\begin{tikzcd}
(W_5^*\times(W_4\times((W_3^*\times W_2)\oplus(W_3^*\times W_1)))) \arrow[r] \arrow[d, "\psi\ \text{bilinear map}"'] & (W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1)))) \arrow[dl, "\phi\ \text{Linear map}"] \\
\underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1))) &
\end{tikzcd}
$$

Diagram 8.

So by universal property of tensor product $W_5^*\times(W_4\times((W_3^*\times W_2)\oplus(W_3^*\times W_1))))$ there exists a unique linear map $\phi: W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus(W_3^*\otimes W_1))))\longrightarrow \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*,Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$. This makes the above diagram commutative:

$$
\begin{tikzcd}
 & K \\
W_5 \arrow[r] \arrow[ur, "W_5^*"] & W_1 \cong K\otimes W_1 \arrow[u, "\text{Proj.}"']
\end{tikzcd}
$$

Diagram 9.

Consider the composition of linear maps where $W_5^*(v)$ is defined as follows:

$F(v) = w_1$, $\exists!\ k \in K$, such that $w_1 \longrightarrow (k,w_1)$ since all maps are linear and k is unique, put $W_5^*(v) = k$ related to w_1.

Define $\zeta: \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*, Hom((W_3,W_2)\oplus Hom(W_3,W_1)))$
$\longrightarrow W_5^* \otimes(W_4\otimes((W_3^*\otimes W_2)\oplus (W_3^*\otimes W_1)))$ by $\zeta(F') = W_5^*(v)w_1$.

Define $W_5^*: W_5 \longrightarrow K$ by $W_5^*(v) = k$, where k is given by $\zeta(F'(v)) = (k, F'(v))$

We can show that W_5^* is linear put $F'(v) = w_1$, for all $F' \in \underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*, Hom((W_3, W_2)\oplus Hom(W_3,W_1)))$, $w_1 \in W_1$ and $w_5^* \in W_5$ and is related to W_1.

$F'(\alpha v_1 + \beta v_2) = \alpha F'(v_1) + \beta F'(v_2)$
$= \alpha k_1 + \beta k_2$
$= \alpha W_5^*(v) + \beta W_5^*(v)$, for all $W_5^* \in W_5$

Where: $w_5^*(v_1) = k_1 \Rightarrow w_5^*(\alpha v_1) = \alpha k_1$,
$w_5^*(\alpha v_1 + \beta v_2) = (\alpha k_1 + \beta k_2)$
$w_5^*(v_2) = k_2 \Rightarrow w_5^*(\beta v_2) = \beta k_2$,

$$\begin{tikzcd} W_5 \arrow[r, "F'"] \arrow[drr, "W_5^*"'] & W_1 \arrow[r, "\text{Linear iso.}"] & W_1\otimes K \arrow[d, "\text{Proj.}"] \\ & & K \end{tikzcd}$$

Diagram 10.

Clear F' is a linear and $\zeta^{-1} = \phi$, thus ζ is linear map.
Related between the $TCoA$ of reductive Lie groups of G on $\underset{K}{\mathrm{Hom}}(W_5,\mathrm{Hom}(W_4^*, Hom((W_3, W_2)\oplus Hom(W_3,W_1)))$ and $TCoA$ of reductive Lie groups of G on $W_5^*\otimes(W_4\otimes((W_3^*\otimes W_2)\oplus (W_3^*\otimes W_1)))$ up to the representation given above:

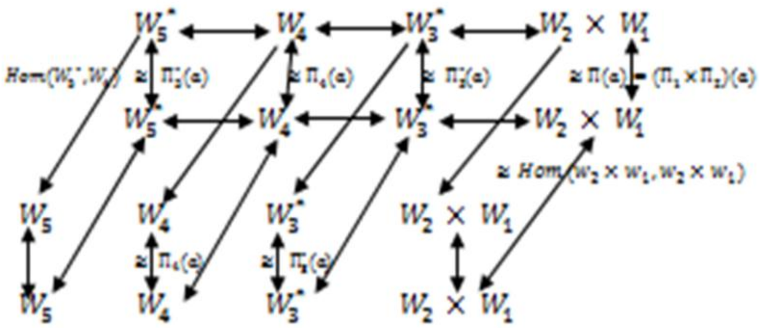

Diagram 11.

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أنواع معينة لفعل زمر لي المركبة

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المستخلص :

الهدف الرئيسي من البحث هو الحصول على فعل بصفات جديدة في زمر لي المركبة من بديهية $Schure$ التي درست وركزت على فعل جبر لي لتمثيليين احدهما عادي والأخر ثنائي ، والشئ المهم والممتع في العمل هو التركيز على بعض الأفعال لزمرة لي المركبة. في هذة الدراسة قمنا بتحليل مفاهيم من فعل زمر لي المركبة على فضاءات $\boldsymbol{Hom}$ وتعريف الضرب التنسوري لتمثيلات اثنتين في زمر لي وركزنا على فعل زمر لي على فضاءات $\boldsymbol{Hom}$، باستخدام التكافؤ

$$Hom(w_2, w_1) \cong w_2^* \otimes w_1$$

بين فضاءات $\boldsymbol{Hom}$ والضرب التنسوري للحصول على فعل زمر لي المركبة على الضرب التنسوري.
الفعل الثاني هو بصيغة تمثيلات ملساء للمجموعة $\boldsymbol{G}$. هذا الفعل هو فعل ثلاثي لزمر لي المركبة $\boldsymbol{G}$ ويرمز لها (فعل زمر لي المركبة $\boldsymbol{TAC}$)على

$$Hom(Hom((w_5 \oplus w_4), w_3^*), Hom(w_2 \oplus w_1, w^*)).$$

وهذا TAC هوتمثيلات ملساء في G ، أن النظريات المقدمة في هذا البحث انشات وبرهنت وجهزت ببعض النتائج كملاحظات و رسوم.