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Trivial Extension of Armendariz Rings and Related Concepts

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Abstract.

This paper investigate the possibility of inheriting the properties of the ring R to the trivial extension ring $T(R,R)$ and present the relationship between the trivial extension $T(R,R)$ and many types of rings. The concepts of 2-primal, reversible, central reversible, semicommutative, nil–semicommutative, π −Armendariz and central π −Armendariz rings are studied with trivial extension $T(R,R)$ and some characterizations of π-Armendariz and central π −Armendariz rings are given.

Keywords. Trivial extension**,** Armendariz**,** α-Armendariz, nil-Armendariz, central $\pi -$ Armendariz .

Mathematics subject classification: 11T55,13F20,11CXX.

1.Introduction.

Throughout this paper, all rings are commutative with identity unless otherwise stated. For a ring R, we denote by $C(R)$ the center of a ring R. A ring R which has no non zero nilpotent elements is called reduced. In (1997) the concept of Armendariz rings is introduced by Rage and Chhawchharia [1]. An Armendariz ring (ARM ring, for short) R is a ring that satisfies if $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ satisfy $f(x)g(x)=0$ implies that $s_i t_j=0$ for each i, j. Rege and Chhawchharia in [1] showed that every reduced ring is ARM. Consequently, this class of rings are related to nilpotent elements. Anderson and Camillo [2] proved that if $n \geq 2$, then $R[x]/x^n$ is an ARM ring if and only if R is reduced. A ring R is said to reversible if $ab=0$ then $ba=0$, for all $,b \in R$. According to shin [3], semicommutative rings introduced as a generalization of commutative rings. A ring R that satisfies $st=0$ implies $sRt=0$ for each $s,t \in R$ is called semicommutative if. Clearly every reduced ring is reversible and every reversible ring is semicommutative but the converse is not true in general [4]. The set of all nilpotent elements in R, the set of all nilpotent polynomials in $R[x]$ and the intersection of all prime ideal are denoted by $N(R)$ ($N(R[x])$ and $P(R)$, respectively). Due to Birkenmeier et al. [5], a ring R is said to be 2-primal if $N(R)=P(R)$. Every semicommutative ring is 2-primal , Semicommutative rings also studied under the name zero insertive by Habeb [6]. Many of authors have been written on ARM property [7] and [8]. A ring R is said to be π-Armendariz (π-ARM, for short) if for any two polynomials $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ such that $f(x)g(x) \in N(R[x])$, then $s_i t_j \in N(R)$ for each i,j [9].

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Each ARM (2-primal) ring is π- ARM, but the converse may not true [9]. Recall that a ring R is said to be weak Armendariz (WARM, for short) if whenever $f(x) = \sum_{i=0}^{m} s_i x^i$ and $g(x) = \sum_{j=0}^{m} t_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ then $s_i t_j \in N(R)$ for each i,j [10]. It is easy to see that every ARM ring is WARM, but the convers is not true in general.

This further encourages to the study of nilpotent elements, which is a generalization of ARM rings which have been studied in [10]. The ARM feature of a ring was generalized to one of skew polynomial as in [11] and [12]. Let $\alpha: R \to R$ be an endomorphism for a ring R. The ring that produced by giving the polynomial ring over R with the multiplication $xr = \alpha(r)x$, for all $r \in R$ is called skew polynomial ring $R[x;\alpha]$ of R. Some properties of skew polynomial rings have been studied in [13] and [14]. As a generalization of the notion of ARM rings, the concepts of α-Armendariz(α −ARM, for short) rings and α-skew Armendariz (α −SARM, for short) are introduced in [11] and [12] respectively. A ring R is called α-SARM (respectively, α-ARM) if for any two polynomials $f(x) = \sum_{i=0}^{m} s_i x^i$, $g(x) = \sum_{j=0}^{n} t_j x^j \in R[x;\alpha]$ such that $f(x)g(x) = 0$ then $s_i \alpha^i(t_j) = 0$ (respectively, $s_i t_j = 0$) for each i, j [11] (Respectively [12]). Agayev et. al. [15] introduced the concept of central Armendariz rings (CARM, for short) which is a generalization of the concept of ARM rings. The notion of CARM rings lies exactly between the notions of abelian rings and ARM rings. A ring R is said to be CARM if there exists any two polynomials $f(x)=\sum_{i=0}^{m} s_i x^i$ and $g(x)=\sum_{j=0}^{m} t_j x^j \in R[x]$ such that $f(x)g(x) = 0$, then $s_i t_j \in C(R)$ for each i,j. As a generalization of CARM rings, Abdali in [16] introduced and studied the sense of central π-ARM rings(Cπ-ARM, for short). A ring R is called Cπ-ARM if for all $f(x) = \sum_{i=0}^{m} s_i x^i$, $g(x)=\sum_{j=0}^{n} t_j x^j \in R[x]$ such that $f(x)g(x) \in N(R[x])$ then $s_i t_j \in C(R)$ for each i,j. Note that every CARM ring is Cπ-ARM.

The main idea is to set M may be ($M = R$) in a commutative ring T=(R,M)=$R \oplus M$ (if R is commutative) so that the structure of M as an R-module is essentially the same as that of M as an T-module, that is, as an ideal of T [17], and [18]. The advantage of the trivial extension is:

(a) Transfer results relating to modules to the ideal case including the R-module$R$, (b) Extending results from rings to modules, (c) It is easier to find counterexamples of rings especially those with zero divisors.

Generally, this paper studied and explained the relationship between some types of rings and some related concepts with trivial extension ring $T(R,R)$.

Moreover, we extended the terms of domain, central reversible, 2-primal, symmetric, reversible, semicommutative, nil– semicommutative, π −ARM, central reduced, central reversible and Cπ −ARM to the trivial extension $T(R,R)$. We showed that if $R[x]$ is domain, then $T(R,R)$ is α-ARM. Also if R is a central reversible ring, then $T = T(R,R)$ is nil-ARM. In addition, R is 2-primal ring iff $T = T(R,R)$ is ARM ring according to specific conditions and characterizations of π-ARM and Cπ −ARM rings are given.

There is a considerable number of authors showed interest in studying trivial extension rings related or not to the family of ARM rings. Kim and Lee in [7] studied trivial extension rings of ARM rings. In 2006, Liu and Zhao show that trivial extension of WARM ring is also WARM [10]. The property of α-ARM has been studied in [12]. Also the trivial extension of central reduced rings has been discussed in [19]. The π -regular rings fact of trivial extension has recently been explained in more details by Abduldaim in [20].

It is worth to mention that many suggestions could be put to the study trivial extensions of some new generalizations like α-skew π-McCoy rings [21]. Moreover, the trivial extension rings can be employed in the field of cryptography as in [22] and [23].

The structure of this paper is as follows. Section 2 is devoted to recall previous known definitions and information about the trivial extension. In section 3, the paper presents the relationship between the trivial extension rings and some kind of rings. Several examples are given to clarify the ideas used within the section.

2. Preliminaries

In this section we survey known results concerning $T = T(R,M) = R \oplus M$. The theme throughout is how properties of T=$T(R,M)$=$R \oplus M$ are related to those of R and M([17], [18]).

Let M be an (R,R)-bimodule. Recall that the trivial extension of R by M (also
called the idealization of M over R) is given to be the set $T = T(R,M)$ of all pairs (r,m) where $r \in R$, $m \in M$, that is:

$T = T(R,M)$=$R \oplus M$=$\{(r,m) | r \in R, m \in M\}$

With addition defined component wise as

$$(r_1,m_1) + (r_2,m_2) = (r_1 + r_2, m_1 + m_2)$$

And multiplication defined according to the rule

$$(r_1,m_1)(r_2,m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

For all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Clearly $T = T(R,M)$ forms a ring (even an R-algebra) and it is commutative if and only if R is commutative. Note that $T(R,0) \cong R$ via $r \to (r,0)$, then R can be embedded into $T(R,M)$, this means M is identified

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with $T(0,M) = \{(0,m)|m \in M\}$ becomes a nonzero nilpotent ideal of $T(R,M)$ of index 2, which explains the term idealization. If N is a submodule of M, then $T(0,N)$, is an ideal of $T(R,M)$, and that $(T(R,M)/T(0,M) \cong R$. The ring $T = T(R,M)$ has identity element (1,0) and any idempotent element of the trivial extension ring $T(R,R)$ is of the form $(e,0)$ where $e^2 = e \in R$. In fact, there is another realization of the trivial extension.

Let $T = T(R,M) \cong \left\{\begin{pmatrix} r & m \\ 0 & r\end{pmatrix} | \ r \in R, m \in M\right\}$, then T is a subring of the ring of 2×2 matrices over with the usual matrix operations and T is a commutative ring with identity. If $M = R$, then $T = T(R,R) \cong R[x]/< x^2 >$where $R[x]$ denote the ring of all polynomials over R and $< x^2 >$ is the ideal generated by$< x^2 >$. Recall that a ring R is said to be semiprime such that $sRs = 0$ then $s = 0$ for $s \in R$, [16].

Proposition 2.1 [24, Proposition 2.18] Let R be a semiprime ring R. Then the following are equivalent:

(1) R is reduced

(2) R is symmetric

(3) R is reversible

(4) R is semicommutative

(5) R is nil-semicommutative

(6) R is 2-primal

Corollary 2.2 [24, Corollary 2.19] Let R be a Von Neumann regular ring R. Then the following are equivalent:

(1) R is reduced
(2) R is symmetric
(3) R is reversible
(4) R is semicommutative
(5) R is nil-semicommutative
(6) R is 2-primal

Theorem 2.3 [2, Theorem 5]: Let R be a ring and $n \geq 2$ a natural number. Then $R[x]/(x^n)$ is ARM if and only if R is reduced.

3. Main Results

It is known that if R is a domain, then $R \oplus R$ is ARM [1]. In addition, R is an α–SARM ring for any endomorphism α of a domain ring R [11, Proposition10]. Hong et al. [12, Example 1.9] illustrate that a domain may not be an α–ARM ring for an arbitrary endomorphism α.

Theorem 3.1 Let $R[x]$ be a domain. Then $T = T(R,R)$ is α-ARM.

Proof Suppose that $f(x) = \sum_{i=0}^{m} A_i\, x^i$, $g(x) = \sum_{j=0}^{n} B_j\, x^j$ in $T = T(R,R)[x,\alpha]$ with $f(x)g(x)$=0 where $A_i = (a_i, u_i)$, $B_j = (b_j, v_j)$ for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $f_0(x) = \sum_{i=0}^{m} a_i\, x^i$, $f_1(x) = \sum_{i=0}^{m} u_i\, x^i$, $g_0(x) = \sum_{j=0}^{n} b_j\, x^j$ and $g_1(x) = \sum_{j=0}^{n} v_j\, x^j$ are elements in $R[x]$. In the other words $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$. So, we have

$f(x)g(x)$=(0,0)=($f_0(x)\, g_0(x)$, $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$) … (1)

And this implies that $f_0(x)\, g_0(x)$=0 and $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0.

Since $R[x]$ is domain, then

(a) Either $f_0(x)$ =0 which means that a_i=0 for all $0 \leq i \leq m$ and by Eq. (1) we have $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0=$f_1(x)\, g_0(x)$= 0 in $R[x,\alpha]$.

Again since $R[x]$ is domain, then either $f_1(x)$ =0 or $g_0(x)$=0, thus $A_i\, B_j$=0 , $0 \leq i \leq m$, $0 \leq j \leq n$.

(b) Or $g_0(x)$= 0 which means that b_j=0 for $0 \leq j \leq n$ and by Eq. (1) we have $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0=$f_0(x)\, g_1(x)$= 0 in $R[x,\alpha]$.

Since $R[x]$ is domain, so we get either $f_0(x) = 0$ or $g_1(x) = 0$. The two cases (a) and (b) yields $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, thus $A_i\, B_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $0 \leq i \leq m$, $0 \leq j \leq n$.

Hence $T = T(R,R)$ is α–ARM ring.

Using Theorem 3.1 to get the following corollary.

Corollary 3.2 Let $R[x]$ be a domain ring. Then T= $T(R,R)$ is α –SARM ring.

Proof Since $R[x]$ is domain, then T= $T(R,R)$ is α –ARM by Theorem 3.1 and by [12, Theorem 1.8] T= $T(R,R)$ is α–SARM ring.

Kose et al. in [25] introduced the definition of central reversible rings. A ring R is said to be central reversible if for each $s,t \in R$ such that $st = 0$ then $ts \in C(R)$. Clearly, every reversible ring is central reversible, but the converse need not be true reversible. The concept of nil-ARM (n-ARM, for short) rings is introduced by Ramon Antoine in 2008 [26]. A ring R is called n-ARM if $f(x), g(x) \in R[x]$ achieves

$f(x)g(x) \in N(R)[x]$ then $a_i b_j \in N(R)$ for all i,j.

Our next result is to determine conditions in case the trivial extension of a ring is n-ARM.

Theorem 3.3 Let R be a central reversible ring, then T= $T(R,R)$is n-ARM.

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Proof Since R is central reversible, so by [25, Theorem 2.19] R is 2- primal, and by [9, Proposition 1.3], $\frac{R}{P(R)}[x] \cong \frac{R[x]}{P(R)[x]}$ is reduced which implies that R is ARM, hence R is n- ARM by [26, Proposition 2.7] and by [26, Proposition 4.1] we get $T=T(R,R)$ is n- ARM .

Using Theorem 3.3 to get the following corollaries:

Corollary 3.4 Let R be a central reversible ring, then $T=T(R,R)$ is WARM.

Proof Since R is central reversible, thus by Theorem 3.4 $T=T(R,R)$ is n- ARM.

Also since each n – ARM ring is WARB, so we get $T=T(R,R)$ is WARM.

Corollary 3.5 Let R be a central reversible ring, then $T=T(R,R)$ is π-ARM.

Proof Since R is a central reversible ring, by Corollary 3.4, $T=T(R,R)$ is WARM and using [16, Proposition 2.2.12], implies that $T=T(R,R)$ is π-ARM.

Recall that if every nilpotent element of R is central, then a ring R is called central reduced [19]. The following proposition explain the relationship between 2-primal and ARM rings on the one hand with trivial extension on the other.

Proposition 3.6 Let R be a semiprime ring, then R is 2-primal ring iff $T=T(R,R)$ is ARM ring.

Proof Firstly, assume that R is 2 – primal ring. By Proposition 2.1 and since R is semiprime, then R is reduced ring and so by [1, Proposition 2.5] we get $T=T(R,R)$ is ARM ring.

Conversely, suppose that $T=T(R,R)$ is ARM ring. By Theorem 2.3, we have R is reduced ring which implies that R is central reduced and so by using [19, Theorem 2.15] implies that R is a 2- primal ring.

As proof of Proposition 3.6, the next corollary can be obtained

Corollary 3.7 Let R be a von Neumann regular ring, then R is 2-primal iff $T=T(R,R)$ is ARM ring.

Proposition 3.8 Let R be a semiprime ring, then R is symmetric iff $T=T(R,R)$ is ARM ring.

Proof We have R is symmetric ring, since R is semiprime by Proposition 2.1, R is reduced ring and by [1, Proposition 2.5], $T=T(R,R)$ is ARM.

Conversely, suppose that $T=T(R,R)$ is ARM ring, by Theorem 2.3, we have R is reduced, thus by Proposition 2.1 we get R is symmetric.

In a manner comparable to the proof of Proposition 3.8, the next corollaries can be obtained.

Corollary 3.9 Let R be a von Neumann regular ring, then R is symmetric iff $T=T(R,R)$ is ARM ring.

Corollary 3.10 Let R be a semiprime ring, then R is reversible iff $T=T(R,R)$ is ARM ring**.**

Corollary 3.11 Let R be a von Neumann regular ring R , R is reversible iff $T=T(R,R)$ is ARM ring.

Corollary 3.12 Let R be a semiprime ring, then R is semicommutative iff $T=T(R,R)$ is ARM ring.

Corollary 3.13 Let R be a von Neumann regular ring, then R is semicommutative iff $T=T(R,R)$ is ARM ring.

Corollary 3.14 Let R be a semiprime ring, then R is nil – semi commutative iff $T(R,R)$ is ARM ring.

Corollary 3.15 Let R be a Von Neumann regular ring, then R is nil – semicommutative iff $T(R,R)$ is ARM ring.

The next proposition illustrates the relationship between reduced α-ARM and WARM ring with trivial extension.

Proposition 3.16 Let R be a reduced α- ARM, then $T=T(R,R)$ is WARM ring.

Proof Suppose that R is reduced α-ARM, we have to prove that $T=T(R,R)$ is WARM. Let $f(x)=\sum_{i=0}^{m}(a_i, c_i)\,x^i$ and $g(x)=\sum_{j=0}^{n}(b_j, d_j)\,x^j \in T=T(R,R)$ [x] such that $f(x)\,g(x)=0$ Where $f(x)=\left(f_0(x), f_1(x)\right)$ and $g(x)=\left(g_0(x), g_1(x)\right)$, $f_0(x)=\sum_{i=0}^{m} a_i\,x^i \in R[x, \alpha]$,

$f_1(x)=\sum_{i=0}^{m} c_i\,x^i \in R[x, \alpha]$, $g_0(x)=\sum_{j=0}^{n} b_j\,x^j \in R[x, \alpha]$ and $g_1(x)=\sum_{j=0}^{n} d_j\,x^j \in R[x, \alpha]$.

$f(x)\,g(x)=\left(f_0(x)\,g_0(x), f_0(x)\,g_1(x)+f_1(x)g_0(x)\right)=(0,0)$ …(2),

i.e. $f_0(x)g_0(x)=0$ …(3)

And $f_0(x)g_1(x)+f_1(x)g_0(x)=0$ … (4)

In $R[x,\alpha]$, since R is reduced $\alpha-$ ARM ring, then by [12, Proposition 1.7] R is $-rigid$, and by [11, Proposition 3] yields $R[x, \alpha]$ is reduced, so from Eq .(2) and by [27, Proposition 1.6], we get $g_0(x)f_0(x)=0$. Multiply Eq. (4) from the right hand by $f_0(x)$ to get

$f_0(x)g_1(x)f_0(x)+f_1(x)g_0(x)f_0(x)=0$, so we get $f_1(x)g_0(x)f_0(x)=0$ and since R is reduced, then by [27, Proposition 1.6] $T=T(R,R)$ is reversible, so we have $f_0(x)f_0(x)g_1(x)=0$ in $R[x,\alpha]$ which implies that $(f_0(x)g_1(x))^2=0$. Since $R[x,\alpha]$ is reduced, then $f_0g_1(x)=0$. …(5)

Substitute Eq. (5) in Eq. (4) to get

$f_1(x)g_0(x)=0$. …(6)

Since R is reduced $\alpha-$ARM and from Eq. (3) , Eq. (5) and Eq. (6), then

$a_ib_j=0$ (resp. $a_id_j=0$ and $c_ib_j=0$) for all i and j this means $(a_i, c_i)\,(b_j, d_j)=0$ for all i and j . Therefor $(a_i, c_i)\,(b_j, d_j) \in N(R)$ so $T(R,R)$ is WARM.

Recall that a ring R is called α-skew π-ARM (α-Sπ-ARM, for short) ring if two polynomials $f(x)=\sum_{i=0}^{m} a_ix^i$, $g(x)=\sum_{j=0}^{n} b_jx^j \in R[x;\alpha]$ such that $f(x)g(x) \in N(R[x;\alpha])$, then $a_i\alpha^i(b_i) \in N(R)$ for each $0 \le i \le m$, $0 \le j \le n$, [16].

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We must remember that if α is an endomorphism of a ring R, then map $\bar{\alpha}: R[x] \to R[x]$ given by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{n} \alpha(a_i) x^i$ is an endomorphism of R[x], and it is extended of α.

Proposition 3.17 Let T= $T(R,R)$ be $\bar{\alpha}-$ ARM ring. Then T= $T(R,R)$ is $\bar{\alpha}-$S $\pi -$ ARM.

Proof Let $f(x)$= $\sum_{i=0}^{m}(a_i, r_i)\, x^i$ and $g(x) = \sum_{j=0}^{n}(b_j, v_j)\, x^j \in T$= $T(R,R)[x,\bar{\alpha}]$, where $f(x) = (f_0(x), f_1(x))$ and $g(x)$= $(g_0(x), g_1(x))$, $f_0(x)$= $\sum_{i=0}^{m} a_i\, x^i$, $f_1(x)$= $\sum_{i=0}^{m} r_i\, x^i$, $g_0(x)$=$\sum_{j=0}^{n} b_j\, x^j$ and $g_1(x)$= $\sum_{j=0}^{n} x^j$ in $R[x, \alpha]$.

Since T is $\bar{\alpha}-$ ARM, then $f(x)\, g(x)$=0 in T=$T(R,R)[x,\bar{\alpha}]$, and this means $f(x)\, g(x) \in N(T)$=$N(T(R,R)\,[x , \alpha])$, so $f(x)\, g(x) = (f_0(x)\, g_0(x), f_0(x)\, g_1(x) +$
$f_1(x)\, g_0(x))$=0 which give

$f_0(x)\, g_0(x) = 0$ in $R[x,\alpha]$ …. (7)

And $f_0(x)\, g_1(x)$+ $f_1(x)\, g_0(x)$=0 in $R[x,\alpha]$ … (8)

Note that R is $\alpha-$rigid by [12, Proposition 2.4], and $R[x,\alpha]$ is reduced by [11, Proposition 3].

From Eq. (7), and [27, Proposition 1.6], then $g_0(x)\, f_0(x)$=0 … (9)

Multiply Eq. (8) on the right-hand side by $f_0(x)$ to get

$f_0(x)\, g_1(x)\, f_0(x)$+$f_1(x)\, g_0(x)\, f_0(x)$=0

By Eq. (9), the last equation become $f_0(x)\, g_1(x)\, f_0(x)$=0 and by [27, Proposition 1.6], $f_0(x)\, f_0(x)\, g_1(x)$=$f_0^{\,2}(x)\, g_1(x)$=0.

Since $R[x,\alpha]$ is reduced, then$f_0(x) g_1(x)$=0. … (10)

By substituting Eq. (10) in Eq. (8), we get $f_1(x)\, g_0(x)$=0 . …(11)

Now, from Eq. (3), (4) and Eq. (5) and since R is $\alpha-$ ARM, then $a_i b_j$=0 , $a_i v_j$ and $r_i b_j$=0 for each $i = 1,2,\dots m,\ j = 1,2,\dots n.$

By [12, Theorem 1.8], we get $a_i\alpha^i(b_j)$=0, $a_i\alpha^i(v_j)$=0 and $r_i\alpha^i(b_j)$=0 for each $i = 1,2,\dots m,\ j = 1,2,\dots n$.Therefore it follows that

$(a_i, r_i)\,\bar{\alpha}^i\,(b_j, v_j) = (0,0) \in N(T(R,R))$, hence $T(R,R)$ is $\bar{\alpha}-$S $\pi -$ARM ring.

As an example of the concept of $\pi -$ARM rings in the sense of trivial extension, we have the following:

Theorem 3.18 Let R be $\pi -$ARM ring. Then the trivial extension ring T= $T(R,R)$=$R \oplus R$ is π-ARM.

Proof Let $f(x) = \begin{pmatrix} a_0 & u_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & u_1 \\ 0 & a_1 \end{pmatrix} x + \cdots + \begin{pmatrix} a_n & u_n \\ 0 & a_n \end{pmatrix} x^n = \begin{pmatrix} f(x) & l(x) \\ 0 & f(x) \end{pmatrix}$,

$$g(x) = \begin{pmatrix} b_0 & v_0 \\ 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & v_1 \\ 0 & b_1 \end{pmatrix} x + \cdots + \begin{pmatrix} b_n & v_n \\ 0 & b_n \end{pmatrix} x^n = \begin{pmatrix} g(x) & h(x) \\ 0 & g(x) \end{pmatrix} \in T[x]$$

Where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $l(x) = u_0 + u_1 x + \cdots + u_n x^n$,
$g(x) = b_0 + b_1 x + \cdots + b_m x^m$, $h(x) = v_0 + v_1 x + \cdots + v_m x^m$ are in $R[x]$. To prove that $T = T(R,R)$ is π-ARM, assume that $f(x)g(x) \in N(T[x])$, which means that $f(x)g(x) = \begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix} \in N(T[x])$, then

$\begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix}^n = \begin{pmatrix} f_1(x)g_1(x)^n & * \\ 0 & f_1(x)g_1(x)^n \end{pmatrix}$=0, for some positive integer n.

So, we get $(f_1(x)g_1(x))^n = 0$, hence $f_1(x)g_1(x) \in N(R[x])$. Since R is π- ARM, then $a_i b_j \in N(R)$, $0 \le i \le n$,
$0 \le j \le m$, i.e, hence there exists some positive integer p_{ij} such that $(a_i b_j)^{pij} = 0$.Take $p = max\{p_{ij}\}, 0 \le i \le n, 0 \le j \le m$. Then

$\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^{pn} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}^n = 0.$ Thus $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix} \in N(R \oplus R)$,

$\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^p = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$,

and $\begin{pmatrix} a_i b_j & a_i v_j + u_i b_j \\ 0 & a_i b_j \end{pmatrix}^{p+1} = 0.$ Hence, T= $T(R,R)$=$R \oplus R$ is π-ARM.

Lemma 3.19 Let R be a central reduced ring. Then $T(R,R)$ is C $\pi -$ARM.

Proof Since R is central reduced, then by [19, Theorem 2.34], $T(R,R)$ is CARM and by using [16, Proposition 3.5.1] this implies that R is a $T(R,R)$ is C$\pi -$ ARM.

As a special case of Lemma 3.19, the following propositions can be obtained.

Lemma 3.20 Let R be a semiprime nil-semicommutative ring. Then $T(R,R)$is C $\pi -$ARM.

Proof By Proposition 2.1, R is reduced. Since every reduced is central reduced and by Lemma 3.19, then $T(R,R)$is C$\pi -$ARM.

Lemma 3.21 Let R be a domain ring. Then $T(R,R)$ is C$\pi -$ARM.

Proof Since R is domain, then by [19, Proposition 2.7] and Lemma 3.19 we have R is C$\pi -$ARM.

The following propositions explain the relationship between C$\pi -$ARM with trivial extension on the one hand and some kinds of rings on the other.

Proposition 3.22 Let R be a central reversible semiprime ring. Then $T(R,R)$ is C$\pi -$ARM.

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Proof Since R is central reversible, then by [16, Theorem 1.4.20 (1)] and by Proposition 2.1, R is reduced and by Lemma 3.19 the proof is complete.

New fact of commutative ring and $C\pi$ −ARM ring with trivial extension can be obtained.

Proposition 3.23 Let R be a ring. Then R is commutative iff $T(R,R)$ is $C\pi$ −ARM.

Proof Assume that R is commutative, then by [19, Proposition 2.29], $T(R,R)$is central reduced. Let $f,g \in T(R,R)$ such that $f=(f_1,f_2)$, $g=(g_1,g_2)$ and $fg \in N(R[x])$. Since $T(R,R)$ is central reduced, then by [19, Theorem 2.31], R is nARM. That is, $fg \in N(R[x]) \subseteq N(R)[x]$ which implies that $a_ib_j \in N(R)$. Again, since R is central reduced, then $a_ib_j \in C(R)$. Hence $T(R,R)$ is $C\pi$ −ARM.

Conversely, suppose that $T(R,R)$ is $C\pi$ −ARM. To prove that R is commutative, let $a,b \in R$ and $f(x)=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}, g(x)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \in T(R,R)[x]$. Since $T(R,R)$ is $C\pi$ −ARM and $f(x)g(x)=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} \in N(T(R,R)[x])$, then $\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix} \in C(T(R,R))$ and this implies that

$\begin{pmatrix}0 & ab\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & ba\\ 0 & 0\end{pmatrix}$, that is $ab=ba$, for all $a,b \in R$. Hence is commutative.

The next theorem explains new fact of $C\pi$ −ARM ring with trivial extension.

Theorem 3.24 Let R be a $C\pi$ −ARM that satisfies the property: if $f(x)g(x)$ is a central nilpotent element and $f(x)$ is a central element of $R[x]$, then $g(x) \in N(R[x])$. Then $T(R,R)$is $C\pi$ −ARM.

Proof Let $f(x)=\begin{pmatrix}a_0 & a_0'\\ 0 & a_0\end{pmatrix}+\begin{pmatrix}a_1 & a_1'\\ 0 & a_1\end{pmatrix}x+\cdots+\begin{pmatrix}a_n & a_n'\\ 0 & a_n\end{pmatrix}x^n=\begin{pmatrix}f_1(x) & f_2(x)\\ 0 & f_1(x)\end{pmatrix}$,

$g(x)=\begin{pmatrix}b_0 & b_0'\\ 0 & b_0\end{pmatrix}+\begin{pmatrix}b_1 & b_1'\\ 0 & b_1\end{pmatrix}x+\cdots+\begin{pmatrix}b_m & b_m'\\ 0 & b_m\end{pmatrix}x^m$

$=\begin{pmatrix}g_1(x) & g_2(x)\\ 0 & g_1(x)\end{pmatrix} \in T(R,R)[x]$ such that

$f(x)g(x)=\begin{pmatrix}f_1(x)g_1(x) & f_1(x)g_2(x)+f_2(x)g_1(x)\\ 0 & f_1(x)g_1(x)\end{pmatrix} \in N(T(R,R)[x])$ …(12)

By some computations on Eq. (1), we can conclude that $f_1(x)g_1(x) \in C(R[x])$ and this can be reduced Eq. (12) to

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=\big(f(x)g(x)\big)^n$$

$$=\begin{pmatrix}(f_1(x)g_1(x))^n & \sum_{i+j=n-1}\big(f_1(x)g_1(x)\big)^i\big(f_1(x)g_2(x)+f_2(x)g_1(x)\big)\big(f_1(x)g_1(x)\big)^j\\ 0 & (f_1(x)g_1(x))^n\end{pmatrix}$$

$$=\begin{pmatrix}(f_1(x)g_1(x))^n & n\big(f_1(x)g_1(x)\big)^{n-1}(f_1(x)g_2(x)+f_2(x)g_1(x))\\ 0 & (f_1(x)g_1(x))^n\end{pmatrix}$$ … (13)

Eq. (13) implies that $(f_1(x)g_1(x))^n=0$ and since R is $C\pi$ −ARM, then$a_ib_j \in C(R)$. Also, $n\big(f_1(x)g_1(x)\big)^{n-1}\big(f_1(x)g_2(x)+f_2(x)g_1(x)\big)=0$ and using [19, Lemma 2.33] implies that $nf_1(x)g_1(x)\big(f_1(x)g_2(x)+f_2(x)g_1(x)\big)$ is central nilpotent element of $R[x]$.

Hence, by hypothesis we get the required result.

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التوسيع التافهة لحلقات ارميندرايز ومفاهيم ذات صلة

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المستخلص :

هذا البحث يحقق في إمكانية انتقال خصائص الحلقة R إلى حلقة التوسيع التافهة $T(R,R)$ وعرض العلاقة بين حلقة التوسيع التافهة$T(R,R)$ والعديد من أنواع الحلقات. ان مفاهيم برايمل من النمط ٢، رفيرسبل، رفيرسبل المركزية، شبه الابدالية، شبه الابدالية الصفرية، أرمندريزمن النمط π و أرمندريزالمركزية من النمط π قد تم دراستها مع حلقة التوسيع التافهة $T(R,R)$ وبعض المكافئات لحلقات أرمندريز من النمط π وحلقات أرمندريز المركزية من النمط π.