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The new exponential identities

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ABSTRACT: We have obtained new exponential identities. By ten original propositions we have proved them.

Keywords: Identities, Pascal's triangle, Binomial coefficients.

Mathematics subject classification: 11D61.

1. Introduction

Pascal's triangle can be arranged in a triangular array of numbers, as follows:

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
 \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 \binom{4}{0} & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & \binom{4}{4} \\
 \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\end{array}
$$

$$\binom{n}{0} \quad \binom{n}{1} \cdots \binom{n}{k-1} \quad \binom{n}{k} \cdots \binom{n}{n-1} \quad \binom{n}{n}$$

$$\binom{n+1}{0} \quad \binom{n+1}{1} \cdots \binom{n+1}{k} \cdots \binom{n+1}{n} \quad \binom{n+1}{n+1}$$

Where $n \geq k$.

It has the following properties.

- The first number and the last number in each row is 1.
- Every other number in the array can be obtained by adding the two numbers appearing directly above it. This property is equivalent to the following identity:

$$\binom{n}{n-1} + \binom{n}{k} = \binom{n+1}{k} \qquad (1.1)$$

- The numbers equidistant from the ends are equal. This property is equivalent to the following identity:

$$\binom{n}{k} = \binom{n}{n-k} \qquad (1.2)$$

Now since the numbers appearing in Pascal's triangle are the binomial coefficients, and here is some of identities satisfied by them.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{1.3}$$

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1} \tag{1.4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \tag{1.5}$$

See [3] for more details.
Can we obtain new identities?
By using the identities above. This paper has answered this question by ten original propositions.

2. Notation and Definitions

We denote the set of natural numbers $\mathbb{N} := \{1,2,3,\dots\}$. By $\mathbb{Z}$ we denote the set of integers numbers. By $\mathbb{C}$ we denote the complex numbers. The set of $\mathbb{C}^*$ is defined by $\mathbb{C}^* := \{ z \in \mathbb{C},\ z \neq 0\}$. The set of all nonzero polynomials over the set $\mathbb{C}$ with indeterminate z is denoted by $\mathbb{C}[z]$. Thus $\mathbb{C}[z] \coloneqq \{f(z): f(z)$ is a polynomial , $f(z) \neq 0\}$.

Definition 2.1. (See [1]). A number P is called a *composite prime*, and $P \in \mathbb{Z}$, if $P = p_1 \cdot p_2 \cdots p_i$. Where $p_1, p_2, \dots, p_i$ are distinct primes.

Definition 2.2. (See [2]). We call $\mathbb{A}$ a *set of basic numbers*.

$$\mathbb{A} := \{p, P \in \mathbb{Z} :\ p \text{ is a prime number},\ P \text{ is a composite prime}\}.$$

Definition 2.3. (See [2]). A number a is called a *basic number* if $a \in \mathbb{A}$.

Definition 2.4 (See [2]). A polynomial $P(z)$ is called a *composite primary polynomial*, and $P(z) \in \mathbb{C}[z]$, if $P(z) = cp_1(z) \cdot p_2(z) \cdot p_i(z)$. Where $p_1(z), p_2(z), \dots, p_i(z)$ are irreducible distinct polynomials and $c \neq 0$ is a constant.

Definition 2.5. (See [2]). We call $\mathbb{A}[z]$ a *set of basic polynomials*.

$$\mathbb{A}[z] \coloneqq \{ p(z), P(z) \in \mathbb{C}[z]:\ p(z) \text{ is a irreducible polynomial},\ P(z) \text{ is a composite primary polynomial} \}.$$

Definition 2.6. (See [2]). A polynomial $A(z)$ is called a *basic polynomial* if $A(z) \in \mathbb{A}[z]$.

Definition 2.7. We will define two types powers triangle:

- A *powers triangle of number* can be obtained by z^r (or a) , where $z \in \mathbb{C}^*$ and $r \in \mathbb{N}$, as follows:

$$\begin{array}{ccccccccccc}
 & & & & 1 & z^{r\binom{0}{0}} & 1 & & & & \\
 & & & 1 & z^{r\binom{1}{0}} & & z^{r\binom{1}{1}} & 1 & & & \\
 & & 1 & z^{r\binom{2}{0}} & & z^{r\binom{2}{1}} & & z^{r\binom{2}{2}} & 1 & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 & 1 & z^{r\binom{n}{0}} & \dots & z^{r\binom{n}{k-1}} & & z^{r\binom{n}{k}} & \dots & z^{r\binom{n}{n}} & 1 & \\
1 & z^{r\binom{n+1}{0}} & & \dots & & z^{r\binom{n+1}{k}} & & \dots & & z^{r\binom{n+1}{n+1}} & 1
\end{array}$$

Or

$$\begin{array}{ccccccccccc}
 & & & & 1 & a^{\binom{0}{0}} & 1 & & & & \\
 & & & 1 & a^{\binom{1}{0}} & & a^{\binom{1}{1}} & 1 & & & \\
 & & 1 & a^{\binom{2}{0}} & & a^{\binom{2}{1}} & & a^{\binom{2}{2}} & 1 & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 & 1 & a^{\binom{n}{0}} & \dots & a^{\binom{n}{k-1}} & & a^{\binom{n}{k}} & \dots & a^{\binom{n}{n}} & 1 & \\
1 & a^{\binom{n+1}{0}} & & \dots & & a^{\binom{n+1}{k}} & & \dots & & a^{\binom{n+1}{n+1}} & 1
\end{array}$$

It has the following interesting properties:

- The first number and the last number in each row is 1.
- Every other number in the array can be obtained by multiplying the two numbers appearing directly above it.
- The numbers equidistant from the ends are equal.

- A *powers triangle of polynomial* can be obtained by $f(z)^r$ $\big($or $A(z)\big)$. By using the symbol $f(z)$ $\big($or $A(z)\big)$ instead of z (or a), likewise, we define a powers triangle of polynomial.

3. The Results

We have proved the following Results:

Proposition 3.1.

$$z^{r\binom{n+1}{k}} = z^{r\binom{n}{n-1}+r\binom{n}{k}}. \tag{3.1}$$

Corollary. If $z^r = a$, then

$$a^{\binom{n+1}{k}} = a^{\binom{n}{n-1}+\binom{n}{k}}. \tag{3.2}$$

Proposition 3.2.

$$z^{r\binom{n}{k}} = z^{r\binom{n}{n-k}} . \tag{3.3}$$

Corollary. If $z^r = a$, then

$$a^{\binom{n}{k}} = a^{\binom{n}{n-k}} . \tag{3.4}$$

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Proposition 3.3.

$$\prod_{k=0}^{n} z^{r\binom{n}{k}} = z^{r2^n}. \quad (3.5)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} a^{\binom{n}{k}} = a^{2^n}. \quad (3.6)$$

Proposition 3.4.

$$\prod_{k=0}^{n} \left(z^{r\binom{n}{k}}\right)^k = z^{rn2^{n-1}}. \quad (3.7)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} \left(a^{\binom{n}{k}}\right)^k = a^{n2^{n-1}}. \quad (3.8)$$

Proposition 3.5.

$$\prod_{k=0}^{n} \left(z^{r\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.9)$$

Corollary. If $z^r = a$, then

$$\prod_{k=0}^{n} \left(a^{\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.10)$$

Proposition 3.6.

$$f(z)^{r\binom{n+1}{k}} = f(z)^{r\binom{n}{n-1}+r\binom{n}{k}}. \quad (3.11)$$

Corollary. If $f(z)^r = A(z)$, then

$$A(z)^{\binom{n+1}{k}} = A(z)^{\binom{n}{n-1}+\binom{n}{k}}. \quad (3.12)$$

Proposition 3.7.

$$f(z)^{r\binom{n}{k}} = f(z)^{r\binom{n}{n-k}}. \quad (3.13)$$

Corollary. If $f(z)^r = A(z)$, then

$$A(z)^{\binom{n}{k}} = A(z)^{\binom{n}{n-k}}. \quad (3.14)$$

Proposition 3.8.

$$\prod_{k=0}^{n} f(z)^{r\binom{n}{k}} = f(z)^{r2^n}. \quad (3.15)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} A(z)^{\binom{n}{k}} = A(z)^{2^n}. \quad (3.16)$$

Proposition 3.9.

$$\prod_{k=0}^{n} \left(f(z)^{r\binom{n}{k}}\right)^k = f(z)^{rn2^{n-1}}. \quad (3.17)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} \left(A(z)^{\binom{n}{k}}\right)^k = A(z)^{n2^{n-1}}. \quad (3.18)$$

Proposition 3.10.

$$\prod_{k=0}^{n} \left(f(z)^{r\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.19)$$

Corollary. If $f(z)^r = A(z)$, then

$$\prod_{k=0}^{n} \left(A(z)^{\binom{n}{k}}\right)^{(-1)^k} = 1. \quad (3.20)$$

Now, here are some examples to show the results.

Example 3.1. If a powers triangle is

$$\begin{array}{ccccccccccc} &&&&1&4&1&&&& \\ &&&1&4&&4&1&&& \\ &&1&4&&4^2&&4&1&& \\ &1&4&4^3&&&4^3&4&1&& \\ 1&4&4^4&&4^6&&4^4&4&1&& \\ 1&4&4^5&4^{10}&&4^{10}&4^5&4&1&& \end{array}$$

Compute

a)

$$\prod_{k=0}^{5} 4^{\binom{5}{k}}.$$

b)

$$\prod_{k=0}^{5} \left(4^{\binom{5}{k}}\right)^k.$$

c)

$$\prod_{k=0}^{5} \left(4^{\binom{5}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{5} 2^{2\binom{5}{k}} = 2^{64}.$$

b)

$$\prod_{k=0}^{5} \left(2^{2\binom{5}{k}}\right)^k = 2^{160}.$$

c)

$$\prod_{k=0}^{5} \left(2^{2\binom{5}{k}}\right)^{(-1)^k} = 1.$$

Example 3.2. If a powers triangle is

$$\begin{array}{c} 1 \quad 3 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 3 \quad 3^2 \quad 3 \quad 1 \\ 1 \quad 3 \quad 3^3 \quad 3^3 \quad 3 \quad 1 \\ 1 \quad 3 \quad 3^4 \quad 3^6 \quad 3^4 \quad 3 \quad 1 \end{array}$$

Compute

d)

$$\prod_{k=0}^{4} 3^{\binom{4}{k}}.$$

e)

$$\prod_{k=0}^{4} \left(3^{\binom{4}{k}}\right)^k.$$

f)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{(-1)^k}.$$

Solution:

d)

$$\prod_{k=0}^{4}3^{\binom{4}{k}} = 3^{16}.$$

e)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{k} = 3^{32}.$$

f)

$$\prod_{k=0}^{4}\left(3^{\binom{4}{k}}\right)^{(-1)^k} = 1.$$

Example 3.3. Compute

a)

$$\prod_{k=0}^{5}(4z^2+2z+1)^{\binom{5}{k}}.$$

b)

$$\prod_{k=0}^{5}\left((4z^2+2z+1)^{\binom{5}{k}}\right)^{k}.$$

c)

$$\prod_{k=0}^{5}\left((4z^2+2z+1)^{\binom{5}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{5}(2z+1)^{2\binom{5}{k}} = (2z+1)^{64}.$$

b)

$$\prod_{k=0}^{5}\left((2z+1)^{2\binom{5}{k}}\right)^{k} = (2z+1)^{160}.$$

c)

$$\prod_{k=0}^{5}\left((2z+1)^{2\binom{5}{k}}\right)^{(-1)^k} = 1.$$

Example 3.4. Compute

a)

$$\prod_{k=0}^{4}(2z+1)^{\binom{4}{k}}.$$

b)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{k}.$$

c)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{(-1)^k}.$$

Solution:

a)

$$\prod_{k=0}^{4}(2z+1)^{\binom{4}{k}} = (2z+1)^{16}.$$

b)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{k} = (2z+1)^{32}.$$

c)

$$\prod_{k=0}^{4}\left((2z+1)^{\binom{4}{k}}\right)^{(-1)^k} = 1.$$

4. Proof of the Results

Proof of Proposition 3.1. Since $z = z$, now (1.1) leads to

$$z^{r\binom{n+1}{k}} = z^{r\binom{n}{n-1}+r\binom{n}{k}}.$$

□

Proof of Proposition 3.2. Since $z = z$, now (1.2) leads to

$$z^{r\binom{n}{k}} = z^{r\binom{n}{n-k}}.$$

□

Proof of Proposition 3.3. By definition 2.7, in row n

$$\prod_{k=0}^{n} z^{r\binom{n}{k}} = z^{r\binom{n}{0}}\cdot z^{r\binom{n}{1}}\cdots z^{r\binom{n}{n}}$$

$$= z^{r\left(\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}\right)} \qquad \text{[By (1.3)]}$$

$$= z^{r2^n}.$$ □

Proof of Proposition 3.4. We expand the left-hand side of (3.7)

$$\prod_{k=0}^{n}\left(z^{r\binom{n}{k}}\right)^{k} = \left(z^{r\binom{n}{0}}\right)^{0}\left(z^{r\binom{n}{1}}\right)^{1}\left(z^{r\binom{n}{2}}\right)^{2}\cdots\left(z^{r\binom{n}{n}}\right)^{n}$$

$$= z^{r\left(0\binom{n}{0}+1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}\right)} \qquad \text{[By (1.4)]}$$

$$= z^{rn2^{n-1}}.$$ □

Proof of Proposition 3.5. We expand the left-hand side of (3.9)

$$\prod_{k=0}^{n}\left(z^{r\binom{n}{k}}\right)^{(-1)^k}$$
$$=\left(z^{r\binom{n}{0}}\right)^{1}\left(z^{r\binom{n}{1}}\right)^{-1}\left(z^{r\binom{n}{2}}\right)^{1}\cdots\left(z^{r\binom{n}{n}}\right)^{(-1)^n}$$
$$=z^{r\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n}\right)} \qquad \text{[By (1.5)]}$$
$$=1.$$ □

Proof. By using the symbol $f(z)$ instead of z, likewise, we prove propositions 3.6, 3.7, 3.8, 3.9 and 3.10.
□

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To my lovely wife Areefa and my son Qys. Thank you. Without you, I would have never achieved my paper.

6. References

[1] **Mohammed Abdulla Saeed Salem.,** “the divisors sequences with finite differences”, *Journal for Algebra and Number Theory Academia*, Mili Publications, **4 (2):** pp. 31-39 (2014).
[2] **Mohammed Abdulla Saeed Salem.,** “The special case for divisors sequences with finite differences”, *Journal for Algebra and Number Theory Academia*, Mili Publications, **7 (2):** pp.11-36 (2017).
[3] **Seymour Lipschutz**, *Schaum's Outlines of Theory and Problems of Finite Mathematics*, first Edition, McGraw-Hill, (1966), pp. 141-144.

المتطابقات الأسية الجديدة

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كلمات مفتاحية: متطابقات، مثلث باسكال، معاملات ذات الحدين.