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A Class of Small Injective Modules

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Abstract:

Let R be a ring. In this paper, a right R-module M is defined to be AS-injective if $\text{Ext}^1(R/K\ ,M)=0$, for any annihilator-small right ideal K of R. We characterize rings over which every right module is AS-injective. Conditions under which the class of AS-injective right R-modules (ASI_R) is closed under quotient (resp. pure submodules, direct sums) are given. Finally, we study the definability of the class ASI_R.

Keywords: Injective module, Definable class, a-small right ideal, Pure submodule.

Mathematics subject classification: 13C11,16D50,16D10.

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1. Introduction

Throughout R is an associative ring with identity and all modules are unitary R-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right R-module (resp. right R-homomorphism). We use R-Mod (resp. Mod-R) to denote to the class of left (resp. right) R-modules. If $Y \subseteq R$, then $r(Y) = \{r \in R \mid Yr = 0\}$ (resp. $l(Y) = \{r \in R \mid rY = 0\}$)stands for the right (resp. left) annihilator of Y in R. We will use M^* to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a module M. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a class of right (resp. left) R-modules. A pair $(\mathcal{F}, \mathcal{G})$ is called almost dual pair if the class $\mathcal{G}$ is closed under direct products and summands, and for any left R-module M, $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{G}$ [11, p. 66]. An exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ of right R-modules is said to be pure if the sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$ is exact, for every finitely presented right R-module N and we called that $\alpha(A)$ is a pure submodule of B [18]. A right R-module M is called FP-injective if every monomorphism $\alpha: M \to N$ is pure. A right R-module M is called pure injective if M is injective with respect to all pure short exact sequences [18]. Recall that a subclass $\mathcal{G}$ of Mod-R is called definable if it is closed under pure submodules, direct limits and direct products [14]. A right ideal X of a ring R is called small in R if $X + Y \neq R$, for any proper right ideal Y of R [8]. A right R-module M is called small injective if $\mathrm{Ext}^1(R/K, M) = 0$, for any small right ideal K of R. A right ideal I of R is called annihilator-small (a-small) and denoted by $I \subseteq^a R_R$ if for any right ideal K of R with $I + K = R$, then $l(K) = 0$ [13].

The sum of all the annihilator-small right ideals of a ring R is called the right AS-ideal of a ring R and denoted by A_r [13].

We refer the reader to [1, 7, 8, 14, 18], for general background materials.

In section 2 of this paper, we introduce the class of AS-injective modules. This class of modules lies between injective modules and small injective modules. We first characterize rings over which every module is AS-injective. Over a commutative ring R, we prove the equivalence of the following statements: (1) $A_r = 0$. (2) Every module is AS-injective. (3) Every principal a-small right ideal of R is AS-injective. (4) Every simple module is AS-injective and $A_r \subseteq^a R_R$. Conditions under which the class of AS-injective right R-modules (ASI_R) is closed under quotient are given. For instance, we prove that the following statements are equivalent: (1) The class ASI_R is closed under quotient. (2) If $K \subseteq^a R$, then K is projective. (3) ASI_R contains all sums of any two AS-injective submodules of any module. Also, we show that the class ASI_R is closed under pure submodules if and only if all a-small right ideals in R are finitely generated if and only if all FP-injective modules are AS-injective. Finally, we give conditions such that any direct sum of modules in the class ASI_R is also belong to ASI_R. For instance, we prove that if $A_r \subseteq^a R_R$, then the following are equivalent. (1) A_r is a noetherian module. (2) The class ASI_R is closed under direct sums.

Section 3 studies the definability of the class ASI_R. It is shown that the following assertions are equivalent: (1) ASI_R is definable. (2) The class ASI_R is closed under pure submodules and pure homomorphic images.

(3) Every a-small right ideal in R is finitely presented. (4) A module $M \in ASI_R$ if and only if $M^{**} \in ASI_R$. Finally, we prove that if the class ASI_R is a definable, then the following are equivalent. (1) The class of flat left R-modules and the class $\{M \in R\text{-Mod} \mid M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in ASI_R\}$ are coincide. (2) Each module in ASI_R is FP-injective. (3) Each pure-injective module in ASI_R is injective.

2. *AS*-Injective modules

Definition 2.1. A module M is said to be annihilator-small injective (shortly, AS-injective), if $\mathrm{Ext}^1(R/K, M) = 0$, for any annihilator-small right ideal K of R; equivalently, if K is any annihilator-small right ideal in R, then any R-homomorphism $f: K \longrightarrow M$ extends to R_R. A ring R is said to be right AS-injective if R_R is AS-injective.

We will use ASI_R to denote to the class of AS-injective right R-modules.

Examples 2.2.

(1) It is clear that AS-injectivity implies small injectivity, but $\mathbb{Z}$ is a small injective $\mathbb{Z}$-module [17] and clearly, it is not AS-injective. Thus the class of small injective modules contains properly the class of AS-injective modules.

(2) All injective modules are AS-injective and generally the converse is not true, for example, let $\{F_i\}_{i\in I}$ be a family of fields and let $R = \prod_{i\in\Lambda} F_i$ be the ring product of F_i, for all $i \in \Lambda$, where addition and multiplication are define componentwise and let $K = \bigoplus_{i\in\Lambda} F_i$. If Λ is infinite, then K_R is not itself injective by [8, p. 140], but K_R is AS-injective, since $A_r = 0$. Therefore, AS-injective module is a proper generalization of injective modules.

Hence $Inj_R \subsetneq ASI_R \subsetneq SI_R$, where Inj_R (resp. SI_R) is the class of injective (resp. small injective) right R-modules.

Remarks 2.3.

(1) The two classes Inj_R and ASI_R are coinciding, when R is an integral domain, since all proper right ideals are a-small in any integral domain.

(2) All finitely generated $\mathbb{Z}$-modules are not AS-injective and this follows from (1) and the fact that every non-trivial finitely generated $\mathbb{Z}$-module is not injective [7, p.31]. Also, we have from [17, Theorem 2.8] that any $\mathbb{Z}$-module is small injective.

(3) From (1) and [9, p.410], we have that any ring R is a field if and only if it is an AS-injective integral domain.

Proposition 2.4. The class of AS-injective modules (ASI_R) is closed under direct summands, direct products and isomorphic copies.

Proof. Clear. □

Theorem 2.5. Consider the following conditions for a ring R.

(1) $A_r = 0$.

(2) All modules are AS-injective.

(3) All principal a-small right ideals of R are AS-injective.

(4) All principal a-small right ideals of R are direct summand in R_R.

(5) All simple modules are AS-injective and $A_r \subseteq^a R_R$.

Then (1) and (5) are equivalent and (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). Moreover, if R is commutative, then (4) implies (1).

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (1) $\Longrightarrow$ (5) are obvious.

(3) $\Longrightarrow$ (4). Let $aR \subseteq^a R$, where $a \in R$. By hypothesis, aR is AS-injective and so there is a homomorphism $\alpha: R \to aR$ such that $\alpha i = I_{aR}$, where $I_{aR}: aR \to aR$ is the identity homomorphism and $i: aR \to R$ is the inclusion mapping. Thus aR is a direct summand in R_R.

(4) $\Longrightarrow$ (1). Let R be a commutative ring. Assume that $A_r \neq 0$, thus there is $(0 \neq) a \in A_r$. By hypothesis, $A_r \subseteq^a R_R$ and hence Lemma 1 in [13] implies $aR \subseteq^a R_R$. By hypothesis, aR is a direct summand in R_R and hence there exist a right ideal K with $aR \oplus K = R_R$. Since $aR \subseteq^a R_R$, $r(K) = 0$. Since $aR + K = R$, we have $r(aR \cap K) = r(aR) + r(K)$ and hence $r(aR) = R$. Thus $aR = 0$, a contradiction. Therefore, $A_r = 0$.

(5) $\Longrightarrow$ (1)Assume that $A_r \neq 0$, thus there is $(0 \neq) a \in A_r$. If $A_r + r(a) \neq R$, then $A_r + r(a) \subseteq C$, for some maximal right ideal C of R. Thus R/C is a simple right R-module. By hypothesis, R/C is an AS-injective module. Define $\alpha: aR \to R/C$ by $\alpha(ar) = r + C$. Clearly, α is a well-defined right R-homomorphism. By AS-injectivity, there exist $b \in R$ with $1 + C = ba + C$ and hence $1 - ba \in C$. Since $a \in A_r$ and A_r is a two sided ideal (by [13, Theorem 9 (1)]), we have $ba \in C$. Thus $C = R$, a contradiction. Therefore, $A_r + r(a) = R$. Since $A_r \subseteq^a R_R$ (by hypothesis), $l(r(a)) = 0$, so that $r(l(r(a))) = R$. By [1, Proposition 2.15, p.37], $r(a) = R$ and hence $a = 0$, a contradiction. Thus $A_r = 0$. □

Recall that a ring R is called regular if for any $x \in R$, there is an element $y \in R$ such that $x = xyx$ [8]

Corollary 2.6. If R is a commutative regular ring, then every module is AS-injective and $A_r = 0$.

Proof. By [8, Theorem 10.4.9, p. 262] and Theorem 2.5. □

It is not true in general that if $K \subseteq^a R_R$, then K is a projective right R-module, for example, if $R = Z_4$ and $K = 2R$, then $K \subseteq^a R_R$ but it is not projective right R-module.

Theorem 2.7. For a ring R, the following are equivalent.

(1) If $K \subseteq^a R_R$, then K is projective.

(2) The class ASI_R is closed under quotient.

(3) ASI_R contains all quotients of injective modules.

(4) ASI_R contains all sums of any two AS-injective submodules of any module.

(5) ASI_R contains all sums of any two injective submodules of any module.

Proof. (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) are obvious.

(1) $\Longrightarrow$ (2) Let $\alpha: N \longrightarrow M$ be any epimorphism, where N is an AS-injective module and M is any module. Let $\lambda: K \longrightarrow M$ be any homomorphism, where $K \subseteq^a R_R$. By hypothesis, K is projective and hence there is a homomorphism $\beta: K \longrightarrow N$ such that $\alpha\beta = \lambda$. By AS-injectivity of N, there is a homomorphism $\gamma: R \to N$ with $\gamma i = \beta$, where $i: K \to R$ is the inclusion mapping. Put $\varphi = \alpha\gamma: R \longrightarrow M$, so that $\varphi i = \alpha\gamma i = \alpha\beta = \lambda$ and hence M is an AS-injective module.

(3) $\Longrightarrow$ (1) Let $K \subseteq^a R_R$. Let $\alpha: E \to N$ be an epimorphism (where E is an injective module) and $\beta: K \to N$ a homomorphism. By hypothesis, $N \in ASI_R$ and hence there is a homomorphism $\lambda: R \longrightarrow N$ with $\lambda i = \beta$, where $i: K \to R$ is the inclusion mapping. By projectivity of R_R, there is a

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homomorphism $\gamma: R \to E$ such that $\alpha\gamma = \lambda$. Let $\tilde{\alpha}: K \longrightarrow E$ be the restriction of γ over K. Clearly, $\alpha\tilde{\alpha} = \beta$ and hence from Proposition 5.2.10 in [2, p.148] we get that K is projective.

(2) $\Longrightarrow$ (4) Let M_1 and M_2 be two AS-injective submodules of module M. By Proposition 2.4, $M_1 \oplus M_2 \in ASI_R$. Since $M_1 + M_2$ is a homomorphic image of $M_1 \oplus M_2$, we have $M_1 + M_2 \in ASI_R$, by hypothesis.

(5) $\Longrightarrow$ (3). By similar argument as in the proof of Theorem 2.14 ((6) $\Longrightarrow$ (3)) in [12]. □

Proposition 2.8. For a ring R, consider the following conditions.

(1) Every module is AS-injective.

(2) R_R is AS-injective and the class ASI_R is closed under quotient.

(3) For any $x \in R$, if $xR \subseteq^a R_R$, then there is $y \in R$ such that $x = xyx$.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and if R is commutative, then (3) implies (1).

Proof. (1) $\Longrightarrow$ (2). Clear.

(2) $\Longrightarrow$ (3). Let $x \in R$ such that $xR \subseteq^a R_R$. Since ASI_R is closed under quotient (by hypothesis), xR is projective, by Theorem 2.7. Define $\alpha: R \to xR$ by $\alpha(r) = xr$, for all $r \in R$. Clearly, α is an epimorphism, so that there is a homomorphism $f: xR \to R$ with $\alpha f(a) = a$, for all $a \in xR$. Since R_R is AS-injective (by hypothesis), there is a homomorphism $g: R \to R$ such that $gi = f$, where $i: xR \to R$ is the inclusion mapping. Thus $x = \alpha\big(f(x)\big) = \alpha\big(g(x)\big) = xyx$, where $y = g(1) \in R$.

(3) $\Longrightarrow$ (1). Suppose that R is a commutative ring. Let $xR \subseteq^a R_R$, where $x \in R$. By hypothesis, there is $y \in R$ with $x = xyx$. Let $e = xy$. Clearly, e is an idempotent of R and $xR = eR$, so that xR is a direct summand of R_R. Therefore, the result follows by Theorem 2.5. □

Proposition 2.9. For a ring R, the following are equivalent.

(1) All a-small right ideals in R are finitely generated.

(2) The class ASI_R is closed under pure submodules.

(3) All FP-injective modules are AS-injective.

Proof. (1) $\Longrightarrow$ (2). Let $M \in ASI_R$ and K a pure submodule of M. Let $I \subseteq^a R_R$, thus the hypothesis implies that I is finitely generated and so R/I is a finitely presented. Hence the sequence $\mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/I, M/K) \longrightarrow 0$ is exact. By [6, Theorem XII.4.4 (4), p. 491], the exact sequence $\mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, M/K) \longrightarrow \mathrm{Ext}^1(R/I, K) \longrightarrow \mathrm{Ext}^1(R/I, M)$ and so $\mathrm{Ext}^1(R/I, K) = 0$. Thus, $K \in ASI_R$ and hence the class ASI_R is closed under pure submodules.

(2) $\Longrightarrow$ (3). If M is any FP-injective module, then M is a pure submodule of an AS-injective module. By hypothesis, $M \in ASI_R$.

(3) $\Longrightarrow$ (1). Let $I \subseteq^a R_R$ and $\alpha: I \to M$ a homomorphism, where M is an FP-injective module. By hypothesis, M is AS-injective and hence α extends to R_R. By [4], I is finitely generated. □

Corollary 2.10. If each a-small right ideal in a ring R is finitely generated, then the class ASI_R is closed under direct sums.

Proof. Let $\{M_i \mid i \in I\}$ be a subclass of ASI_R. By Proposition 2.4, $\prod_{i\in I} M_i \in ASI_R$. By [14, Proposition 2.1.10, p. 57], $\bigoplus_{i\in I} M_i$ is a pure submodule in $\prod_{i\in I} M_i$ and hence $\bigoplus_{i\in I} M_i \in ASI_R$, by Proposition 2.9. □

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Theorem 2.11. For a ring R, consider the following conditions.

(1) A_r is a noetherian module.

(2) The class ASI_R is closed under direct sums.

(3) $M^{\mathbb{N}}$ is AS-injective, for any AS-injective module M_R.

(4) $M^{\mathbb{N}}$ is AS-injective, for any injective module M_R.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and if $A_r \subseteq^a R_R$, then (4) $\Longrightarrow$ (1).

Proof. (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(1) $\Longrightarrow$ (2). By [13, Theorem 9(1)] and Corollary 2.10.

(4) $\Longrightarrow$ (1). Let $A_r \subseteq^a R_R$ and let $K_1 \subseteq K_2 \subseteq \cdots$ be a chain of right ideals of R with $K_i \subseteq A_r$. Let $E = \bigoplus_{i=1}^{\infty} E_i$, where $E_i = E(R/K_i)$. For every $i \geq 1$, put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1 \\ j\neq i}}^{\infty} E_j\right)$, thus M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus \left(\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{\infty} E_j\right)$ is AS-injective. By Proposition 2.4, E is AS-injective. Define $\alpha: \bigcup_{i=1}^{\infty} K_i \longrightarrow E$ by $\alpha(x) = (x + K_i)_i$. Clearly, α is a well-defined homomorphism. By hypothesis, $A_r \subseteq^a R_R$ and hence Lemma 1 in [13] implies that $\bigcup_{i=1}^{\infty} K_i \subseteq^a R_R$. Thus α extends to a homomorphism $\beta: R \longrightarrow E$ and hence $\beta(R) \subseteq \bigoplus_{i=1}^{n} E(R/K_i)$ for some $n \in \mathbb{N}$, since R is finitely generated. Then $\alpha(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{j=1}^{n} E(R/K_i)$.

So, if $a \in \bigcup_{i=1}^{\infty} K_i$, then $a \in K_m$ for all $m > n$, and hence $\bigcup_{i=1}^{\infty} K_i = K_{n+1}$. Therefore, the chain $K_1 \subseteq K_2 \subseteq \cdots$ terminates at K_{n+1} and hence A_r is a noetherian module. $\square$

Corollary 2.12. If $A_r \subseteq^a R_R$, then the following are equivalent.

(1) A_r is a noetherian module.

(2) Direct sum of injective modules is AS-injective.

Lemma 2.13. If R satisfies ACC (ascending chain condition) on a-small right ideals of R, then $A_r \subseteq^a R_R$

Proof. Let $\mathcal{H} = \{K \,|\, K \subseteq^a R_R\}$. Thus $\mathcal{H}$ has a maximal element, say N (by Zorn's lemma). Since $A_r = \sum_{K\in\mathcal{H}} K$, it follows that $A_r = N$ and so $A_r \subseteq^a R_R$. $\square$

Proposition 2.14. For a ring R, the following are equivalent.

(1) R satisfies ACC on a-small right ideals.

(2) A_r is a noetherian R-module.

(3) $M^{\mathbb{N}}$ is AS-injective, for any injective module M_R and $A_r \subseteq^a R_R$.

Proof. (1) $\Longrightarrow$ (2). Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of right ideals of R in A_r. By Lemma 2.13, $A_r \subseteq^a R_R$. By [13, Lemma 1], N_i are a-small right ideals. By hypothesis, the chain $N_1 \subseteq N_2 \subseteq \cdots$ terminates and hence A_r is a noetherian R-module.

(2) $\Longrightarrow$ (1). By [13, Theorem 9(1)].

(2) $\Longrightarrow$ (3). By Theorem 2.11 and Lemma 2.13.

(3) $\Longrightarrow$ (2). By Theorem 2.11. $\square$

3. Definability of the class ASI_R

For any class $\mathcal{G}$ of right R-modules, we will set $\mathcal{G}^+ = \{M \in \text{Mod-}R \,|\, M$ is a pure submodule of a module in $\mathcal{G}\}$ and $\mathcal{G}^\ominus = \{M \in R\text{-Mod} \,|\, M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{G}\}$.

Proposition 3.1. The pair $((ASI_R)^\ominus, ASI_R)$ is an almost dual pair over a ring R.

Proof. By proposition 2.4, the class ASI_R is closed under direct summands and direct products. By [11, Proposition 4.2.11, p.72], the pair $((ASI_R)^\ominus, ASI_R)$ is an almost dual pair over a ring R. $\square$

Corollary 3.2. Consider the following conditions for the class ASI_R over a ring R.

(1) The class ASI_R is definable.

(2) $(ASI_R, (ASI_R)^{\ominus})$ is an almost dual pair over a ring R.

(3) $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$.

(4) $(ASI_R)^{**} \subseteq ASI_R$.

(5) The class ASI_R is closed under pure homomorphic images.

Then $(1) \Leftrightarrow (2)$, $(1) \Longrightarrow (3)$, $(1) \Longrightarrow (5)$ and $(3) \Leftrightarrow (4)$. Moreover, if all a-small right ideals in R are finitely generated, then all five conditions are equivalent.

Proof. $(1) \Leftrightarrow (2)$. By Proposition 3.1 and [11, Proposition 4.3.8, p. 89].

$(1) \Longrightarrow (3)$. Since ASI_R is a definable class, it is closed under pure submodules and hence $(ASI_R)^+ = ASI_R$. Since $((ASI_R)^{\ominus}, ASI_R)$ is an almost dual (by Proposition 3.1), it follows from [11, Theorem 4.3.2, p.85], that $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$.

$(1) \Longrightarrow (5)$. By [14, 3.4.8, p. 109].

$(3) \Leftrightarrow (4)$. By Proposition 3.1 and [11, Theorem 4.3.2, p.85].

$(4) \Longrightarrow (1)$ and $(5) \Longrightarrow (1)$. Suppose that all a-small right ideals in R are finitely generated. By Proposition 2.9, the class ASI_R is closed under pure submodules and hence $(ASI_R)^+ = ASI_R$. Thus the results follow from [11, Theorem 4.3.2, p.85]. □

Corollary 3.3. If every AS-injective modules is pure-injective, then the following statements are equivalent for a class ASI_R over a ring R.

(1) ASI_R is definable.

(2) ASI_R is closed under direct sums.

(3) $(ASI_R)^+ = ASI_R$.

(4) Each a-small right ideal in R is finitely generated.

Proof. The equivalence of $(1), (2)$ and (3) follows from Proposition 3.1 and [11, Theorem 4.5.1, p.103].

$(1) \Leftrightarrow (4)$. By Proposition 3.1, Proposition 2.9 and [11, Theorem 4.5.1, p.103]. □

Lemma 3.4. A left R-module $M \in (ASI_R)^{\ominus}$ if and only if $\mathrm{Tor}_1(R/I, M) = 0$, for any a-small right ideal I of a ring R.

Proof. Let M be a left R-module and $I \subseteq^a R_R$. By [5, Theorem 3.2.1, p.75], $\mathrm{Ext}^1(R/I, M^*) \cong (\mathrm{Tor}_1(R/I, M))^*$, so that $\mathrm{Tor}_1(R/I, M) = 0$ if and only if $M^* \in ASI_R$. Hence $({}_RASF, ASI_R)$ is an almost dual, where ${}_RASF = \{M \in R\text{-Mod} \mid \mathrm{Tor}_1(R/I, M) = 0$, for any a-small right ideal I of $R\}$. By [11, Proposition 4.2.11, p.72], $(ASI_R)^{\ominus} = {}_RASF$. □

A right R-module M is called n-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, with each F_i is a finitely generated free right R-modules [3].

Theorem 3.5. The following statements are equivalent for a class ASI_R over a ring R.

(1) ASI_R is definable.

(2) The class ASI_R is closed under pure submodules and pure homomorphic images.

(3) Every a-small right ideal in R is finitely presented.

(4) A module $M \in ASI_R$ if and only if $M^* \in (ASI_R)^{\ominus}$.

(5) A module $M \in ASI_R$ if and only if $M^{**} \in ASI_R$.

Proof. $(1) \Longrightarrow (2)$. By [14, 3.4.8, p. 109].

$(2) \Longrightarrow (3)$. Let M be any FP-injective module, thus there is a pure exact sequence $0 \to M \xrightarrow{i} E \xrightarrow{\pi} E/M \to 0$, where E is an injective right R-module. By hypothesis, $E/M \in ASI_R$.

Let $K \subseteq^a R_R$, thus $\text{Ext}^1(R/K, E/M) = 0$. By [6, Theorem 4.4 (4), p. 491], the sequence $0 = \text{Ext}^1(R/K, E/M) \longrightarrow \text{Ext}^2(R/K, M) \longrightarrow \text{Ext}^2(R/K, E) = 0$ is exact and hence $\text{Ext}^2(R/K, M) = 0$. By [13, Theorem 4.4 (3), p. 491], the sequence $0 = \text{Ext}^1(R, M) \longrightarrow \text{Ext}^1(K, M) \longrightarrow \text{Ext}^2(R/K, M) = 0$ is exact, so that $\text{Ext}^1(K, M) = 0$. By hypothesis, ASI_R is closed under pure submodules, so that K is finitely generated by Proposition 2.9 and hence [4, Proposition, p. 361] implies that K is finitely presented.

(3) $\Longrightarrow$ (1). Let $M \in ASI_R$. Let $K \subseteq^a R_R$, thus K is finitely presented (by hypothesis) and hence there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} K \longrightarrow 0$, where F_1, F_2 are finitely generated free right R-modules. Let $\beta = i\alpha_1$,where $i: K \longrightarrow R$ is the inclusion mapping, thus the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/K \longrightarrow 0$ is exact, where $\pi: R \longrightarrow R/K$ is the natural epimorphism. Hence R/K is a 2-presented module, so that from [3, Lemma 2.7 (2)] we have $\text{Tor}_1(R/K, M^*) \cong (\text{Ext}^1(R/K, M))^* = 0$. By Lemma 3.4, $M^* \in (ASI_R)^{\ominus}$ and hence $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$. By hypothesis, every a-small right ideal in R is finitely generated, so that ASI_R is closed under pure submodules by Proposition 2.9. By Theorem 3.2, ASI_R is a definable class.

(1) $\Longrightarrow$ (4). By Corollary 3.2, $(ASI_R, (ASI_R)^{\ominus})$ is an almost dual pair and hence a module $M \in ASI_R$ if and only if $M^* \in (ASI_R)^{\ominus}$.

(4) $\Longrightarrow$ (5). By hypothesis, $(ASI_R)^* \subseteq (ASI_R)^{\ominus}$. By Corollary 3.2, $(ASI_R)^{**} \subseteq ASI_R$. Hence for any right R-module M, if $M \in ASI_R$, then $M^{**} \in ASI_R$. Conversely, if $M^{**} \in ASI_R$, then $M^* \in (ASI_R)^{\ominus}$. By hypothesis, $M \in ASI_R$.

(5) $\Longrightarrow$ (1). Let N be a FP-injective module, thus there is a pure exact sequence $0 \to N \to E \to E/N \to 0$, where E is an injective right R-module. By [18, 34.5, p.286], the sequence $0 \to N^{**} \to E^{**} \to (E/N)^{**} \to 0$ is split. By hypothesis, $E^{**} \in ASI_R$ and hence $N^{**} \in ASI_R$. By hypothesis, $N \in ASI_R$ so that ASI_R is closed under pure submodules by Proposition 2.9. Thus ASI_R is definable class by Corollary 3.2. □

Note that if the class ASI_R is closed under pure submodules, then $(ASI_R)^+ = ASI_R$. Thus we have the following corollary.

Corollary 3.6. The class ASI_R is a definable if and only if it is closed under pure submodules and the class $(ASI_R)^+$ is a definable.

Corollary 3.7. If the class ASI_R is a definable, then the following are equivalent.

(1) The class of flat left R-modules and the class $(ASI_R)^{\ominus}$ are coincide.

(2) Every module in ASI_R is FP-injective.

(3) Every pure-injective module in ASI_R is injective.

Proof. (1) $\Longrightarrow$ (2). Let $M \in ASI_R$, thus $M^* \in (ASI_R)^{\ominus}$ by Corollary 3.2. By hypothesis, M^* is a flat left R-module and hence Proposition 3.54 in [15, p.136] implies that M^{**} is injective. Since M is a pure submodule in M^{**}, we have M is FP-injective by [18, 35.8, p.301].

(2) $\Longrightarrow$ (3). Let M be any pure-injective module in ASI_R. Let $\mathcal{E}: 0 \to M \longrightarrow M' \longrightarrow M'' \to 0$ be any exact sequence. By hypothesis, M is FP-injective. By [16, Proposition 2.6], the sequence $\mathcal{E}$ is pure and hence pure-injectivity of M implies that the sequence $\mathcal{E}$ is split by [18, 33.7, p. 279] . Therefore, M is injective.

$(3) \Longrightarrow (1)$. Let M be a flat left R-module, thus $\mathrm{Tor}_1(N, M) = 0$, for any right R-module N. By Lemma 3.4, $M \in (ASI_R)^{\ominus}$. Conversely, if $M \in (ASI_R)^{\ominus}$, then $M^* \in ASI_R$. By [14, Proposition 4.3.29, p. 149], M^* is a pure injective module. By hypothesis, M^* is injective and hence M is flat by [10, Theorem, p.239]. □

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صنف من المقاسات الأغمارية الصغيرة

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المستخلص:

لتكن R حلقة. في هذا البحث المقاس الايمن M على الحلقة R عرف ليكون اغماري من النمط $AS-$ اذا كان $\mathrm{Ext}^1(R/K\,,M)=0$ لأي مثالي صغير- مبطل ايمن K من الحلقة R. تشخيص الحلقات التي يكون كل مقاس معرف عليها هو اغماري من النمط $AS-$. الشروط التي بموجبها يكون صنف المقاسات الأغمارية من النمط $AS-$ اليمنى على الحلقة R (ASI_R) مغلق تحت القسمة (بالنسبة الى: المقاسات الجزئية النقية، الجمع المباشر) قد اعطيت. اخيراً، ندرس قابلية التعريف للصنف (ASI_R).