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A new subclasses of meromorphic univalent functions associated with a differential operator

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Abstract

In this paper we have introduced and studied some new subclasses of meromorphic univalent functions which are defined by means of a differential operator. We have obtained numerous sharp results including coefficient conditions, extreme points, distortion bounds and convex combinations for the above classes of meromorphic univalent functions.

Keywords: Univalent Functions, Meromorphic Functions, Differential Operator, Distortion Inequality, Extreme Points.

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1. Introduction

Let $\mathfrak{H}$ denote the class of functions which are analytic in the punctured disk $\mathcal{U}^* = \{z: 0 < |z| < 1\}$ of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_0 > 0. \qquad (1.1)$$

Suppose that $\mathfrak{H}^*$denote the subclass of $\mathfrak{H}$ consisting of functions that are univalent in $\mathcal{U}^*$.

Further $\mathfrak{H}_m^*$ denote subclass of $\mathfrak{H}^*$ consisting of functions f of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N} \qquad (1.2)$$

Definition: A function $f \in \mathfrak{H}_m^*$ is said to be meromorphic starlike of order α in $\mathcal{U}^*$ if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > -\alpha, z \in \mathcal{U}^*, 0 \le \alpha < 1. \quad (1.3)$$

On the other hand, a function $f \in \mathfrak{H}_m^*$ is said to be meromorphic convex of order α in $\mathcal{U}^*$ if it satisfies the inequality

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\alpha, z \in \mathcal{U}^*, 0 \le \alpha < 1. \quad (1.4)$$

Various subclasses of $\mathfrak{H}$ have been introduced and studied by many authors see [1], [2], [5], [7], [8], [16],[17],[19], [20], [21] and [23] In recent years, some subclasses of meromorphic functions associated with several families of integral operators and derivative operators were introduced and investigated see [7] [8], [18] and [4],[15]. The first differential operator for meromorphic function was introduced by Fraisin and Darus [10]. Ghanim and Darus introduced a differential operator [11]:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = zf'(z) + \frac{2a_0}{z},$$

$$I^2 f(z) = z\left(I^1 f(z)\right)' + \frac{2a_0}{z},$$

$$I^k f(z) = z\left(I^{(k-1)} f(z)\right)' + \frac{2a_0}{z},$$

where $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, z \in \mathcal{U}^*$.

For a function f in $\mathfrak{H}_m^*$, from definition of the differential operator $I^k f(z)$, we easily see that

$$I^k f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} n^k a_{m+n} z^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}, k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, z \in \mathcal{U}^*. \qquad (1.5)$$

By using the operator I^k, some authors have established many subclasses of meromorphic functions, for example [9], [11],[12] and [13]. With the help of the differential operator I^k, we define the following new class of meromorphic univalent functions and obtain some interesting results.

Let $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$, denote the family of meromorphic univalent functions f of the form (1.2) such that

$$\left|\frac{z^2\left(I^k f(z)\right)' + a_0}{\vartheta z^2\left(I^k f(z)\right)' - a_0 + (1+\vartheta)\eta a_0}\right| < \theta, \qquad (1.6)$$

For $0 \le \eta < 1, 0 < \theta \le 1$, $0 \le \vartheta \le 1$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

For a given real number $z_0 (0 < z_0 < 1)$. Let $\mathfrak{H}_{mi} (i = 0, 1)$be a subclass of $\mathfrak{H}_m^*$satisfying the condition $z_0 f(z_0) = 1$ and $-{z_0}^2 f'(z_0) = 1$ respectively.

Let

$$\mathfrak{H}_{mi,k}^*(\eta, \theta, \vartheta, z_0) = \mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta) \cap \mathfrak{H}_{mi}, (i = 0, 1). \qquad (1.7)$$

For other subclasses of meromorphic univalent functions, one may refer to the recent work of Aouf [2], Aouf and Darwish [3], Cho et al [8], Joshi et al [14], Srivastava and Owa [21] and [22]. Also we prove a necessary and sufficient condition for a subset C of the real interval $[0, 1]$ should satisfy the property $\cup_{z_r \in C} \mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_r)$ and $\cup_{z_r \in C} \mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_r)$ each constitute a convex family.

2. Coefficient Inequalities

In this section, we provide a necessary and sufficient condition for a function f meromorphic univalent in $\mathcal{U}^*$ to be in $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$, $\mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_0)$.

Theorem 2.1: A function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m,k}^*(\eta, \theta, \vartheta)$ if and only if

$$\sum_{n=0}^{\infty} n^k (m+n)(1 + \vartheta\theta) a_{m+n} \le \theta a_0 (1 - \eta)(1 + \vartheta), \quad (2.1)$$

where $0 \le \eta < 1, 0 < \theta \le 1$, $0 \le \vartheta \le 1$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

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The result is sharp for the function given by

$$f(z)=\frac{a_0}{z}+\frac{\theta a_0(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)}z^{m+n},\quad n\geq 1 \qquad (2.2)$$

Proof: Assume that the condition (2.1) is true. We must show that $f\in\mathfrak{H}^*_{m,k}(\eta,\theta,\vartheta)$ or equivalently prove that

$$\left|\frac{z^2\left(I^kf(z)\right)'+a_0}{\vartheta z^2\left(I^kf(z)\right)'-a_0+(1+\vartheta)\eta a_0}\right|<\theta,$$

$$\left|\frac{z^2\left(I^kf(z)\right)'+a_0}{\vartheta z^2\left(I^kf(z)\right)'-a_0+(1+\vartheta)\eta a_0}\right|=$$

$$\left|\frac{a_0+\left(-a_0+\sum_{n=1}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}\right)}{\vartheta\left(-a_0+\sum_{n=1}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}\right)-a_0+(1+\vartheta)\eta a_0}\right|$$
$$=$$
$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}}{\vartheta\left(-a_0+\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}\right)-a_0+(1+\vartheta)\eta a_0}\right|$$
$$\leq$$
$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}}{\vartheta\left(-a_0+\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}\right)-a_0+(1+\vartheta)\eta a_0}\right|<\theta.$$

The last inequality is true by (2.1).

Conversely, suppose that $f\in\mathfrak{H}^*_{m,k}(\eta,\theta,\vartheta)$. We must show that the condition (2.1) holds true. We have

$$\left|\frac{z^2\left(I^kf(z)\right)'+a_0}{\vartheta z^2\left(I^kf(z)\right)'-a_0+(1+\vartheta)\eta a_0}\right|<\theta.$$

Thus

$$\left|\frac{\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}}{\vartheta\left(-a_0+\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}\right)-a_0+(1+\vartheta)\eta a_0}\right|<\theta.$$

Since $Re(z)<|z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}}{\vartheta\left(-a_0+\sum_{n=0}^{\infty}(m+n)n^ka_{m+n}z^{m+n+1}\right)-a_0+(1+\vartheta)\eta a_0}\right\}<\theta.$$

Now, choosing values of z on the real axise and allowing $z\to 1$ from the left through real values, the last inequality immediately yields the desired condition in (2.1).

Finally, it is observed that the result is sharp for the function given by

$$f(z)=\frac{a_0}{z}+\frac{\theta a_0(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)}z^{m+n},\quad n\geq 1.$$

Theorem 2.2: A function $f(z)\in\mathfrak{H}^*_m$ defined by equation (1.2) is in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k(m+n)(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)}+z_0^{\,m+n+1}\right]a_{m+n}\leq 1,\qquad (2.3)$$

where $0\leq\eta<1, 0<\theta\leq 1, \quad 0\leq\vartheta\leq 1,\ k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $z\in\mathcal{U}^*$.

The result is sharp for the function given by

$$f(z)=\frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z\left[n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{\,m+n+1}\right]},\quad m\in\mathbb{N}, n\geq 1 \qquad (2.4)$$

Proof: Assume that $f\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$, then

$$f(z_0)=\frac{a_0}{z_0}+\sum_{n=0}^{\infty}a_{m+n}z_0^{\,m+n},\quad a_0>0, a_{m+n}>0, m\in\mathbb{N}$$

$$z_0f(z_0)=a_0+\sum_{n=0}^{\infty}a_{m+n}z_0^{\,m+n+1},\quad a_0>0, a_{m+n}>0, m\in\mathbb{N}$$

$$1=a_0+\sum_{n=0}^{\infty}a_{m+n}z_0^{\,m+n+1},\quad a_0>0, a_{m+n}>0, m\in\mathbb{N}$$

$$a_0=1-\sum_{n=0}^{\infty}a_{m+n}z_0^{\,m+n+1},\quad a_0>0, a_{m+n}>0, m\in\mathbb{N}, \qquad (2.5)$$

Subisstituting equation (2.5) in inequality (2.1), we get

$$\sum_{n=0}^{\infty}n^k(m+n)(1+\vartheta\theta)a_{m+n}\leq\theta\left(1-\sum_{n=0}^{\infty}a_{m+n}z_0^{\,m+n+1}\right)(1-\eta)(1+\vartheta),$$

$$\sum_{n=0}^{\infty}n^k(m+n)(1+\vartheta\theta)a_{m+n}+\sum_{n=0}^{\infty}\theta(1-\eta)(1+\vartheta)a_{m+n}z_0^{\,m+n+1}\leq\theta(1-\eta)(1+\vartheta)$$

Thus,

$$\sum_{n=0}^{\infty}\left[\frac{n^k(m+n)(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)}+z_0^{\,m+n+1}\right]a_{m+n}\leq 1.$$

Hence the proof is complete.

Theorem 2.3: A function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}(m+n)\left[\frac{n^k(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} - z_0^{m+n+1}\right]a_{m+n} \le 1, \quad (2.6)$$

where $0 \le \eta < 1, 0 < \theta \le 1, \quad 0 \le \vartheta \le 1, \ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

The result is sharp for the function given by

$$f(z) = \frac{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z(m+n)[n^k(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)z_0^{m+n+1}]}, \quad m \in \mathbb{N}, n \ge 1 \quad (2.7)$$

Proof: Assume that $f \in \mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$f(z_0) = \frac{a_0}{z_0} + \sum_{n=0}^{\infty} a_{m+n} z_0^{m+n}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$-z_0^2 f'(z_0) = a_0 + \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$1 = a_0 + \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$

$$a_0 = 1 - \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}, \quad (2.8)$$

subistituting equation (2.8) in equation (2.1), we get

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)a_{m+n}, \le \theta\left(1 - \sum_{n=0}^{\infty}(m+n)a_{m+n}z_0^{m+n+1}\right)(1-\eta)(1+\vartheta)$$

and,

$$\sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)a_{m+n} + \sum_{n=0}^{\infty}\theta(m+n)(1-\eta)(1+\vartheta)a_{m+n}z_0^{m+n+1} \le \theta(1-)(1-\eta)(1+\vartheta)$$

Thus,

$$\sum_{n=0}^{\infty}(m+n)\left[\frac{n^k(1+\vartheta\theta)}{\theta(1-\eta)(1+\vartheta)} - z_0^{m+n+1}\right]a_{m+n} \le 1.$$

Hence the proof is complete.

From Theorem 2.2 and Theorem 2.3, we have the following results:

Corollary 2.1: If a function $f(z) \in \mathfrak{H}_m^*$ defined by (1.2) is in the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$a_{m+n} \le \frac{\theta(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+n+1}}, \quad (2.9)$$

where $0 \le \eta < 1, 0 < \theta \le 1 \ 0 \le \vartheta \le 1, \ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

Corollary 2.2: If a function $f(z) \in \mathfrak{H}_m^*$ defined by (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$a_{m+n} \le \frac{\theta(1-\eta)(1+\vartheta)}{(m+n)[n^k(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)z_0^{m+n+1}]}, \quad (2.10)$$

where $0 \le \eta < 1, 0 < \theta \le 1 \ 0 \le \vartheta \le 1, \ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $z \in \mathcal{U}^*$.

3. Covering theorems

In this section, distortion theorems will be considered and covering property for functions in the classes $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta,\theta,\vartheta,z_0)$ will also be given.

Theorem 3.1: If a function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, then

$$|f(z)| \ge \frac{m(1+\vartheta\theta)-\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+1}]},$$

where $0 \le \eta < 1, 0 < \theta \le 1 \ 0 \le \vartheta \le 1, \ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, and $0 < |z| < 1$.

The result is sharp with the extremal function f given by

$$f(z) = \frac{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)r^{m+1}}{r[m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+1}]}.$$

Proof: Since $f \in \mathfrak{H}_{m0,k}^*(\eta,\theta,\vartheta,z_0)$, by Theorem 2.2 we have

$$m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+1}\sum_{n=0}^{\infty} a_{m+n} \le \sum_{n=0}^{\infty} n^k(m+n)(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+n+1}a_{m+n} \le \theta(1-\eta)(1+\vartheta),$$

$$\sum_{n=0}^{\infty} a_{m+n} \le \frac{\theta(1-\eta)(1+\vartheta)}{m(1+\vartheta\theta)+\theta(1-\eta)(1+\vartheta)z_0^{m+1}},$$

Also we have

$$a_0 = 1 - \sum_{n=0}^{\infty} a_{m+n} {z_0}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N},$$
$$\geq \frac{m(1+\theta\vartheta)}{m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}},$$

Thus from the above equation we obtain

$$|f(z)| = \left| \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n} \right|, a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$
$$\geq \frac{a_0}{r} - r^m \sum_{n=0}^{\infty} a_{m+n}$$
$$\geq \frac{m(1+\vartheta\theta) - \theta(1-\eta)(1+\vartheta) r^{m+1}}{r[m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Hence the proof is complete.

Theorem 3.2: If a function $f(z) \in \mathfrak{H}_m^*$ defined by equation (1.2) is in the class $\mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_0)$, then

$$|f(z)| \leq \frac{m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) r^{m+1}}{r[m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}]},$$

where $0 \leq \eta < 1, 0 < \theta \leq 1$ $0 \leq \vartheta \leq 1$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $0 < |z| = r < 1$.

The result is sharp with the extremal function f given by

$$f(z) = \frac{m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) r^{m+1}}{rm[(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Proof: Since $f \in \mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_0)$ by Theorem 2.3 we have

$m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1} \sum_{n=0}^{\infty} a_{m+n} \leq \sum_{n=0}^{\infty} n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1} a_{m+n} \leq \theta(1-\eta)(1+\vartheta),$

$$\sum_{n=0}^{\infty} a_{m+n} \leq \frac{\theta(1-\eta)(1+\vartheta)}{m(1+\vartheta\theta) - \theta(1-\eta)(1+\vartheta){z_0}^{m+1}}.$$

Also we have

$$a_0 = 1 + \sum_{n=0}^{\infty} (m+n) a_{m+n} {z_0}^{m+n+1}, \quad a_0 > 0, a_{m+n} > 0, m \in \mathbb{N},$$
$$\leq \frac{(1+\theta\vartheta)}{(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}}.$$

Thus from the above equation we obtain

$$|f(z)| = \left| \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n} \right|, a_0 > 0, a_{m+n} > 0, m \in \mathbb{N}$$
$$\leq \frac{a_0}{r} + r^m \sum_{n=0}^{\infty} a_{m+n}$$
$$\leq \frac{m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) r^{m+1}}{rm[(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}.$$

Hence the proof is complete.

Corollary 3.1: The disk $0 < |z| < 1$ is mapped onto a domain that contains the disk $|w| < \frac{m(1+\vartheta\theta) - \theta(1-\eta)(1+\vartheta) r^{m+1}}{[m(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+1}]}$ by any function $f \in \mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_0)$.

4. Extreme Points

The extreme points of the class $\mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_0)$ and $\mathfrak{H}_{m1,k}^*(\eta, \theta, \vartheta, z_0)$ are given by the following theorem.

Theorem 4.1: Let $f_0(z) = \frac{1}{z}$,

and

$$f_{m+n}(z) = \frac{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) z^{m+n+1}}{z[n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]},$$
$n \geq 0$

then $f(z)$ is in the class $\mathfrak{H}_{m0,k}^*(\eta, \theta, \vartheta, z_0)$, if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \gamma_n f_{m+n}(z)$ where $\gamma_n \geq 0, \gamma_i = 0$ $(i = 1, 2, \ldots, m-1, m \geq 2)$ and $\sum_{n=0}^{\infty} \gamma_n = 1$.

Proof: Suppose

$$f(z) = \sum_{n=0}^{\infty} \gamma_n f_{m+n}(z)$$
$$= \frac{\gamma_0}{z} + \sum_{n=0}^{\infty} \frac{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta) z^{m+n+1} \gamma_{m+n}}{z[n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}$$
$$= \frac{1}{z}\left[\gamma_0 + \sum_{n=0}^{\infty} \frac{n^k (m+n)(1+\vartheta\theta)\gamma_{m+n}}{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}\right]$$
$$+ \sum_{n=0}^{\infty} \frac{\theta(1-\eta)(1+\vartheta)\gamma_{m+n} z^{m+n+1}}{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}.$$

Then, we have

$$\sum_{n=0}^{\infty} \frac{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}{\theta(1-\eta)(1+\vartheta)} \times \left(\frac{\theta(1-\eta)(1+\vartheta)\gamma_{m+n}}{n^k (m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}} \right),$$

$$\sum_{n=0}^{\infty} \gamma_{m+n} = 1 - \gamma_0 \le 1.$$

Now, we have

$$z_0 f_{m+n}(z_0) = 1.$$

Thus,

$$z_0 f(z_0) = \sum_{n=0}^{\infty} \gamma_{m+n} z_0 f_{m+n}(z_0) = \sum_{n=0}^{\infty} \gamma_{m+n} = 1.$$

This implies that $f \in \mathfrak{H}_{m0,k}$.

Therefore $f \in \mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$.

Conversely, suppose $f \in \mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$. Since

$$a_{m+n} \le \frac{\theta(1-\eta)(1+\vartheta)}{n^k(m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}}, \quad n \ge 0.$$

Set

$$\gamma_{m+n} = \frac{[n^k(m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}{\theta(1-\eta)(1+\vartheta)} a_{m+n}, n \ge 0,$$

and $\gamma_0 = 1 - \sum_{n=0}^{\infty} \gamma_{m+n}$.

Then

$$f(z) = \sum_{n=0}^{\infty} \gamma_n f_n(z).$$

This completes the proof of Theorem 4.1.

Theorem 4.2: Let $f_0(z) = \frac{1}{z}$,

and

$$f_{m+n}(z) = \frac{n^k(m+n)(1+\vartheta\theta) + \theta(1-\eta)(1+\vartheta)z^{m+n+1}}{z(m+n)[n^k(1+\vartheta\theta) - \theta(1-\eta)(1+\vartheta){z_0}^{m+n+1}]}, \quad n \ge 0$$

Then $f(z)$ is in the class $\mathfrak{H}^*_{m1,k}(\eta, \theta, \vartheta, z_0)$, if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \gamma_n f_n(z)$ where $\gamma_n \ge 0, \gamma_i = 0$ $(i = 1, 2, \dots, m-1, m \ge 2)$ and $\sum_{n=0}^{\infty} \gamma_n = 1$.

Corollary 4.1: The extreme points of the class $\mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$ are the functions $f_0(z), f_m, f_{m+1}, f_{m+2}, \dots$ in Theorem 4.1.

Corollary 4.2: The extreme points of the class $\mathfrak{H}^*_{m1,k}(\eta, \theta, \vartheta, z_0)$ are the functions $f_0(z), f_m, f_{m+1}, f_{m+2}, \dots$ in Theorem 4.2.

5. Closure Theorems

Theorem 5.1: The class $\mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$ is closed under convex linear combination

Proof: Suppose that the functions $f, g \in \mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$ defined by

$$f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{m+n} z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

and

$$g(z) = \frac{b_0}{z} + \sum_{n=0}^{\infty} b_{m+n} z^{m+n}, b_0 > 0, b_{m+n} > 0, z \in \mathcal{U}^*$$

respectively, it is sufficient to prove that the function H defined by

$$H(z) = \omega f(z) + (1-\omega) g(z), \quad (0 \le \omega \le 1)$$

is also in the class $\mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$.

Since

$$H(z) = \frac{\omega a_0 + (1-\omega) b_0}{z} + \sum_{n=0}^{\infty} (\omega a_{m+n} + (1-\omega) b_{m+n}) z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

we observe that

$$\sum_{n=0}^{\infty} [n^k(m+n)(1+\vartheta\theta) + {z_0}^{m+n+1}](\omega a_{m+n} + (1-\omega) b_{m+n}) \le \theta(1-\eta)(1+\vartheta),$$

with the aid of theorem 2.2.

Thus $H(z) \in \mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$.

This completes the proof of the theorem.

In a similar manner, by using Theorem 2.3, we can prove the following theorem.

Theorem 5.2: The class $\mathfrak{H}^*_{m1,k}(\eta, \theta, \vartheta, z_0)$ is closed under convex linear combination.

Proof: The proof is similar to that of Theorem 5.1.

Theorem 5.3: Let the function $f_l(z), l = 0, 1, 2, \dots, q$ defined by

$$f_l(z) = \frac{a_{0,l}}{z} + \sum_{n=0}^{\infty} a_{m+n,l} z^{m+n}, a_0 > 0, a_{m+n,l} > 0, z \in \mathcal{U}^*$$

be in the class $\mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$. Then the function

$$\varphi(z) = \sum_{l=0}^{q} c_l f_l(z), \quad (c_l \ge 0)$$

is also in the class $\mathfrak{H}^*_{m0,k}(\eta, \theta, \vartheta, z_0)$, where $\sum_{l=0}^{q} c_l = 1$.

Proof: By Theorem 2.2 and for every $l = 0, 1, 2, \dots, q$ we have

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+z_0{}^{m+n+1}]a_{m+n,l} \le \theta(1-\eta)(1+\vartheta),$$

Then,

$$\varphi(z)=\sum_{l=0}^{q}c_l\left(\frac{a_{0,l}}{z}+\sum_{n=0}^{\infty}a_{m+n,l}z^{m+n}\right),\quad (c_l\ge 0)$$

$$=\frac{c_l a_{0,l}}{z}+\sum_{n=0}^{\infty}\left(\sum_{l=0}^{q}c_l a_{m+n,l}\right)z^{m+n}.$$

Since

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+z_0{}^{m+n+1}]\left(\sum_{l=0}^{q}c_l a_{m+n,l}\right),$$

$$=\sum_{l=0}^{q}c_l\left(\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta)+z_0{}^{m+n+1}]\,a_{m+n,l}\right),$$

$$\le\left(\sum_{l=0}^{q}c_l\right)\theta(1-\eta)(1+\vartheta),$$

$$=\theta(1-\eta)(1+\vartheta),$$

Then, $\varphi(z)\in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$.

Theorem 5.4: Let the function $f_l(z), l = 0, 1, 2, \dots, q$ defined by

$$f_l(z)=\frac{a_{0,l}}{z}+\sum_{n=0}^{\infty}a_{m+n,l}z^{m+n}, a_0>0, a_{m+n,l}>0, z\in\mathcal{U}^*$$

be in the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$. Then the function

$$\varphi(z)=\sum_{l=0}^{q}c_l f_l(z),\quad (c_l\ge 0)$$

is also in the class $\mathfrak{H}^*_{m1,k}(\eta,\theta,\vartheta,z_0)$, where $\sum_{l=0}^{q}c_l=1$.

Proof: The proof is similar to that of Theorem 5.3.

6. Convex Family

Definition 6.1: The family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is defined by

$$\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)=\cup_{z_r\in C}\,\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_r),$$

where C is a nonempty subset of the real interval $[0,1]$ and $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is defined by a convex family if the subset C consists of one element only by Theorems 5.1 and 5.3.

Now, we have the following results:

Lemma 6.1: Let $z_1, z_2 \in C$ be two distinct positive numbers and $f(z)\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)\cap\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$, then $f(z)=\frac{1}{z}$.

Proof: Suppose that $f(z)\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)\cap\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$,

we have

$$a_0=1-\sum_{n=0}^{\infty}a_{m+n}z_1{}^{m+n+1}$$

$$=1-\sum_{n=0}^{\infty}a_{m+n}z_2{}^{m+n+1}.$$

Also

$$f(z)=\frac{a_0}{z}+\sum_{n=0}^{\infty}a_{m+n}z^{m+n},\quad a_0>0, a_{m+n}>0, m\in\mathbb{N}$$

Thus,$a_{m+n}\equiv 0, \forall\, n\ge 0$, because $a_{m+n}\ge 0, z_1>0$ and $z_2>0$, hence

$$f(z)=\frac{1}{z}.$$

This completes the proof of the Lemma.

Theorem 6.1: Suppose that $C\subset[0,1]$, then $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is a convex family if and only if C is connected.

Proof: Assume that C is connected and $z_1, z_2\in C$ with $z_1<z_2$.

$$a_0=1-\sum_{n=0}^{\infty}a_{m+n}z_0{}^{m+n+1}$$

$$=1-\sum_{n=0}^{\infty}b_{m+n}z_1{}^{m+n+1}.$$

Suppose that the functions $f\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$ defined by

$$f(z)=\frac{a_0}{z}+\sum_{n=0}^{\infty}a_{m+n}z^{m+n}, a_0>0, a_{m+n}>0, z\in\mathcal{U}^*$$

and $g\in\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$

$$g(z)=\frac{b_0}{z}+\sum_{n=0}^{\infty}b_{m+n}z^{m+n}, b_0>0, b_{m+n}>0, z\in\mathcal{U}^*$$

it is sufficient to prove that the function H defined by

$$H(z)=\omega f(z)+(1-\omega)g(z),\quad (0\le\omega\le 1)$$

that there exists a $z_2(z_0\le z_2\le z_1)$ is also in the class $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$.

Then

$$K(z) = zH(z)$$

$$K(z) = \omega a_0 + (1-\omega)b_0 + \sum_{n=0}^{\infty}(\omega a_{m+n} + (1-\omega)b_{m+n})z^{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

$$= 1 + \omega\sum_{n=0}^{\infty}(z^{m+n} - {z_0}^{m+n})a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}(z^{m+n} - {z_1}^{m+n})\, b_{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

since z is real number, then $K(z)$ is also real number also we have

$K(z_0) \le 1$ and $K(z_1) \ge 1$, there exists $z_2 \in [z_0, z_1]$, such that $K(z_2) = 1$.

Therefore,

$$z_2 H(z_2) = z_2, \quad (z_0 \le z_2 \le z_1)$$

this implies that

$$H(z) \in \mathfrak{H}^*_{m0,k}.$$

We observe that

$$\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_2}^{m+n+1}](\omega a_{m+n} + (1-\omega)b_{m+n})$$

$$= \omega\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_0}^{m+n+1}]\, a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_1}^{m+n+1}]\, b_{m+n}$$

$$+\theta(1-\eta)(1+\vartheta)\omega\sum_{n=0}^{\infty}({z_2}^{m+n+1} - {z_0}^{m+n+1})\, a_{m+n} + \theta(1-\eta)(1+\vartheta)(1-\omega)\sum_{n=0}^{\infty}({z_2}^{m+n+1} - {z_1}^{m+n+1})\, b_{m+n}$$

$$= \omega\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_0}^{m+n+1}]\, a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}[n^k(m+n)(1+\vartheta\theta) + {z_1}^{m+n+1}]\, b_{m+n}$$

$$\le \theta(1-\eta)(1+\vartheta) + (1-\omega)\,\theta(1-\eta)(1+\vartheta) = \theta(1-\eta)(1+\vartheta).$$

With the aid of theorem 2.2.

Thus, $H(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_2)$.

Since z_1 and z_2 are arbitrary numbers, the family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is convex.

Conversely, if the set C is not connected, then there exists z_0, z_1 and z_2 such that $z_0, z_1 \in C$ and $z_2 \notin C$ and $z_0 < z_2 < z_1$.

Now, let $f(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$, and $g(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_1)$

Therefore,

$$K(\omega) = K(z_2,\omega) = 1 + \omega\sum_{n=0}^{\infty}({z_2}^{m+n+1} - {z_0}^{m+n+1})a_{m+n} + (1-\omega)\sum_{n=0}^{\infty}({z_2}^{m+n+1} - {z_1}^{m+n+1})\, b_{m+n}, a_0 > 0, a_{m+n} > 0, z \in \mathcal{U}^*$$

for fixed z_2 and $0 \le \omega \le 1$.

Since $K(z_2, 0) < 1$ and $K(z_2, 1) > 1$, there exists $\omega_0; 0 < \omega_0 < 1$, such that $K(z_2,\omega_0) = 1$ or $z_2 K(z_2) = 1$,

where $K(z) = \omega_0 f(z) + (1-\omega_0)g(z)$.

Therefore $K(z) \in \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,z_0)$

Also $K(z) \notin \mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ using Lemma 6.1.

Since $z_2 \in C$ and $K(z) \ne z$.

Thus the family $\mathfrak{H}^*_{m0,k}(\eta,\theta,\vartheta,C)$ is not convex which is a contradiction.

This completes the proof of theorem.

Conclusion: The main impact of this paper is to is to introduce a new subclasses of meromorphic univalent functions, and study their geometrical properties , like coefficient estimate, distortion theorem, extreme points and convex family.

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حول اصناف جديدة من الدوال احادية التكافؤ الميرومورفية بمؤثر تفاضلي

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المستخلص :

في هذا البحث نعرف وندرس اصناف جديدة $\mathfrak{H}^{*}_{mi,k}(\eta,\theta,\vartheta,z_0)(i=1,2)$ من الدوال احادية التكافؤ الميرومورفية المعرفة بواسطة مؤثر تفاضلي ، ونحصل على العديد من النتائج المهمة مثل متباينة المعاملات، والنقاط القصوى، نظرية البعد، التركيب المحدب للاصناف من الدوال السابقة.