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On the class of multivalent analytic functions defined by differential operator for derivative of first order

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Abstract

In the submitted search ,by making use of Differential operator ,we drive coefficient bounds and some important properties of the subclass $T_j(n,p,q,\alpha,\lambda)$ $(p,j \in N = \{1,2,\dots\}; q,n \in N_0 = N \cup \{0\}; 0 \le \alpha < p-q)$ of analytic and multivalent function with negative coefficients .Distortion property for functions in the class $T_j(n,p,q,\alpha,\lambda)$ are investigated once by using the composition involving an integral operator and certain fractional calculus operator and other once by using the composition involving an integral operator and certain fractional calculus inverse operator .

Keywords. Multivalent function, Coefficient bounds ,Distortion inequality , δ- neighbourhood , Differential operator, integral and fractional operators .

Mathematics Subject Classification: 30C45.

1-Introduction

Let $T(j,p)$ be the class of analytic and multivalent functions $f(z)$ in the open unit disk $U=\{z:z\in C:|z|<1\}$ that defined by

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k \quad (a_k\geq 0; j,p\in N=\{1,2,\dots.\}) . \quad \textbf{(1)}$$

Let $T(j,p)$ the class consists of function of the form

$$f^q(z)=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}a_kz^{k-q} \quad (a_k\geq 0\,;\,q\in N_0=N\cup\{0\}\,;\,p>q) . \quad \textbf{(2)}$$

The Differential operator for a function in $T(j,p)$ is define by

$$D_p^n(f^q(z))=\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^n a_kz^{k-q} \quad (p,j\in N=\{1,2,\dots\}\,;q,n\in N_0\,;\,p>q) . \quad \textbf{(3)}$$

The operator D_p^n was studied by M.K. Aouf [5] and Altintas et al. [7] , earlier by Owa [13] ,Yamakawa[9] ,Owa [12] ,Srivastava Owa[4] .
It is easy to see that

$$D_p^{n+1}f^q(z)=\frac{z}{(p-q)}(D_p^nf^q(z))' \quad \textbf{(4)}$$

By using the operator $D_p^nf^q(z)$ Given by **(3)** ,a function $f(z)$ belonging to $T_j(n,p,q,\alpha,\lambda)$ if and only if

$$R\left(\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda\,D_p^{n+1}(f^q(z))}\right)>\alpha \quad (p\in N\,;q,n\in N_0=N\cup\{0\}\,;\,p>q) , \quad \textbf{(5)}$$

for some α $(0\leq\alpha<p)$ and for all $z\in U$.

Next the following earlier investigations by Osman Altintas,Huseyin Irmak and H.M.Srivastava [6] ,

when $f(z)\in T(j,p)$ we define the δ - neighborhood by

$$N_\delta(f)=\{g:g\in T(j,p),g(z)=z^p-\sum_{k=j+p}^{\infty}b_kz^k \;\; and\; \sum_{k=j+p}^{\infty}k|a_k-b_k|\leq\delta\;\}. \quad \textbf{(6)}$$

So that ,obviously ,

$$N_\delta(h)=\{g:g\in T(j,p),g(z)=z^p-\sum_{k=j+p}^{\infty}b_kz^k \;\; \text{And}\; \sum_{k=j+p}^{\infty}|b_k|\leq\delta\} \quad \textbf{(7)}$$

Where, and in what follows,

$$h(z)=z^p \qquad (k\geq j+p; n,p\in N; q\in N_0=N\cup\{0\}). \quad \textbf{(8)}$$

we using the familiar operator $J_{c,p}$ defined by Bernardi [10], Libera[8] and Srivastave et al. [2] as follows

$$(J_{c,p}f)(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (f(z)\in T(j,p)\,;c>-p\,;p\in N), \quad \textbf{(9)}$$

and fractional calculus operator D_z^μ Srivastave [11] , Srivastava et al.[3] that known as the form

$$D_z^\mu(z^p)=\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p-\mu} \qquad (p>-1;\mu\epsilon R\,) \quad \textbf{(10)}$$

2-Coefficient Inequalities

We drive sufficient condition for $f(z)$ that defined by using differential operator .

Theorem 1. Assume that $f(z)\in T(j,p)$. Then $f(z)\in T_j(n,p,q,\alpha,\lambda)$ if and only if

$$\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left[\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\frac{k-q}{p-q}-1\right)\right)\right]\delta(k,q)a_k<(1-\alpha)\delta(p,q)$$

$$(0\leq\alpha<p-q\,;\,p,j\in N\,;\,q\,,n\in N_0\,;p>q\,) \quad \textbf{(11)}$$

Where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases}p(p-1)\dots..(p-q+1) & q\neq0\\ 1 & q=0\end{cases} \quad \textbf{(12)}$$

Proof. If $f(z)\in T_j(n,p,q,\alpha,\lambda)$, then

$$R\left(\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda\,D_p^{n+1}(f^q(z))}\right)=R\left(\frac{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^{n+1}a_kz^{k-p}}{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[1-\lambda+\lambda(\frac{k-q}{p-q})]a_kz^{k-p}}\right)$$

$$\left\{\frac{\frac{p!}{(p-q)!}-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^{n+1}a_kz^{k-p}}{\frac{p!}{(p-q)!}+\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[\lambda-1-\lambda(\frac{k-q}{p-q})]a_kz^{k-p}}\right\}>\alpha$$

$$(1-\alpha)\frac{p!}{(p-q)!}>\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n\left(\left(\frac{k-q}{p-q}\right)+\lambda\alpha-\alpha-\lambda\alpha\left(\frac{k-q}{p-q}\right)\right)\frac{k!}{(k-q)!}a_kz^{k-p} ,$$

Since $z\to1^-$,we have $\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)a_kz^{k-p}<(1-\alpha)\delta(p,q)$

Conversely ,assume that inequality **(11)** holds true ,since

$R(w)>\alpha$ if and only if $\left|\frac{w-1}{w+(1-2\alpha)}\right|<1$

Since

$$\left|\frac{\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda\,D_p^{n+1}(f^q(z))}-1}{\frac{D_p^{n+1}(f^q(z))}{(1-\lambda)D_p^n(f^q(z))+\lambda\,D_p^{n+1}(f^q(z))}+(1-2\alpha)}\right|$$

$$=\left|\frac{(1-\lambda)D_p^{n+1}(f^q(z))-(1-\lambda)\,D_p^{n}(f^q(z))}{(1+\lambda-2\alpha\lambda)D_p^{n+1}(f^q(z))+(1-2\alpha)-\lambda(1-2\alpha)\,D_p^{n}(f^q(z))}\right|$$

$$=\left|\frac{-\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[\frac{k-q}{p-q}-\lambda\left(\frac{k-q}{p-q}\right)-1+\lambda]a_k z^{k-p}}{2\,\frac{p!}{(p-q)!}(1-\alpha)+\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n[2\alpha+\lambda-2\alpha\lambda-1-\frac{k-q}{p-q}-\lambda\left(\frac{k-q}{p-q}\right)+2\alpha\lambda(\frac{k-q}{p-q})]a_k z^{k-p}}\right|$$

$$\leq\frac{\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n[\left(\frac{k-q}{p-q}\right)-\lambda\left(\frac{k-q}{p-q}\right)-1+\lambda]\frac{k!}{(k-q)!}a_k z^{k-p}}{2\,(1-\alpha)\frac{p!}{(p-q)!}+\sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^n\left[2\alpha+\lambda-2\alpha\lambda-1-\left(\frac{k-q}{p-q}\right)\{1+\lambda-2\alpha\lambda\}\right]\frac{k!}{(k-q)!}a_k z^{k-p}}$$

$$\leq 1\,.$$

Putting $j=1, n=1, q=0$ and $\lambda=0$ in Theorem 1,we have the following corollary :

Corollary 1 . Let the function $f(z)\in T(j,p)$. Then $f(z)\in C(p,\alpha)$ if and only if

$$\sum_{k=1+p}^{\infty}\frac{k}{p}\left[\frac{k}{p}-\alpha\right]a_k<(1-\alpha) \qquad (0\leq\alpha<p\,;p\in N)\ .$$

Not that this result obtained by Salagean et al[1] .

Corollary 2. Assume that the function $f(z)$ defined by **(2)** be in the class $T_j(n,p,q,\alpha,\lambda)$ .Then

$$\sum_{k=j+p}^{\infty}\delta(k,q)a_k<\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}$$

(13)

$(k\geq j+p\ ;p\,,j\in N\,;q\,,n\in N_0\,;\ p>q\,)$.

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)z^k\ \ (k\geq j+p\,;\ p,j\in N\,;q\,,n\in N_0\,;p>q\,)\ . \qquad \textbf{(14)}$$

3-Extreme points

Theorem 2. Let $f_p(z)=z^p$ and $f_k(z)=z^p-\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)z^k$, for $k\geq j+p$ and $p>q$. Then $f(z)\in T_j(n,p,q,\alpha,\lambda)$ if and only if it is of the form

$f(z)=\sum_{k=p}^{\infty}\omega_k f_k(z)$,where $\omega_k\geq 0$ for all $k\geq j+p$ and $\sum_{k=p}^{\infty}\omega_k=1$.

Proof . Suppose that $f(z)=\sum_{k=p}^{\infty}\omega_k f_k(z)=\omega_p f_p(z)+\sum_{k=j+p}^{\infty}\omega_k f_k(z)$

$= z$^$p + ∑$_$(k = j + p\,)$^$∞$▒ 〖ω_k (((1 − α)δ(p, q))/(((k − q)/(p − q))^n (((k − q)/(p − q)) − α(1 + λ(((k − q)/(p − q)) − 1)))δ(k, q))) z^k 〗 .

Then

$$\sum_{k=j+p}^{\infty}\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)\omega_k\left[\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right]$$

$$=\sum_{k=j+p}^{\infty}\omega_k(1-\alpha)\delta(p,q)$$

$$=(1-\omega_p)(1-\alpha)\delta(p,q)$$

$$\leq(1-\alpha)\delta(p,q)\ .$$

Thus ,it follows from Theorem **(1)** that $f(z)\in T_j(n,p,q,\alpha,\lambda)$.

Conversely ,suppose that $f(z)\in T_j(n,p,q,\alpha,\lambda)$,since

$$a_k<\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left[\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right]\delta(j+p,q)}\ ,k\geq j+p\ .$$

We define

$$\omega_k=\frac{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}{(1-\alpha)\delta(p,q)}a_k$$

$(k\geq j+p)$.

And $\omega_p=1-\sum_{k=j+p}^{\infty}\omega_k$ by simple calculation ,we get

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_k z^k$$

$$=z^p-\sum_{k=j+p}^{\infty}\left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)}\right)\omega_k z^k$$

$$=\omega_p z^p-\sum_{k=j+p}^{\infty}\omega_k f(z)\ .$$

Thus we get the result .

4-Neighbouhoods for the function class $T_j(n,p,q,\alpha,\lambda)$

In this section ,we conclude the neighborhood properties for each of the following slightly mutated function in the class $T_j(n,p,q,\alpha,\lambda,\gamma)$ .Our first implication relation including the δ- neighbourhood $N_\delta(h)$ is given below in the following Theorem .

Theorem 3. If $f(z)$ belonging to $T_j(n,p,q,\alpha,\lambda)$,then

$$T_j(n,p,q,\alpha,\lambda)\subset N_\delta(h)\ . \qquad \textbf{(15)}$$

Where $h(z)$ is defined as **(8)** and

$$\delta:=\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}$$

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Proof . For $f(z) \in T(j,p)$,Theorem **(1)** ,immediately yields .

$$\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)\sum_{k=j+p}^{\infty}a_k \leq (1-\alpha)\delta(p,q) ,$$

so that $\delta(j+p,q)\sum_{k=j+p}^{\infty}ka_k \leq \left(\frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)$.

Thus ,we have

$$\sum_{k=j+p}^{\infty}ka_k \leq \frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)} := \delta .$$

This complete the proof .

A function $f(z) \in T(j,p)$ is said to be in the class $T_j(n,p,q,\alpha,\lambda,\gamma)$ if there exists another function $g(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < p-\gamma \quad (z \in U : 0 \leq \gamma < p).$$

Theorem 4. Let $g(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ .Suppose also that

$$\gamma = p - \frac{\delta}{j+p}\left[\frac{(j+p)!\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!-(1-\alpha)\delta(p,q)(j+p-q)!}\right] \quad \textbf{(16)}$$

Then

$$N_\delta(g) \subset T_j(n,p,q,\alpha,\lambda,\gamma)$$

Proof . Suppose that $f(z) \in N_\delta(g)$,we then find from **(6)** that

$$\sum_{k=j+p}^{\infty}k|a_k-b_k| \leq \delta$$

Which readily implies the following coefficient inequality

$$\sum_{k=j+p}^{\infty}|a_k-b_k| \leq \frac{\delta}{(j+p)} \quad ,(j,p \in N; p > q)$$

Next ,since $g(z) \in T_j(n,p,q,\alpha,\lambda)$,we have

$$\sum_{k=j+p}^{\infty}b_k \leq \frac{(1-\alpha)\delta(p,q)((j+p-q)!)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=j+p}^{\infty}|a_k-b_k|}{1-\sum_{k=j+p}^{\infty}b_k}$$

$$\leq \frac{\delta}{(j+p)}\left(\frac{1}{1-\frac{(1-\alpha)\delta(p,q)(j+p-q)!}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)(j+p)!}}\right)$$

$$= \frac{\delta\ (j+p)!\left(\frac{j}{p-q}+1\right)^n\left[\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right]}{\left((j+p)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\right)(j+p)!-((1-\alpha)\delta(p,q))(j+p-q)!}$$

$$= p-\gamma$$

Provided that γ is given properly by **(16)** .Thus we have $f(z) \in T_j(n,p,q,\alpha,\lambda,\gamma)$ for every γ given by **(16)**.This obviously completes the proof of Theorem **(4)**.

5- Properties involving the operator $J_{c,p}$ and D_z^{μ}

Lemma 1.[6] let the function $f(z) \in T(j,p)$,then

$$D_z^{\mu}\left(J_{c,p}f(z)\right) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\left(\frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k-\mu+1)}\right)a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; j,p \in N) \quad \textbf{(17)}$$

And

$$J_{c,p}\left(D_z^{\mu}f(z)\right) = \frac{(c+p)\Gamma(p+1)}{(p-\mu+c)\Gamma(p-\mu+1)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\left(\frac{(c+p)\Gamma(k+1)}{(k-\mu+c)\Gamma(k-\mu+1)}\right)a_k z^{k-\mu}$$

$$(\mu \in R ;\ c > -p ;\ j,p \in N) \quad \textbf{(18)}$$

Provided that there are no zeros appear in the denominators in (17) and (18) .This in general ,the operators $J_{c,p}$ and D_z^{μ} are non-commutative .

So as to give growth and distortion properties for functions in the class $T_j(n,p,q,\alpha,\lambda)$ including the operators $J_{c,p}$ and D_z^{μ} ,we find it to be convenient to use the order operation exhibited by **(18)** and **(19)** as we shown in the following Theorems .

Theorem 5 .If $f(z)$ is in the class $T_j(n,p,q,\alpha,\lambda)$, then

$$\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p+\mu} \leq \left|D_z^{-\mu}\left(J_{c,p}f(z)\right)\right|$$

$$\leq \left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p+\mu}$$

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$(z \in U\,;\ 0 \le \alpha < p-q\,;\ \mu > 0\,;\ n, q \in N_0\,;\ j, p \in N, c > -p\,;\ p > q\,)$
(19)

The result is sharp for the function give by
$$J_{c,p}(f(z)) = z^p - \left(\frac{(c+p)(1-\alpha)\delta(p,q)}{(j+p+c)\left(\frac{j}{p-q}+1\right)^n\left(\frac{j}{p-q}(1-\alpha\lambda)+(1-\alpha)\right)}\right) z^{j+p}$$
(20)

Proof . It follows from Theorem **(1)** that

$$\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)\sum_{k=j+p}^{\infty} a_k \le$$
$$\sum_{k=j+p}^{\infty}\left[\frac{k-q}{p-q}\right]^n\left(\left(\frac{k-q}{p-q}\right)-\alpha\left(1+\lambda\left(\left(\frac{k-q}{p-q}\right)-1\right)\right)\right)\delta(k,q)a_k$$
$$\le (1-\alpha)\delta(p,q)\ .$$

Which readily yields

$$\sum_{k=j+p}^{\infty} a_k \le \frac{(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\ .$$
(21)

Assumed that the function defined in U by
$$F(z) = \left(\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)}\right) z^{-\mu} D_z^{-\mu}(J_{c,p}f(z))$$
$$= z^p - \sum_{k=j+p}^{\infty}\left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(k+1)}{(k+c)\Gamma(p+1)\Gamma(k+\mu+1)}\right) a_k z^k$$
$$= \delta(p,q)z^{p-q} - \sum_{k=j+p}^{\infty}\theta(k)a_k z^{k-q} \qquad (z \in U)\ ,$$
(22)

if we set $\theta(k) = \frac{(p+c)\Gamma(p+\mu+1)\Gamma(k+1)}{(k+c)\Gamma(p+1)\Gamma(k+\mu+1)}$ $(k \ge j+p; j, p \in N)$. **(23)**

Then it is easily seen that $\theta(k)$ is decreasing function of k when $\mu > 0$,and hence
$$0 < \theta(k) \le \theta(j+p) = \frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)} \qquad (c > -p : \mu > 0;\ j, p \in N))\ .$$ **(24)**

Where
$$D_z^{-\mu}(J_{c,p}f(z)) = z^{p+\mu} - \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right) a_k z^{j+p+\mu}$$

By using **(21)** and **(24)**, we deduce that
$$|z|^p - \theta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k \le |F(z)| \le$$
$$|z|^p + \theta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k$$

That is
$$|z|^p - \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right)|z|^{j+p} \le |F(z)| \le$$
$$|z|^p + \left(\frac{(p+c)\Gamma(p+\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)}\right)|z|^{j+p}$$
Which yields inequality **(19)**

Theorem 6 . If $f(z)$ is in $T_j(n, p, q, \alpha, \lambda)$, then

$$\left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p-\mu} \le |D_z^{\mu}(J_{c,p}f^q(z))|$$
$$\le \left\{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \left(\frac{(c+p)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(j+p+\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^j\right\}|z|^{p-\mu}$$
$(z \in U; 0 \le \alpha < p-q; 0 \le \mu \le 1; n, q \in N_0; j, p \in N, c > -p; p > q\,)$. **(25)**

The result is sharp for the function give by **(20)** .

Proof . It follows from Theorem **(1)** that

$$\sum_{k=j+p}^{\infty} k a_k \le \frac{(j+p)(1-\alpha)\delta(p,q)}{\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)\delta(j+p,q)} \qquad (0 \le \alpha < p: 0 \le \mu \le 1; j, p \in N)\ ,$$ **(26)**

suppose that the function defined in U as follows
$$G(z) = \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu} D_z^{\mu}(J_{c,p}f(z))$$

$$= z^p - \sum_{k=j+p}^{\infty}\left(\frac{(p+c)\Gamma(p-\mu+1)\Gamma(k)}{(k+c)\Gamma(p+1)\Gamma(k-\mu+1)}\right) k a_k z^k$$

$$= z^p - \sum_{k=j+p}^{\infty}\vartheta(k) k a_k z^k \qquad (z \in U)\ ,$$
(27)

if we set $\vartheta(k) = \frac{(p+c)\Gamma(p-\mu+1)\Gamma(k)}{(k+c)\Gamma(p+1)\Gamma(k-\mu+1)}$ $(k \ge j+p; 0 \le \mu < 1; j, p \in N)$. **(28)**

Then it is easily seen that $\vartheta(k)$ is decreasing function of k when $\mu < 1$,and hence

$$0 < \vartheta(k) \leq \vartheta(j+p) = \frac{(p+c)\Gamma(p-\mu+1)\Gamma(j+p)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)}$$
$(c > -p\,;\ p\,,j \in N\,; 0 \leq \mu < 1)$. **(29)**

By using **(26)** and **(29)**, we deduce that

$$|z|^p - \vartheta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} k a_k \leq |F(z)| \leq |z|^p + \vartheta(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} k a_k$$

That is

$$|z|^p - \left(\frac{(c+p)\Gamma(p-\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^{j+p} \leq |F(z)| \leq$$

$$|z|^p + \left(\frac{(c+p)\Gamma(p-\mu+1)\Gamma(j+p+1)(1-\alpha)\delta(p,q)}{(j+p+c)\Gamma(p+1)\Gamma(j+p-\mu+1)\left(\frac{j}{p-q}+1\right)^n\left(\left(\frac{j}{p-q}(1-\alpha\lambda)\right)+(1-\alpha)\right)}\right)|z|^{j+p}$$

.
Which yields inequality **(25)** .

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حول اصناف الدوال التحليلية المتعددة التكافوء المعرفة بواسطة المؤثر التفاضلي للمشتقة من الرتبة الاولى

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المستخلص :

في البحث المقدم .وباستخدام العامل التفاضلي ، وجدنا حدود المعامل ، بعض الخواص المهمة لدوال التحليلية المتعددة التكافوء لمعاملات سالبه للصنف الجزئي $T_j(n,p,q,\alpha,\lambda)$ حيث $(p,j \in N=\{1,2,\dots\}; q,n \in N_0 = N \cup \{0\}; 0 \le \alpha < p-q)$ لدوال .قدمنا خاصية النشوء لتلك الدوال باستخدام التركيب المتضمن العامل التفاضلي والعامل الكسري الحسابي *مرة ومرة اخرى استخدمنا* التركيب المتضمن العامل التفاضلي والعامل الكسري الحسابي المعكوس .