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A Class of Meromorphic p-valent Functions

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Abstract:
In this paper, we define a class of meromorphic p-valent functions and study some properties as coefficient inequality, closure theorem , growth and distortion bounds , arithmetic mean**,** radius of convexity, Convex linear combination and partial sums .

Keywords: Meromorphic p-valent function,Convex function, Integral operator .

Mathematics Subject Classification: 30C45.

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1-Introduction: Let A_p^* denote the class of functions f of the form

$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n\, z^n, (a_n \geq 0, n \geq p, p \in N),$ (1.1)

which are analytic and p-valent in the punctured unit disk $U^*=\{z\in \mathbb{C}: 0 < |z| < 1\}$. Jum-Kim Srivastara [2] defined an integral operator $I_p^\sigma f(z)$ for $f \in A_p^*$ as follows

$I_p^\sigma f(z) = \frac{1}{z^{p+1}\Gamma(\sigma)} \int_0^z (\log\frac{z}{t})^{\sigma-1} t^p f(z) dt\,, (n \in N).$ (1.2)

If $f(z)$ is of the form (1.1), then

$$I_p^\sigma f(z) = z^{-p} + \sum_{n=p}^{\infty} (\frac{1}{n+p+1})^\sigma\, a_n\, z^n (n \geq p, p \in N). \quad (1.3)$$

In particular, when p=1 we have:

$I_p^\sigma f(z) = z^{-1} + \sum_{n=1}^{\infty} (\frac{1}{n+2})^\sigma\, a_n\, z^n (n \geq p, p \in N).$

Let f and g be analytic in unit disk U, then g is said to be subordinate of f, written as $g \prec f$ or $g(z) \prec f(z)$, if there exists a schwartz function ω which is analytic in U with $\omega(0)=0$ and $|\omega(z)| < 1 (z \in U)$ such that $g(z)= f(\omega(z))$.

In particular, if the function f is univalent in U, we have the following equivalence ([3],[4]).

$$g(z) \prec f(z) (z \in U) \Leftrightarrow g(0) = f(0) \, and \, g(U) \subseteq f(U).$$

Definition(1.1): A function $f \in A_p^*$ is said to be in the class $A_p^*(\sigma, b, x, y)$ of functions of the form (1.1), which satisfies the condition

$$p - \frac{1}{b}\left\{1 + \frac{z^2(I_p^\sigma f(z))''}{z(I_p^\sigma f(z))'} + p\right\} \prec p\frac{1+xz}{1+yz}, \quad (1.4)$$

where

$-1 \leq y \leq x \leq 1, p \in N, \sigma < 0, b$ *non zero complex number*.

We can re-write the condition (1.4) as

$$\left|\frac{z(I_p^\sigma f(z))'' + (1+p)(I_p^\sigma f(z))'}{yz(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp](I_p^\sigma f(z))'}\right| < 1 . (1.5)$$

2.Coefficient inequality:

In the following theorem, we give a sufficient and necessary condition to be the function in the class $A_p^*(\sigma, b, x, y)$.

Theorem (2.1): Let $f \in A_p^*$ be given by (1.1). Then $f \in A_p^*(\sigma, b, x, y)$ if and only if

$$\sum_{n=p}^{\infty} [n(n+p)(1-y) - np|b|\left(\frac{x}{-y}\right)] (\frac{1}{n+p+1})^\sigma a_n \leq p^2|b|(x-y). \quad (2.1)$$

The results is sharp for the function f given by

$f(z) = z^{-p} + \left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y) - np|b|(x-y)]}\right)(n+p+1)^\sigma z^n, (n \geq p, n \in N).$ (2.2)

Proof: Assuming that the inequality (2.1) holds true and $|z| = 1$. Then ,we have

$$\left|z^2(I_p^\sigma f(z)'' + (1+p)z(I_p^\sigma f(z))'\right| - \left|yz^2(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp]z(I_p^\sigma f(z))'\right|$$

$$=\left|\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n z^n\right| - \left|p^2|b|(x-y) + \sum_{n=p}^{\infty}[yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n z^n\right|$$

$$\leq \sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n\, |z|^n - p^2|b|(x-y) - \sum_{n=p}^{\infty}[yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n |z|^n$$

$$=\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n - p^2|b|(x-y) - \sum_{n=p}^{\infty}[yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n \leq 0,$$

by hypothesis.

Hence, by the Maximum Modulus Theorem, we have $f(z) \in A_p^*(\sigma, b, x, y)$.

Conversely, suppose that $f(z) \in A_p^*(\sigma, b, x, y)$. Then from (1.5) ,we have

$$\left|\frac{z^2(I_p^\sigma f(z)'' + (1+p)z(I_p^\sigma f(z))'}{yz^2(I_p^\sigma f(z))'' + [y(1+p(1-b)) + xbp] z(I_p^\sigma f(z))'}\right|$$

$$= \left|\frac{\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n\, z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty} [yn(n+p) + n|b|p(x-y)] (\frac{1}{n+p+1})^\sigma a_n z^n}\right| < 1.$$

Since $\mathrm{Re}(z) \leq |z|$ for all $z(z\in U)$ *, we have*

$$\mathrm{Re}\left(\frac{\sum_{n=p}^{\infty} n(n+p)(\frac{1}{n+p+1})^\sigma a_n z^n}{p^2|b|(x-y) + \sum_{n=p}^{\infty}[yn(n+p) + n|b|p(x-y)](\frac{1}{n+p+1})^\sigma a_n z^n}\right) \leq 1.$$

We choose the value of z on the real and $z \to 1^-$, we get

$$\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|\binom{x}{-y}](\frac{1}{n+p+1})^{\sigma}a_n \le p^2|b|(x-y),$$

which give (2.1).Sharpness of the result follows by setting

$f(z)=z^{-p}+\left(\frac{p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)]}\right)(n+p+1)^{\sigma}z^n,\ (n \ge p, n \in N).$

Corollary (2.1) : Let $f(z) \in A_p^*(\sigma,b,x,y)$ .Then $a_n \le \frac{p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}$, $(n \ge p, n \in N)$.

3. Growth and the Distortion Bounds:

In the following theorems, we obtain the growth and the distortion theorems for the function in the class $A_p^*(\sigma,b,x,y)$.

Theorem (3.1): If the function $f(z)$defined by (1.1) is in the class $A_p^*(\sigma,b,x,y)$, then for $0<|z|=r<1$, we have:

$$r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p \le |f(z)| \le r^{-p}+\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p, \quad (3.1)$$

where equality holds true for the function

$$f(z)=z^{-p}+\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)z^p. \quad (3.2)$$

Proof: Since $f(z) \in A_p^*(\sigma,b,x,y)$ .Then from (2.1)

$2p^2(1-y)-p^2|b|(x-y)(\frac{1}{2p+1})^{\sigma}\sum_{n=p}^{\infty}|a_n| \le \sum_{n=p}^{\infty}[n(n+p)(1-y)-p|b|(x-y)](\frac{1}{n+p+1})^{\sigma}a_n \le p^2|b|(x-y)$,

we conclude that

$$\sum_{n=p}^{\infty}|a_n| \le \frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)} \quad (3.3)$$

Thus for $0<|z|=r<1$,

$$|f(z)| \le |z|^{-p}+\sum_{n=p}^{\infty}a_n|z|^n \le r^{-p}+r^p\sum_{n=p}^{\infty}a_n, \quad (3.4)$$

or

$$|f(z)| \le r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p, \quad (3.5)$$

and

$$|f(z)| \ge |z|^{-p}-\sum_{n=p}^{\infty}a_n|z|^n \ge r^{-p}-r^p\sum_{n=p}^{\infty}a_n,$$

or

$$|f(z)| \ge r^{-p}-\left(\frac{|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-|b|(x-y)}\right)r^p.$$

On using (3.4) and (3.5) inequality (3.1) follows.

Theorem (3.2): If $f \in A_p^*(\sigma,b.x,y)$ then

$$r^{-(p+1)}-\left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-p^2|b|(x-y)}\right)r^{p-1} \le |f(z)'| \le r^{-(p+1)}+\left(\frac{p|b|(x-y)(2p+1)^{\sigma}}{2(1-y)-p^2|b|(x-y)}\right)r^{p-1}.$$

The result is sharp for the function f is given by (1.3)

Proof: The proof is similar to that of Theorem (3.1).

4. Extreme Points

In the next theorems, we obtain extreme points for the class $A_p^*(\sigma,b,x,y)$.

Theorem (4.1): Let $f_{p-1}(z)=z^{-p}$ and $f_n(z)=z^{-p}+\left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma}}{[n(n+p)(1-y)-np|b|(x-y)]}\right)z^n, \quad (4.1)$

for $n \ge p$. Then $f(z) \in A_p^*(\sigma,b,x,y)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=p-1}^{\infty}\mu_n f_n(z), \text{where } \mu_n \ge 0 \text{ and } \sum_{n=p-1}^{\infty}\mu_n=1. \quad (4.2)$$

Proof: Let

$$f(z)=\sum_{n=p-1}^{\infty}\mu_n f_n(z)=z^{-p}+\sum_{n=p}^{\infty}\left(\frac{p^2|b|(x-y)(n+p+1)^{\sigma}\mu_n}{[n(n+p)(1-y)-np|b|(x-y)]}\right)z^n.$$

Then

$$\sum_{n=p}^{\infty}\frac{[n(n+p)(1-y)-np|b|(x-y)]\frac{(\frac{1}{n+p+1})^{\sigma}}{p^2|b|(x-y)}.\ p^2|b|(x-y)}{[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}=\sum_{n=p}^{\infty}\mu_n=1-\mu_{p-1} \le 1.$$

Using Theorem (2.1),we easily get $f(z) \in A_p^*(\sigma,b,x,y)$.

Conversely, let $(z) \in A_p^*(\sigma,b,x,y)$.

From the Theorem (2.1), we have

$$a_n \leq \frac{p^2|b|(x-y)(n+p+1)^\sigma}{[n(n+p)(1-y)-np|b|(x-y)]} \text{ for } n \geq p.$$

Setting

$$\mu_n = \frac{n(n+p)(1-y)-np|b|(x-y)}{p^2|b|(x-y)}.$$

$$\left(\frac{1}{n+p+1}\right)^\sigma \text{ for } n \geq p,$$

$$and\ \mu_{p-1} = 1 - \sum_{n=p}^{\infty} \mu_n.$$

Then

$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n =$
$z^{-p} + \sum_{n=p}^{\infty} \left(\frac{p^2|b|(x-y)(n+p+1)^\sigma \mu_n}{[n(n+p)(1-y)-np|b|(x-y)]}\right) z^n =$
$\mu_{p-1} z^{-p} + \sum_{n=p}^{\infty} \mu_n f_n(z) = \sum_{n=p-1}^{\infty} \mu_n f_n(z)$

This completes the proof.

5. Radius of convexity

In the following theorem, we obtain the radius of convexity for the function in the class $A_p^*(\sigma,b,x,y)$.

Theorem (5.1): Let f the function f (z) defined by (1.1) is in the class $A_p^*(\sigma,b,x,y)$. Then f is meromorphically p-valent convex of order $\lambda(0 \leq \lambda < p)$ in the disk $|z| < r_2$, where $r_2 = r_2(p,\sigma,b,x,y) =$

$$\inf_{n\geq p}\left[\frac{(p-\lambda)[(n+p)(1-y)-p|b|(x-y)]\left(\frac{1}{n+p+1}\right)^\sigma}{(n+2p-\lambda)p|b|(x-y)}\right]^{\frac{1}{n+p}} \quad (5.1)$$

The result is sharp for the function f given by (3.4).

Proof: A function f meromorphic p-valent convex of order λ $(0 \leq \lambda < p)$ if

$$-Re\left\{1 + \frac{f''(z)}{f'(z)}\right\} > \lambda.$$

We must show that

$$\left|\frac{zf''(z)}{f'(z)} + (1+p)\right| < p - \lambda, \quad \text{for} \quad |z| < r_2. \quad (5.2)$$

We have $\left|\frac{zf''(z)}{f'(z)} + (1+p)\right| = \left|\frac{zf''(z)+(1+p)f'(z)}{f'(z)}\right| = \left|\frac{\sum_{n=p}^{\infty} n(n+p)a_n z^{n+p}}{-p+\sum_{n=p}^{\infty} n\, a_n z^{n+1}}\right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum_{n=p}^{\infty} n\, a_n|z|^{n+p}}$.

Thus ,(5.2) will be satisfied if

$$\sum_{n=p}^{\infty} \frac{n(n+2p-\lambda)}{p(p-\lambda)} a_n |z|^{n+p} \leq 1. \quad (5.3)$$

Since $f \in A_p^*(\sigma,b,x,y)$, we have

$$\sum_{n=p}^{\infty} \frac{[n(n+p)(1-y)-np|b|(x-y)]\left(\frac{1}{n+p+1}\right)^\sigma}{p^2|b|(x-y)} a_n \leq 1.$$

Hence,(5.3) will be true if

$$\frac{n(n+2p-\lambda)}{p(p-\lambda)}|z|^{n+p} \leq \left[\frac{n(n+p)(1-y)-np|b|(x-y)\left(\frac{1}{n+p+1}\right)^\sigma}{p^2|b|(x-y)}\right],$$

or equivalently

$$|z| \leq \left(\frac{(p-\lambda)(n+p)(1-y)-np|b|(x-y)\left(\frac{1}{n+p+1}\right)^\sigma}{(n+2p-\lambda)p^2|b|(x-y)}\right)^{\frac{1}{n+p}}, n \geq p$$

which follows the result.

6. Convex linear combination:

Theorem (6.1): The class $A_p^*(\sigma,b,x,y)$ is closed under convex linear combinations.

Proof: Let f_1 and f_2 be the chance elements of $A_p^*(\sigma,b,x,y)$. Then for each t $(0 < t < 1)$ plus $(a_n, b_n \geq 0)$. we show that $(1-t)f_1 + tf_2 \in A_p^*(\sigma,b,x,y)$. Thus we have

$(1-t)f_1 + tf_2 = z^{-p} + \sum_{n=p}^{\infty}[(1-t)a_n + tb_n]z^n$.

Hence

$\sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|(x-y)]\left(\frac{1}{n+p+1}\right)^\sigma[(1-t)a_n + tb_n]$.

$$= (1-t)\sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|\left(\frac{x}{-y}\right)]\left(\frac{1}{n+p+1}\right)^\sigma a_n + t\sum_{n=p}^{\infty} \frac{n(n+p)(1-y)}{-np|b|(x-y)}\left(\frac{1}{n+p+1}\right)^\sigma b_n$$

$\leq (1-t)p^2|b|(x-y) + tp^2|b|(x-y)$
$= p^2|b|(x-y)$.

This completes the proof.

7. The anthmetic mean:

Theorem (7.1): Let the functions f_k sharp by $f_k(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,k}$, $(a_{n,k} \geq 0, n \in N, k = 1,2,\ldots l)$,

be in the class $A_p^*(\sigma,b,x,y)$ for each k = (1,2,3,… l), then the function h sharp by

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} e_n z^n, (e_n \geq 0, n \in N)$$

also belong to the class $A_p^*(\sigma,b,x,y)$, where $e_n = \frac{1}{l}\sum_{n=p}^{\infty} a_{n,k}$, $(n \geq p, p \in N)$.

proof: As $f_k \in A_p^*(\sigma,b,x,y)$, it follows the Theorem (2.1) that

$$\sum_{n=p}^{\infty}[n(n+p)(1-y)-np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}a_{n,k} \le p^2|b|(x-y),$$

for each k = 1,2,3,…. l Hence

$$\sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}e_n$$
$$= \sum_{n=p}^{\infty}[n(n+p)(1-y) - np|b|(x-y)](\frac{1}{n+p+1})^{\sigma}\left(\frac{1}{l}\sum_{k=1}^{l}a_{n,k}\right)$$
$$= \frac{1}{l}\sum_{k=1}^{l}(\sum_{n=p}^{\infty}[n(n+p)(1-y) - np\,|b|(x-y)\,]\left(\frac{1}{n+p+1}\right)a_{n,k})$$
$$\le \frac{1}{l}\sum_{k=1}^{l}p^2\,|b|\,(x-y)$$
$$= p^2|b|\,(x-y).$$

Then $h \in A_p^*\,(\sigma,b,x,y)$.

8. Partial sums

Theorem(8.1): Let $f \in A_p^*(\sigma,b,x,y)$ be assumed by(1.1)and $g \in A_p^*(\sigma,b,x,y)$ be assumed by

$$g(z) = z^{-p} + \sum_{n=p}^{\infty}b_n z^n.$$

We define the partial sums $S_1(z) and\ S_k(z)$ as follows :

$$S_1(z) = z^{-1} and\ \ S_k(z) = z^{-p} + \sum_{n=p}^{k-1}a_n z^n\,,(k \in N|\{1\}). \quad (8.1)$$

Also suppose that

$$\sum_{n=p}^{\infty}c_n a_n \le 1\,, c_n = \frac{[n(n+p)(1-y)-np\,|b|(x-y)](\frac{1}{n+p+1})^{\sigma}}{p^2|b|\,(x-y)}. \quad (8.2)$$

Then, we hav $Re\left\{\frac{f(z)}{S_k(z)}\right\} > 1 - \frac{1}{c_k}$,$(z \in U\,, k \in N)$, (8.3)

and $Re\left\{\frac{S_k(z)}{f(z)}\right\} > \frac{c_k}{1+c_k}$,$(z \in U\,, k \in N)$. (8.4)

Each of the bounds in (8.3) and (8.4) is the best possible for $k \in N$.

Proof: We can see from (8.2) that $c_{n+1} > c_n > 1\,, n = p, p+1, p+2, p+3, \ldots$

Therefore, we have:

$$\sum_{n=p}^{k=1}a_n + c_k\sum_{n=k}^{\infty}a_n \le \sum_{n=p}^{\infty}c_n a_n \le 1. \quad (8.5)$$

By setting

$$g_1(z) = c_k\left[\frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{c_k}\right)\right] = 1 + \frac{c_k\sum_{n=k}^{\infty}a_n z^{n+1}}{1+\sum_{n=p}^{k-1}a_n z^{n+1}}\,, \quad (8.6)$$

and applying (8.5) we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{c_k\sum_{n=k}^{\infty}a_n}{2-2\sum_{n=p}^{k-1}a_n - c_k\sum_{n=k}^{\infty}a_n}, \quad (8.7)$$

which readily yields the assertion (8.3) if ,we take

$$f(z) = z^{-p} - \frac{z^k}{c_k}. \quad (8.8)$$

Then

$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{c_k} \to 1 - \frac{1}{c_k}\ (z \to 1^-)$, which shows that the bound in (8.3) is the best possible for $k \in N$.

Similarly, if we put

$$g_2(z) = (1+c_k)\left[\frac{S_k(z)}{f(z)} - \frac{c_k}{1+c_k}\right] = 1 - \frac{(1+c_k)\sum_{n=k}^{\infty}a_n z^{n+1}}{1+\sum_{n=p}^{k-1}a_n z^{k+1}}, \quad (8.9)$$

and make use of (8.9), we have

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+c)\sum_{n=k}^{\infty}a_n}{2-2\sum_{n=p}^{k-1}a_n + (1-c_k)\sum_{n=k}^{\infty}a_n}, \quad (8.10)$$

which leads us to the assertion (8.4). The bound (8.5) is sharp for each $k \in N$ with the function given by (6.7). The proof of the theorem is complete.

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