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# On Differential Sandwich Theorems of Multivalent Functions Defined by a Linear operar

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**Abstract:**

The main object of the present paper is to derive some results for multivalent analytic functions defined by linear operator by using differential subordination and superordination

**Keywords:** Analytic functions, multivalent functions, Hadamard product, subordination, linear operators.

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## 1. Introduction

Let $A_p$ denote the class of functions $f$ of the form:

$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, $(p \in \mathbb{N} = \{1,2, \dots\};\ z\, U)$, (1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

For two functions $f$ and $g$ are analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, written $f \prec g$, if there exists Schwarz function $w$, analytic in $U$ with $w(0) = 0$ *and* $|w(z)| < 1$ *in* $U$ *such that* $f(z) = g(w(z)), z \in U$ If $g$ is univalent and $g(0) = f(0)$, then $f(u) \subset g(u)$ .

If $f \in A_p$ is given by (1.1) and $g \in A_p$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}.$$

Then Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, b_{p+n} z^{p+n} \quad .$$

The linear operator $J_{\mu,v}^{\lambda,p}(a,c): A_p \to A_p$ defined by

$J_{\mu,v}^{\lambda,p}(a,c)f(z) = \emptyset_{\mu,v}^{\lambda,p}(a,c;z) * f(z), \quad (f \in A_p, z \in U)$, (1.2)

where

$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n (p+1)_n (p+1-\mu+v)_n}{(c)_n (p+1-\mu)_n} z^{p+n}$ (1.3)

and

$$d_n = \begin{cases} 1 & n = 0 \\ d(d+1)(d+2)\dots(d+n-1) & n \in N \end{cases} .$$

For $a \in R$ , $c \in R \setminus z_\circ^-$ , *where* $z_\circ^- = \{0,-1,-2,\dots\}$ ,$0 \le \lambda < 1$ ,$\mu, v \in R$ *and* $\mu - v - p < 1$ *and* $f \in A_p$ .Then linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c): A_p \longrightarrow A_p$ (*see*[9]) is defined by

$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) \coloneqq \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) * f(z)$, (1.4)

where $\psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z)$ is the function defined in terms of the Hadamard product by the following condition:

$\emptyset_{\mu,v}^{\lambda,p}(a,c;z) * \psi_{\mu,v}^{\lambda,p,\alpha}(a,c;z) = \frac{z^p}{(1-z)^{a+p}} \quad (a > -p)$. (1.5)

We can easily find from (1.3) - (1.5) that

$I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(c)_n (p+1-\lambda+v)_n (\alpha+p)_n (p+1-\mu)_n}{(a)_n (p+1)_n (p+1-\mu+v)_n n!} a_{p+n} z^{p+n}$ (1.6)

It is easily verified from (1.6) that

$\boldsymbol{z(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z))' = (\alpha+p) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) - \alpha I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z).}$ ***(1.7)***

Note that the linear operator $I_{\mu,v}^{\lambda,p,\alpha}(a,c)$ unifies many other operators considered earlier. In particular

1) $I_{0,v}^{0,p,\alpha}(a,c) \equiv J_p^a(a,c)$ (*see Cho al.*[5]).
2) $I_{0,v}^{0,p,\alpha}(a,a) \equiv D^{\alpha+p-1}$ (*see Goel and Sohi*[6]).
3) $I_{0,v}^{0,p,1}(p+1-\lambda,1) \equiv \Omega_Z^{(\lambda,P)}$ (*see Srivastava and Aouf*[16]).
4) $I_{0,v}^{0,p,\alpha-1}(a,c) \equiv J_p^{a,\alpha}$(*see Hohlov*[8]).
5) $I_{0,v}^{0,1-\alpha,\alpha}(a,c) \equiv L_P(a,c)$ (*see Saition*[13] ).
6) $I_{0,v}^{0,p,1}(p+\alpha,1) \equiv J_{\alpha,P,\alpha} \in z, \alpha > -p$ (*see Liu an Noor*[10]).

The main object of this idea is to find sufficient conditions for certain normalized analytic functions $\boldsymbol{f}$ to satisfy:

$$\boldsymbol{q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q_2(z),}$$

and

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q_2(z) \ ,$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $U$ with $q_1(0)$and $q_2(0) = 1$.

## 2- Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas .

**Definition 2.1. [11]:** Denote by $Q$ the set of all functions $q$ that are analytic and injective on $\overline{U} \setminus E(q)$, where

$\overline{U} = U \cup \{z \in \partial U\}$, and

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\} \quad (2.1)$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U / E(q)$ .

Further ,let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a), Q(0) \equiv Q_0$ *and* $Q(1) \equiv Q_1$ .

**Lemma 2.1.[1]:** Let $q(z)$ be convex univalent function in $U$, let $\alpha \in \mathbb{C}. \beta \in \mathbb{C} \setminus \{0\}$ and suppose that

$$Re(1 + \frac{zq''(z)}{q'(z)}) > \max\{0, -Re(\frac{\alpha}{\beta})\} .$$

If $p(z)$ is analytic in $U$ and

$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z)$,

then $p(z) \prec q(z)$ and $q$ is the best dominant.

Lemma 2.2. [3]: Let $\boldsymbol{q}$ *be univalent in* $\boldsymbol{U}$*and let* $\emptyset$ *and* $\boldsymbol{\theta}$*be analytic in the domain* $\boldsymbol{D}$*containing* $\boldsymbol{q(U)}$ *with* $\emptyset(\boldsymbol{w}) \neq \boldsymbol{0}$, *when* $\boldsymbol{w} \in \boldsymbol{q(U)}$.

Set $Q(z) = zq'(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$, suppose that

1- $Q$ is starlike univalent in $U$,
2- $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, \quad z \in U.$

If $p$ is analytic in $U$ with $p(0) = q(0), p(U) \subseteq D$ and

$\emptyset(p(z)) + zp'(z)\emptyset(p(z)) \prec \emptyset(q(z)) + zq'(z)\emptyset(q(z))$,

then $p \prec q$ , and $q$ is the best dominant .

**Lemma 2.3.[12]:** Let $q(z)$ be convex univalent in the unit disk $U$ and let $\theta$ and $\emptyset$ be analytic in a domain $D$ containing $q(U)$. Suppose that

$1 - Re\{\frac{\theta'(q(z))}{\emptyset(q(z))}\} > 0$ for $z \in U$,

$2 - zq'(z)\emptyset(q(z))$ is starlike univalent in $z \in U$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\emptyset(p(z))$ is univalent in $U$ , and

$$\theta(q(z)) + zq'(z)\emptyset(q(z)) \prec \theta(p(z)) + zp'(z)\emptyset(p(z)), \quad (2.2)$$

then $q \prec p$ , and $q$ is the best subordinant.

**Lemma 2.4.[12]:**Let $q(z)$ be convex univalent in $U$ and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\mathrm{Re}(\beta) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \beta zp'(z)$ is univalent in $U$ , then

$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z)$,

which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

## 3-Subordination Results

**Theorem 3.1.**Let $q(z)$ be convex univalent in $U$ with $q(0) = 1, \eta, \delta \in \mathbb{C} \setminus \{0\}$. Suppose that

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\delta}{\eta}\right)\right\}. \quad (3.1)$$

If $f \in W$ is satisfies the subordination

$$G(z) \prec q(z) + \frac{\eta}{\delta} zq'(z), \quad (3.2)$$

where

$$G(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \times \left(1 + \eta\left(\frac{(pt_2 - t_2\alpha) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z) + (t_2 - t_1\alpha + t_2 p - pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} \frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + (t_1\alpha - t_1 p) I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right), \quad (3.3)$$

then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

**Proof:** Define a function $k(z)$ by

$$k(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}, \quad (3.5)$$

then the function $k(z)$ is analytic in $U$and $q(0) = 1$, therefore,differentiating (3.5) logarithmically with respect to $z$ and using the identity (1.7) in the resulting equation,

$$G(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \times \left(1 + \eta\left(\frac{(pt_2 - t_2\alpha) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)(z) + (t_2 - t_1\alpha + t_2 p - pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} \frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + (t_1\alpha - t_1 p) I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$

Thus the subordination (3.2 ) is equivalent to

$$k(z) + \frac{\eta}{\delta} zk'(z) \prec q(z) + \frac{\eta}{\delta} zq'(z).$$

An application of Lemma (2.1) with $\beta = \frac{\eta}{\delta}$ and $\alpha = 1$ , we obtain (3.4).

Taking $q(z) = \frac{1+Az}{1+Bz}$ ,$(-1 \leq B < A \leq 1)$, in Theorem (3.1), we obtain the following Corollary.

**Corollary 3.1.** Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$*and* $(-1 \leq B < A \leq 1)$.Suppose that

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\delta}{\eta}\right)\right\}.$$

If $f \in W$ is satisfy the following subordination condition:

$$G(z) \prec \frac{1 + Az}{1 + Bz} + \frac{\eta}{\delta} \frac{(A - B)z}{(1 + Bz)^2} ,$$

where $G(z)$ given by (3.3) , then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec \frac{1+Az}{1+Bz} ,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant .

Taking $A = 1$ *and* $B = -1$ in Corollary (3.1), we get following result.

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**Corollary 3.2.** Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$ and suppose that

$Re\left(\frac{1+z}{1-z}\right) > \max\{0, -Re\left(\frac{\delta}{\eta}\right)\}$.

If $f \in W$ is satisfy the following subordination

$$G(z) \prec \frac{1+z}{1-z} + \frac{\eta}{\delta}\frac{2z}{(1-z)^2},$$

where

$G(z)$ given by (3.3), then

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

**Theorem 3.2.** Let $q(z)$ be convex univalent in unit disk $U$ with $q(0) = 1$, let $\eta, \delta \in \mathbb{C}\{0\}, \gamma, t, \psi, \tau \in \mathbb{C}, f \in W$, and suppose that $f$ and $q$ satisfy the following conditions:

$$Re\left\{\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \quad (3.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0. \quad (3.7)$$

If $r(z) \prec t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)}$, (3.8)

where

$$r(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + t\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right) + t + s_{\delta}(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right), \quad (3.9)$$

then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec q(z)$, and $q(z)$ is best dominant .

**Proof :** Define analytic function $k(z)$ by

$$k(z) = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}. \quad (3.10)$$

Then the function $k(z)$ is analytic in $U$ and $g(0) = 1$,

differentiating (3.10) logarithmically with respect to $z$, we get

$$\frac{zk'(z)}{k(z)} = \delta(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right). \quad (3.11)$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$.

Also , if we let

$$\phi(z) = zq'(z)\phi(q(z)) = s\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)},$$

we find $Q(z)$ is starlike univalent in $U$, we have

$$h'(z) = \psi q'(z) + 2\tau\gamma q(z)q'(z) + s\frac{q'(z)}{q(z)} + sz\frac{q''(z)}{q(z)} - sz\left(\frac{q'(z)}{q(z)}\right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)},$$

hence that

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\psi}{s}q(z) + \frac{2\tau\gamma}{s}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0.$$

By using (3.11), we obtain

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} = \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\left(\psi + \tau\gamma\gamma\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta}\right) + t + \left(s_{\delta}(\alpha+p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right)\right).$$

By using (3.8), we have

$$\psi k(z) + \tau\gamma k^2(z) + s\frac{zk'(z)}{k(z)} \prec \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)} \quad (3.8)$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $k(z) \prec q(z)$ and the function $q(z)$ is the best dominant .

Taking the function $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, in Theorem (3.2) , the condition (3.6) becomes.

$$Re\left(\frac{\psi}{s}\frac{1+Az}{1+Bz} + \frac{2\tau\gamma}{s}\left(\frac{1+Az}{1+Bz}\right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0, \quad (3.12)$$

hence, we have the following Corollary.

**Corollary 3.3.** Let $(-1 \le B < A \le 1), s, \delta \in \mathbb{C} \setminus \{0\}, \gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.12) holds.

If $f \in W$ and

$$r(z) \prec t + \psi\frac{1+Az}{1+Bz} + \tau\gamma\left(\frac{1+Az}{1+Bz}\right)^2 + s\frac{(A-B)z}{(1+Bz)(1+Az)},$$

where $r(z)$ is defined in (3.9), then

$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is best dominant .

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$ $(0 < p \le 1)$, in Theorem (3.2), the condition (3.6) becomes

$$Re\left\{\frac{\psi}{s}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\tau\gamma}{s}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{2z^2}{1-z^2}\right\} 0, (s \in \mathbb{C} \setminus \{0\}), \quad (3.13)$$

hence ,we have the following Corollary .

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**Corollary3.4.** Let $0 < \rho \le 1$ ,$S, \delta \in \mathbb{C} \setminus \{0\}$ $\gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.13) holds. If $f \in W$ and

$$r(z) \prec t + \psi\left(\frac{1+z}{1-z}\right)^{\rho} + \tau\gamma\left(\frac{1+z}{1-z}\right)^{2\rho} + s\frac{2\rho z}{1-z^2},$$

where $r(z)$ is defined in (3.9), then

$$\left(\frac{I_{M,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\rho}, \text{ and } \left(\frac{1+z}{1-z}\right)^{\rho}$$

is the best dominant.

## 4-Superordination Results

**Theorem 4.1.** Let $q(z)$ be convex univalent $U$with $q(0) = 1, \delta \in \mathbb{C} \setminus \{0\}, Re\{\eta\} > 0,$ if $f \in W$ ,such that

$$\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p} \neq 0$$

and

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \mathcal{H}[q(0),1] \cap Q. \qquad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z) + \frac{\eta}{\delta} z q'(z) \prec G(z), \qquad (4.2)$$

holds , then

$$q(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \qquad (4.3)$$

and $q(z)$ is the best subordinant.

**Proof:** Define a function $k(z)$ by

$$k(z) = \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta}. \qquad (4.4)$$

Differentiating (4.4) with respect to $z$ logarithmically, we get.

$$\frac{z\acute{k}(z)}{k(z)} = \delta\left(\frac{t_1\left(z\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right)'\right) + t_2\left(z\left(I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)\right)'\right) -}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right) + t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)} \frac{pt_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + pt_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{t_1\left(I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)\right) + t_2\left(I_{\mu,V}^{\lambda,P,\alpha}(a,c)f(z)\right)}\right) \qquad (4.5)$$

A simple computation and using (1.7) from (4.5), we get

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \times \left(1 + \eta\left(\frac{(pt_2 - \alpha t_2) I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z) + (t_2 - \alpha t_1 + pt_2 - pt_1)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} \frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + (\alpha t_1 + pt_1) I_{\mu,v}^{\lambda,p,\alpha+2}(a,c)f(z)}{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}\right)\right)$$

$$= k(z) + \frac{\eta}{\delta} z k'(z),$$

now , by using Lemma(2.4), we get the desired result .

Taking $q(z) = \frac{1+Az}{1+Bz}(-1 \le B < A \le 1)$, in Theorem (4.1),we get the following Corollary.

**Corollary 4.2.** *Let* $Re\{\eta\} > 0, \delta \in \mathbb{C} \setminus \{0\}$ and $-1 \le B < A \le 1,$ such that

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q.$$

.

If the function $G(z)$ given by (3.3) is univalent in $U$ and $f \in W$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz} + \frac{\eta}{\delta}\frac{(A-B)Z}{(1+BZ)^2} \prec G(z),$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z) + t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^{\delta},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subordinant .

**Theorem 4.2.** Let $q(z)$ be convex univalent in unit disk $U$, Let $\delta, s \in \mathbb{C} \setminus \{0\}. \gamma, t, \psi, \tau \in \mathbb{C}, q(z) \neq 0,$ *and* $f \in W$. Suppose that

$$Re\left\{\frac{q(z)}{s}(2\tau\gamma q(z) + \psi)\right\} q'(z) > 0,$$

*and satisfies the next conditions*

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q, \qquad (4.6)$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0.$$

If the function $r(z)$ is given by (3.9) is univalent in $U$,

$$t + \psi q(z) + \tau\gamma q^2(z) + s\frac{zq'(z)}{q(z)} \prec \mathrm{r}(z) \qquad (4.7)$$

implies

$$q(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^{\delta},$$

*and* $q(z)$ is the best subordinant.

**Proof:** Let the function $k(z)$ defined on $U$ by (3.14).

Then a computation show that

$$\frac{zk'(z)}{k(z)} = \delta(\alpha + p)\left(\frac{I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)}{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)} - 1\right), \qquad (4.8)$$

by setting $\theta(w) = t + \psi\omega + \tau\gamma\omega^2$ *and* $\emptyset(w) = \frac{s}{\omega}$,it can be easily observed that$\theta(w)$is analytic in $\mathbb{C}$ ,$\phi(w)$is analytic in $\mathbb{C} \setminus \{0\}$ *and that* $\emptyset(w) \neq 0$ $(W \in \mathbb{C} \setminus \{0\})$.

Also , we get $Q(z) = zq'(z)\emptyset(q(z)) = s\frac{zq'(z)}{q(z)}$, it observed that $Q(z)$ is starlike univalent in $U$ .

Since $q(z)$ is convex , it follows that

$$Re\left(\frac{z\theta'(q(z))}{\phi(q(z))}\right) = Re\{\frac{q(z)}{s}\left(2\tau\gamma q(z)\right) + \psi\}\acute{q}(z) > 0.$$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$$\theta\left(q(z) + zq'(z)\phi(q(z))\right) = \theta\left(k(z) + zk'(z)\phi(k(z))\right),$$

thus , by applying Lemma (2.3), the proof is completed.

## 5.Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich Theorem.

**Theorem 5.1.** Let $q_1$ and $q_2$ be convex univalent in $U$ with $q_1(0) = q_2(0) = 1$ and $q_2$ satisfies (3.1). Suppose that $Re\{\eta\} > 0, \eta, \delta \in \mathbb{C} \setminus \{0\}$.
If $f \in W$, such that

$$\left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^\delta \in \mathcal{H}[q(0),1] \cap Q,$$

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \frac{\eta}{\delta}zq_1'(z) \prec G(z) \prec q_2(z) + \frac{\eta}{\delta}zq_2'(z), \quad (5.1)$$

then

$$q_1(z) \prec \left(\frac{t_1 I_{\mu,v}^{\lambda,p,\alpha+1}(a,c)f(z)+t_2 I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{(t_1+t_2)z^p}\right)^\delta \prec q_2(z),$$

where $q_1$ and $q_2$ are respectively , the subordinant and the best dominant of (5.1).
Combining Theorem (3.2) with Theorem (4.2), we obtain the following sandwich Theorem.

**Theorem 5.2.** Let $q_i$ be two convex univalent functions in $U$ , such that $q_i(0) = 1$, $q_i(0) \neq 0$ (i=1,2).Suppose that $q_1$ and $q_2$satisfies (3.8) and (4.8), respectively.
If $f \in W$ and suppose that $f$ satisfies the next conditions:

$$\left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^\delta \in \mathcal{H}[Q(0),1] \cap Q,$$

and

$$\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p} \neq 0,$$

and $r(z)$ is univalent in $U$ , then

$$t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)} \prec t + \psi q_1(z) + \tau\gamma q_1^2(z) + s\frac{zq_1'(z)}{q_1(z)},$$

implies

$$q_1(z) \prec \left(\frac{I_{\mu,v}^{\lambda,p,\alpha}(a,c)f(z)}{z^p}\right)^\delta \prec q_2(z),$$

and $q_1$and $q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

## References:

[1]R. M. Ali, V.Ravich and , K. G. subramanian, Differential sandwich theorem for certain analytic functions, for East J.Math. Sci.,(2004), 15,87-94.
[2]R. M. Ali, V.Ravichandran and N.Seenivasagan, on subordination and superordination of the multiplier transformation for meromorphic functions, Bull. Malays . Math.Sci.Soc., 33 (2010),311-324.
[3] T. Bulbocaco, Differential Subordinations and Superordinations, Recent Results, House of Scientific Bookpubl.,Cluj-napoca, (2005),
[4]T. Bulboacă, Classes of first order differential superordinations, Demonstration math., 35(2) (2002), 287-292 .
[5]N.E. Cho, O.H. Kwon and H.M. Srivastava , Inclusion and argument Properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal . Appl., 292(2004), 470-483.
[6]R.M. Goel and N.S. sohi, Anew criterion for P-valent functions, Proc. Amer .Math. Soc., 78(1980), 535-357.
[7]S.P. Goyal, P. Goswami and H. Silverman, Subordination and superordivation results for a class of analytic multivalent functions, Int. J. Math. Sci., Article ID 561638, 1-12, (2008).
[8] Yu. E. Hohlov, Operators and operations in the class of univalent functions, Izv, Vvssh.
Ucebn. Zaved. Math., 10(1987), 83-89.
[9]R.Kargar , A.Bilavi, S. Abdolahi and S.Maroufi, A class of multivalent analytic functions defined by anew linear operator , J. Math. Com P.Sci.,8 (2014),326-334.
[10]J.L.Liu and k.l.Noor, some properties of Noor integral operator , J.Natur. Geom., 21(2002), 81-90.
[11]S. S. Miller and P.T. Mocanu , Differential subordinations :Theory and Applications, Series on Monographs and Text Books in Pure and Applied Mathematics, 225, Marcel Dekker, NewYork and Basel, (2000).
[12]S. S. Miller and P.T. Mocanu , Subordinations of differential superodinations , complex variables, 48(10) (2003) , 815-826 .
[13]H.Saition, A linear operator and its applications of first order differential subordinations, Math. Japon., 44(1996), 31-38.

[14]T.N. Shanmugam, S. Shivasubramaniam and H. Silverrman, On sandwich theorems for classes of analytic functions, Int. J. Math. Sci.,Article ID29684(2006).,1-13.

[15]T.N.Shanmugam, V.Rvichandran and S.Sivasubramanian, Differential sandwich theorems for subclasses of analytic functions , Aust.J.Math.Anal. App1.,3, Article 8(2006).,1.11.

[16]H.M. Srivastava and M.K.Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients , Int. J. Math. App1., 171(1992),1-13.

[17]N. Tuneski, On certain sufficient conditions for starlikeness, Internat. J.Math.Sci.23(8) (2000),521-527.

# على نظريات الساندويتش التفاضلية من وظائف متعددة التكافؤ المحددة من قبل المشغل الخطي

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**المستخلص :**
ا لهدف الرئيسي من هذا البحث هو استخلاص بعض النتائج للوظائف التحليلية متعددة التكافؤ التي يحددها المشغل الخطي باستخدام التبعية التفاضلية والإخضاع .