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Approximaitly Quasi-Prime Submodules and Some Related Concepts

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Abstract:

"Let R be a commutative ring with identity and B is a left unitary R-module. A proper submodule E of B is called a quasi-prime submodule, if whenever $rsb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E$ or $sb \in E$". As a generalization of a quasi-prime submodules, in this paper we introduce the concept of approximaitly quasi-prime submodules, where a proper submodule E of B is an approximaitly quasi-prime submodule, if whenever $rsb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$, where $soc(B)$ is the intersection of all essential submodules of B. Many basic properties, characterization and examples of this concept are given. Furthermore, we study the behavior of approximaitly quasi-prime submodules under R-homomorphisms. Finally, we introduced characterizations of approximaitly quasi-prime submodule in class of multiplication modules.

Keywords. Prime submodules, Quasi-prime submodules, Approximaitly prime submodules, Socle of submodules, Multiplication Modules, Approximaitly quasi-prime submodules.

1. Introduction

A quasi-prime submodule was introduced and studied in 1999 by [1] as a generalization of a prime submodule, where a proper submodule E of an R-module B is called a prime, if whenever $rb \in E$, where $r \in R$, $b \in B$ implies that either $b \in E$ or $r \in [E:_R B]$ where $[E:_R B] = \{r \in R: rB \subseteq E\}$ [3], Recently several generalizations of quasi-prime submodules were introduced such as "Weakly quasi-prime, Nearly quasi-prime, WE-quasi-prime, Weakly quasi 2-absorbing, Nearly quasi 2-absorbing, and Pseudo quasi 2-absorbing submodules see [14,12,6,7,8,9]". In this paper, we give another generalization of a quasi-prime submodule, where a proper submodule E of B is an approximaitly quasi-prime submodule, if whenever $rsb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. The concept of approximaitly quasi-prime submodule is also, generalization of the concept of approximaitly prime submodules which appear in [10], also generalization of prime submodules. Recall that an R-module B is multiplication if every submodule E of B is of the form $E = IB$ for some ideal I of R, in particular $E = [E:_R B]B$ [4]. Let E and D be a submodule of a multiplication R-module B with $E = IB$ and $D = JB$ for some ideals I and J of R, then $ED = IJB$ that is $ED = ID$. In particular $EB = IBB = IB = E$. Also for any $b \in B$, we define $Eb = E\langle b\rangle = Ib$ [15].

2. Basic properties of Approximaitly Quasi-Prime Submodules

In this part of the paper we introduce the definition of approximaitly quasi-prime submodule, and give some basic properties, examples and characterizations of this concept.

Definition 2.1 A proper submodule E of B is said to be an approximaitly quasi-prime submodule(for short app-quasi-prime), if whenever $rsb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. An ideal I of a ring R is an app-quasi-prime ideal of R if and only if I is an app-quasi-prime submodule of an R-module R.

Remark 2.2 It is clear that every quasi-prime submodule is an app-quasi-prime, but the convers is not true in general, the following example explain that:

Example 2.3 Let $B = Z_{12}$, $R = Z$ and $E = \langle\bar{0}\rangle$, and $soc(Z_{12}) = \langle\bar{2}\rangle$. E is an app-quasi-prime submodule of B since if $rsb \in E$, where $r, s \in Z$, $b \in Z_{12}$, implies that either $rb \in E + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $sb \in E + \langle\bar{2}\rangle = \langle\bar{2}\rangle$. But E is not a quasi-prime submodule of B, since $3.4.\bar{2} \in E$, but neither $3.\bar{2} \in E$ nor $4.\bar{2} \in E$.

The following proposition gives a characterization of app-quasi-prime submodules.

Proposition 2.4 Let B be an R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $IJD \subseteq E$, where I, J are ideals in R and D is a submodule of B, implies that either $ID \subseteq E + soc(B)$ or $JD \subseteq E + soc(B)$.

Proof ($\Longrightarrow$) Suppose that $IJD \subseteq E$, where I, J are ideals in R and D is a submodule of B, and with $ID \nsubseteq E + soc(B)$ and $JD \nsubseteq E + soc(B)$. So there exists $d_1, d_2 \in D$ and $r \in I$, $s \in J$ such that $rd_1 \notin E + soc(B)$ and$sd_2 \notin E + soc(B)$. Since E is an app-quasi-prime submodule of B and $rsd_1 \in E$ and $rd_1 \notin E + soc(B)$ implies that $sd_1 \in E + soc(B)$. Also $rsd_2 \in E$ and $sd_2 \notin E + soc(B)$ implies that $rd_2 \in E + soc(B)$. It follows that either $ID \subseteq E + soc(B)$ or $JD \subseteq E + soc(B)$.

($\Longleftarrow$) Assume that $rsb \in E$, where $r, s \in R$, $b \in B$ implies that $(r)(s)(b) \subseteq E$, so by hypothesis either $(r)(b) \subseteq E + soc(B)$ or $(s)(b) \subseteq E + soc(B)$. Thus either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Hence E is an app-quasi-prime submodule of B.

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As direct application of proposition 2.4 , we get the following corollaries.

Corollary 2.5 Let B be an R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $rsD \subseteq E$, where $r,s \in R$ and D is a submodule of B, implies that either $rD \subseteq E + soc(B)$ or $sD \subseteq E + soc(B)$.

Corollary 2.6 Let B be an R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $rIb \subseteq E$, where $r \in R$, $b \in B$ and I is a ideal of R, implies that either $rb \in E + soc(B)$ or $Ib \subseteq E + soc(B)$.

Corollary 2.7 Let B be an R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $IJb \subseteq E$, where I,J are ideals in R, and $b \in B$, implies that either $Ib \subseteq E + soc(B)$ or $Jb \subseteq E + soc(B)$.

Proposition 2.8 Let B be an R-module, and E be a proper submodule of B with $soc(B) \subseteq E$. Then E is an app-quasi-prime submodule of B if and only if $[E + soc(B): b]$ is a prime ideal of R for each $b \in B$.

Proof ($\Longrightarrow$) Let $rs \in [E + soc(B): b]$, where $r,s \in R$, implies that $rsb \in E + soc(B)$. But $soc(B) \subseteq E$, it follows that $E + soc(B) = E$, hence $rsb \in E$. But E is an app-quasi-prime submodule of B, implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Thus either $r \in [E + soc(B): b]$ or $s \in [E + soc(B): b]$.

($\Longleftarrow$) Suppose that $rsb \in E$, where $r,s \in R$, $b \in B$ implies that either $rbs \in E \subseteq E + soc(B)$, it follows that $rbs \in E + soc(B)$, hence $rs \in [E + soc(B): b]$. But $[E + soc(B): b]$ is a prime ideal of R, implies that either $r \in [E + soc(B): b]$ or $s \in [E + soc(B): b]$, it follows that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$.

Proposition 2.9 Let B be an R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if $[E:_B rs] \subsetneq [E + soc(B):_B r] \cup [E + soc(B):_B s]$ for all $r,s \in R$.

Proof ($\Longrightarrow$) Let $b \in [E:_B rs]$, implies that $rsb \in E$. But E is an app-quasi-prime submodule of B, then either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. It follows that either $b \in [E + soc(B):_B r]$ or $b \in [E + soc(B):_B s]$. Thus $[E:_B rs] \subsetneq [E + soc(B):_B r] \cup [E + soc(B):_B s]$

($\Longleftarrow$) Now, let $rsb \in E$, where $r,s \in R$, $b \in B$, then $b \in [E:_B rs] \subsetneq [E + soc(B):_B r] \cup [E + soc(B):_B s]$, implies that $b \in [E + soc(B):_B r]$ or $b \in [E + soc(B):_B s]$. Hence $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Thus E is an app-quasi-prime submodule of B.

Proposition 2.10 Let B be an R-module, and E be a proper submodule of B such that E is an app-quasi-prime submodule of B. Then $[E:_R rsb] \subsetneq [E + soc(B):_R rb] \cup [E + soc(B):_R sb]$ for all $r,s \in R$, $b \in B$.

Proof Let $x \in [E:_R rsb]$, where $r,s \in R$, $b \in B$, implies that $rs(xb) \in E$. But E is an app-quasi-prime submodule of B, then either $r(xb) \in E + soc(B)$ or $s(xb) \in E + soc(B)$, it follows that either $x \in [E + soc(B):_R rb]$ or $x \in [E + soc(B):_R sb]$. Hence $x \in [E + soc(B):_R rb] \cup [E + soc(B):_R sb]$. Thus $[E:_R rsb] \subsetneq [E + soc(B):_R rb] \cup [E + soc(B):_R sb]$.

Remark 2.11 Let B be an R-module, and E is an app-quasi-prime submodule of B, it is not necessary that $[E: B]$ is an app-quasi-prime ideal of R. For example in a Z-module Z_{12} , $\langle \bar{0} \rangle$ is an app-quasi-prime submodule, but $[\langle \bar{0} \rangle: Z_{12}] = 12Z$ is not app-quasi-prime ideal of Z-module Z. Since $2.2.3 \in 12Z$, but $2.3 \notin 12Z + soc(Z) = 12Z + 0 = 12Z$.

Proposition 2.12 Let B be an R-module, and E is an app-quasi-prime submodule of B with $soc(B) \subseteq E$. Then $[E: B]$ is an app-quasi-prime ideal of R.

Proof Let $rst \in [E: B]$, where $r, s, t \in R$, implies that $rs(tB) \subseteq E$. Thus since E is an app-quasi-prime submodule of B, then by Corollary 2.5 either $r(tB) \subseteq E + soc(B)$ or $s(tB) \subseteq E + soc(B)$. But $soc(B) \subseteq E$, implies that $E + soc(B) = E$. Hence either $r(tB) \subseteq E$ or $s(tB) \subseteq E$. That is either $rt \in \{E: B] \subseteq [E: B] + soc(R)$ or $st \in \{E: B] \subseteq [E: B] + soc(R)$. Therefore $[E: B]$ is an app-quasi-prime ideal of R.

Recall that an R-module B is faithful if $ann_R B = (0)$ [4].

Before we introduce the converse of Proposition 2.12 we recall the following lemmas:

Lemma 2.13 [4, Coro. 2.14] Let B be a faithful multiplication R-module then $soc(R)B = soc(B)$.

Recall that an R-module B is a non-singular provided that $Z(B) = B$, where $Z(B) = \{b \in B : bI = 0$ for some essential ideal I of $R\}$ [5].

Lemma 2.14 [5, Coro. 1.26] If B is a non-singular R-module, then $soc(R)B = soc(B)$.

Proposition 2.15 Let B be a faithful multiplication R-module and E is a proper submodule of B. If $[E: B]$ is an app-quasi-prime ideal of R, then E is an app-quasi-prime submodule of B.

Proof Suppose that $rsb \in E$, where $r, s \in R$, $b \in B$ implies that $rs(b) \subseteq E$. Since B be a multiplication, then $(b) = JB$ for some ideal J of R. That is $rsJB \subseteq E$, it follows that $rsJ \subseteq [E: B]$. But $[E: B]$ is an app-quasi-prime ideal of R, then by Corollary 2.5 either $rJ \subseteq [E: B] + soc(R)$ or $sJ \subseteq [E: B] + soc(R)$, it follows that either $rJB \subseteq [E: B]B + soc(R)B$ or $sJB \subseteq [E: B]B + soc(R)B$. Hence, by Lemma 2.13 $soc(R)B = soc(B)$, and since B is multiplication $[E: B]B = E$. therefore either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Hence E is an app-quasi-prime submodule of B.

Proposition 2.16 Let B be a non-singular multiplication R-module and E is a proper submodule of B. If $[E: B]$ is an app-quasi-prime ideal of R, then E is an app-quasi-prime submodule of B.

Proof Follows in similar way of Proposition 2.15 and using Lemma 2.14 .

Lemma 2.17 [13, Coro. Of Theo. 9] Let I and J be ideals of a ring R, and B is a finitely generated multiplication R-module. Then $IB \subseteq JB$ if and only if $I \subseteq J + annB$.

Proposition 2.18 Let B be a faithful finitely generated multiplication R-module. If J is an app-quasi-prime ideal of R, then JB is an app-quasi-prime submodule of B.

Proof Suppose that $rsb \in JB$, where $r, s \in R$, $b \in B$. Then $rs(b) \subseteq JB$. Since B is a multiplication, implies that $(b) = IB$ for some ideal I of R. Thus $rsIB \subseteq JB$. But B is a finitely generated, so by Lemma 2.17 $rsI \subseteq J + annB$. But B is faithful, it follows that $annB = (0)$, hence $rsI \subseteq J$, but J is an app-quasi-prime ideal of R, then by Corollary 2.5 either $rI \subseteq J + soc\ (R)$ or $sI \subseteq J + soc(R)$, it follows that either $rIB \subseteq JB + soc\ (R)B$ or $sIB \subseteq JB + soc(R)B$. But by Lemma 2.13 $soc(R)B = sov(B)$. Hence $rIB \subseteq JB + soc\ (B)$ or $sIB \subseteq JB + soc(B)$, it follows that either $rb \in JB + soc\ (B)$ or $sb \in JB + soc(B)$. Therefor JB is an app-quasi-prime submodule of B.

Proposition 2.19 Let B be a finitely generated multiplication non-singular R-module, and J is an app-quasi-prime ideal of R with $annB \subseteq J$. Then JB is an app-quasi-prime submodule of B.

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Proof Similar steps of Proposition 2.18 and using Lemma 2.14 and the condition $annB \subseteq J$ implies that $J + annB = J$.

Remark 2.20 The intersection of any two app-quasi-prime submodules of an R-module B not necessary app-quasi-prime submodule of B, the following example shows that.

Example 2.21 Let B be the Z-module Z and $E = 2Z$ and $D = 3Z$. It is clear that E and D are app-quasi-prime submodules of B, but $E \cap D = 6Z$ is not app-quasi-ptime submodule of B since $2.3.1 \in Z$, but $2.1 \notin 6Z + soc(B)$ and $3.1 \notin 6Z + soc(B)$, where $soc(B) = (0)$.

Proposition 2.22 Let B be an R-module, and E, D are app-quasi-prime submodules with $E \subseteq soc(B)$ and $D \subseteq soc(B)$. Then $E \cap D$ is an app-quasi-prime submodule of B.

Proof Suppose that $rsb \in E \cap D$, where $r, s \in R$, $b \in B$, then $rsb \in E$ and $rsb \in D$. Since both E and D are app-quasi-prime submodules of B, so either $rb \in E + soc\ (B)$ or $sb \in E + soc(B)$ and either $rb \in D + soc\ (B)$ or $sb \in D + soc(B)$. But $E \subseteq soc(B)$ and $D \subseteq soc(B)$, it follows that $E + soc(B) = soc(B)$ and $D + soc(B) = soc(B)$ and $E \cap D \subseteq soc(B)$, implies that $E \cap D + soc(B) = soc(B)$. Thus we have either $rb \in E \cap D + soc\ (B)$ or $sb \in E \cap D + soc(B)$. That is $E \cap D$ is an app-quasi-prime submodule of B.

Lemma 2.23 [11, Lemma 2.3.15] Let B be an R-module, and E, D, F are submodules of B with D is contained in F then $(E + D) \cap F = (E \cap F) + (D \cap F) = (E \cap F) + D$.

Lemma 2.24 [2, Coro. 9.9] Let B be an R-module, and E submodule of B, then $soc(E) = E \cap soc(B)$.

Proposition 2.25 Let B be an R-module, and E, D are two submodules of B with D is not contained in E and $soc(B) \subseteq D$. If E is an app-quasi-prime submodule of B, then $E \cap D$ is an app-quasi-prime submodule of D.

Proof Since D is not contained in E, then $E \cap D$ is a proper submodule of D. Now , let $rsb \in E \cap D$ where $r, s \in R$, $b \in D \subseteq B$, then $rsb \in E$ and $rsb \in D$. But E is an app-quasi-prime submodule of B, then either $rb \in E + soc\ (B)$ or $sb \in E + soc(B)$, since $b \in D$, it follows that either $rb \in (E + soc\ (B)) \cap D$ or $sb \in (E + soc(B)) \cap D$. Since $soc(B) \subseteq D$, then by Lemma 2.23

We have $\big(E + soc(B)\big) \cap D = (E \cap D) + soc(B) \cap D$, and by Lemma 2.24 we have $soc(B) \cap D = soc(D)$. Hence either $rb \in E \cap D + soc\ (D)$ or $sb \in E \cap D + soc(D)$. Thus $E \cap D$ is an app-quasi-prime submodule of D.

Proposition 2.26 Let $f \in Hom\ (B, B')$ be an R-epimorphism, and E is an app-quasi-prime submodule of B'. Then $f^{-1}(E)$ is an app-quasi-prime submodule of B.

Proof It is clear that $f^{-1}(E)$ is a proper submodule of B. Now, let $rsb \in f^{-1}(E)$, where $r, s \in R$, $b \in B$, implies that $rsf(b) \in E$. since E is an app-quasi-prime submodule of B', so either $rf(b) \in E + soc(B')$ or $sf(b) \in E + soc(B')$. Hence either $rb \in f^{-1}(E) + f^{-1}\big(soc(B')\big) \subseteq f^{-1}(E) + soc(B)$ or $sb \in f^{-1}(E) + f^{-1}\big(soc(B')\big) \subseteq f^{-1}(E) + soc(B)$. That is either $rb \in f^{-1}(E) + soc(B)$ or $sb \in f^{-1}(E) + soc(B)$. Therefore $f^{-1}(E)$ is an app-quasi-prime submodule of B.

Proposition 2.27 Let $f \in Hom\ (B, B')$ be an R-epimorphism, and E be an app-quasi-prime submodule of B with $Ker\ f \subseteq E$. Then $f(E)$ is an app-quasi-prime submodule of B'.

Proof $f(E)$ is a proper submodule of B'. If not, suppose that $f(E) = B'$, let $b \in B$, then $f(b) \in B' = f(E)$, implies that $f(b) = f(e)$ for some $e \in E$, it follows tat $f(b - e) = 0$, so $b - e \in Ker\ f \subseteq E$, hence $b \in E$, that is $E = B$ contradiction. Now let

$rsb' \in f(E)$, where $r,s \in R$, $b' \in B'$. Since f is an epimorphism, then $f(b) = b'$ for some $b \in B$, thus $rsf(b) \in f(E)$, it follows that $rsf(b) = f(e)$ for some $e \in E$. That is $f(rsb - e) = 0$, so $rsb - e \in Ker\, f \subseteq E$, implies that $rsb \in E$. But E is an app-quasi-prime submodule of B, then either $rb \in E + soc(B)$ or $sb \in E + soc(B)$, it follows that either $rf(b) \in f(E) + f(soc(B)) \subseteq f(E) + soc(B')$ or $sf(b) \in f(E) + f(soc(B)) \subseteq f(E) + soc(B')$.
Hence either $rb' \in f(E) + soc(B')$ or $sb' \in f(E) + soc(B')$. Thus $f(E)$ is an app-quasi-prime submodule of B'.

Proposition 2.28 Let B, B' be $R - modules$, and E be a proper submodule of B', such that $E + soc(B')$ is a quasi-prime submodule of B', with $Hom_R\,(B, E + soc(B'))$ is a proper submodule of $Hom_R\,(B, B')$. Then $Hom_R\,\big(B, E + soc(B')\big)$ is app-quasi-prime submodule of $Hom_R\,(B, B')$.

Proof Suppose that $rsf \in Hom_R\,\big(B, E + soc(B')\big)$ where $r,s \in R$, $f \in Hom_R\,(B, B')$. Then for each $b \in B$, we have $rsf(b) \in E + soc(B')$. But $E + soc(B')$ is quasi-prime submodule of B', then either $rf(b) \in E + soc(B')$ of $sf(b) \in E + soc(B')$. That is $rf \in Hom_R\,\big(B, E + soc(B')\big) \subseteq Hom_R\,\big(B, E + soc(B')\big) + soc\big(Hom_R\,(B, B')\big)$ or $sf \in Hom_R\,\big(B, E + soc(B')\big) \subseteq Hom_R\,\big(B, E + soc(B')\big) + soc(Hom_R\,(B, B'))$. Thus $Hom_R\,(B, E + soc(B')$ is app-quasi-prime submodule of $Hom_R\,(B, B')$.

Proposition 2.29 Let $B = B_1 \oplus B_2$ be an R-module, where B_1, B_2 be modules, and $E = E_1 \oplus E_2$ be submodule of B with E_1, E_2 are submodules of B_1, B_2 respectively with $E \subseteq soc(B)$. If E is an app-quasi-prime submodule of B, then E_1 and E_2 are app-quasi-prime submodules of B_1 and B_2 respectively.

Proof Suppose that $b_1 \in E_1$, where $r,s \in R$, $b_1 \in B_1$, then $rs(b_1, 0) \in E$. Since E is an app-quasi-prime submodule of B, then either $r(b_1, 0) \in E + soc(B)$ or $s(b_1, 0) \in E + soc(B)$. But $E \subseteq soc(B)$, implies that $E + soc(B) = soc(B) = soc(B_1) \oplus soc(B_2)$. If $r(b_1, 0) \in soc(B_1) \oplus soc(B_2)$, implies that $rb_1 \in soc(B_1) \subseteq E_1 + soc(B_1)$ If $s(b_1, 0) \in soc(B_1) \oplus soc(B_2)$, implies that $sb_1 \in soc(B_1) \subseteq E_1 + soc(B_1)$. Hence $rb_1 \in E_1 + soc(B_1)$ or $sb_1 \in E_1 + soc(B_1)$. Therefore E_1 is an app-quasi-prime submodule of B_1
Similarly E_2 is an app-quasi-prime submodule of B_2.

Proposition 2.30 Let $B = B_1 \oplus B_2$ be an R-module, where B_1, B_2 is an R-modules, then.
1) E is an app-quasi-prime submodule of B_1, with $E \subseteq soc(B_1)$ and B_2 is a semi simple if and only if $E \oplus B_2$ is an app-quasi-prime submodule of B.

2) D is an app-quasi-prime submodule of B_2 with $D \subseteq soc(B_2)$ and B_1 is a semi simple if and only if $B_1 \oplus D$ is an app-quasi-prime submodule of B.

Proof
(1) $(\Longrightarrow)$ Let $rs(b_1, b_2) \in E \oplus B_2$, where $r,s \in R$, $(b_1, b_2) \in B$, then $rsb_1 \in E$ and $rsb_2 \in B_2$. But E is an app-quasi-prime submodule of B_1, and $E \subseteq soc(B_1)$, then either $rb_1 \in E + soc(B_1) = soc(B_1)$ or $sb_1 \in E + soc(B_1) = soc(B_1)$. Now since B_2 is a semi simple, then by [2, p221] $soc(B_2) = B_2$. So, if $rb_1 \in E + soc(B_1) = soc(B_1)$ then $(b_1, b_2) \in soc(B_1) \oplus soc(B_2) = soc(B_1 \oplus B_2) \subseteq E \oplus B_2 + soc(B_1 \oplus B_2)$. If if $sb_1 \in E + soc(B_1) = soc(B_1)$ then $s(b_1, b_2) \in soc(B_1) \oplus soc(B_2) = soc(B_1 \oplus B_2) \subseteq E \oplus B_2 + soc(B_1 \oplus B_2)$. Thus $E \oplus B_2$ is an app-quasi-prime submodule of B.

$(\Longleftarrow)$ Let $b_1 \in E$, where $r,s \in R$, $b_1 \in B_1$, then for each $b_2 \in B_2$ $rs(b_1, b_2) \in E \oplus B_2$. But $E \oplus B_2$ is an app-quasi-prime submodule of B, so, either $r(b_1, b_2) \in E \oplus B_2 + soc(B_1 \oplus B_2)$ or $s(b_1, b_2) \in E \oplus B_2 + soc(B_1 \oplus B_2)$. If $r(b_1, b_2) \in E \oplus B_2 + soc(B_1) \oplus soc(B_2)$, since $E \subseteq soc(B_1)$, then $E + soc(B_1) = soc(B_1)$, and $soc(B_2) = B_2$ so, $r(b_1, b_2) \in E \oplus B_2 + (E + soc(B_2)) \oplus B_2$ implies that $r(b_1, b_2) \in (E + soc(B_2)) \oplus B_2$, it follows that $rb_1 \in E + soc(B_1)$. Similarly if $s(b_1, b_2) \in E \oplus B_2 + soc(B_1) \oplus soc(B_2)$, implies that $sb_1 \in E + soc(B_1)$. Therefore E is an app-quasi-prime submodule of B_1.

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(2) In similar way we can prove (2).

Remark 2.31 It is clear that every prime submodule is an app-quasi-prime submodule while the convers is not true in general as the following example shows that.

Example 2.32 Consider the Z-module Z_4, the submodule $E=\langle\bar{0}\rangle$ is an app-quasi-prime submodule of Z_4, since for each $r,s\in Z$, and $b\in Z_4$, with $rsb\in E$, we have either $rb\in E+soc(Z_4)=E+\langle\bar{2}\rangle$ or $sb\in E+soc(Z_4)=E+\langle\bar{2}\rangle$. But $\langle\bar{0}\rangle$ is not prime submodule of Z_4, because $2.\bar{2}\in\langle\bar{0}\rangle$, $2\in Z$, $\bar{2}\in Z_4$, but $\bar{2}\notin\langle\bar{0}\rangle$ and $2\notin[\langle\bar{0}\rangle:Z_4]=4Z$.

Recall that a proper submodule E of an R-module B is called an app-prime submodule of B, if whenever $rb\in E$, with $r\in R$, $b\in B$, implies that either $b\in E+soc(B)$ or $rB\subseteq E+soc(B)$ [10].

Remark 2.33 It is clear that every app-prime submodule is an app-quasi-prime submodule , while the converse is not true in general, as the following example shows that.

Example 2.34 Consider the Z-module $Z\oplus Z$, and $E=(0)\oplus 2Z$, E is not app-prime, since $2(0,1)\in E$, but $(0,1)\notin E+soc(Z\oplus Z)$, and $2\notin[(0)\oplus 2Z+soc(Z\oplus Z):Z\oplus Z]=(0)$. But E is an app-quasi-prime because E is a quasi-prime submodule of $Z\oplus Z$.

Proposition 2.35 Let B be an R-module, and E be a proper submodule of B, with $soc(B)\subseteq E$. Then E is an app-quasi-prime submodule of B if and only if $[E:_R I]$ is an app-quasi-prime submodule of B for every ideal I of R.

Proof ($\Longrightarrow$) Let $rsb\in[E:_R I]$, with $r,s\in R$, $b\in B$, implies that $rsbI\subseteq E$, that is $rsba\in E$ for each $a\in I$. Since E is an app-quasi-prime submodule of B, it follows that either $rba\in E+soc(B)$ or $sba\in E+soc(B)$, but $soc(B)\subseteq E$, implies that $E+soc(B)=E$. Thus either $rba\in E$ or $sba\in E$. That is either $rb\in[E:_R I]\subseteq[E:_R I]+soc(B)$ or $rb\in[E:_R I]\subseteq[E:_R I]+soc(B)$. Hence $[E:_R I]$ is an app-quasi-prime submodule of B.

($\Longleftarrow$) Since $[E:_R I]$ is an app-quasi-prime submodule of B for each ideal I of R, thus put $I=R$, we get $[E:R]=E$ is an app-quasi-prime submodule of B.

Proposition 2.36 Let B be a multiplication R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $FDb\subseteq E$, for some submodules F and D of B and $b\in B$, then either $Fb\subseteq E+soc(B)$ or $Db\subseteq E+soc(B)$.

Proof ($\Longrightarrow$) Suppose that $FDb\subseteq E$, for some submodules F and D of B and $b\in B$. But B is a multiolication then $F=IB$ and $D=JB$ for some ideals I,J of R, thus $FDb=IJb\subseteq E$. But E is an app-quasi-prime submodule of B, then by Corollary 2.7 either $Ib\subseteq E+soc(B)$ or $Jb\subseteq E+soc(B)$. It follows that either $Fb\subseteq E+soc(B)$ or $Db\subseteq E+soc(B)$.

($\Longleftarrow$) Assume that $IJb\subseteq E$, where I,J are ideals in R and $b\in B$. Since B is a multiplication it follows that, $IDb=FDb\subseteq E$, so by hypothesis either $Db\subseteq E+soc(B)$ or $Fb\subseteq E+soc(B)$, that is either $Ib\subseteq E+soc(B)$ or $Jb\subseteq E+soc(B)$. Hence by Corollary 2.7 Then E is an app-quasi-prime submodule of B.

Proposition 2.37 Let B be a multiplication R-module, and E be a proper submodule of B. Then E is an app-quasi-prime submodule of B if and only if whenever $FDL\subseteq E$, for some submodules F,D and L of B, then either $FL\subseteq E+soc(B)$ or $DL\subseteq E+soc(B)$.

Proof ($\Longrightarrow$) Suppose that $FDL\subseteq E$, for some submodules F, D and L of B. But B is a multiolication then $F=IB$ and $D=JB$ for some ideals I,J of R, thus $FDL=IJL\subseteq E$. Since E is an app-quasi-prime submodule of B, then by Proposition 2.4 either $IL \subseteq E + soc(B)$ or $JL \subseteq E + soc(B)$. It follows that either $FL \subseteq E + soc(B)$ or $DL \subseteq E + soc(B)$.

($\Longleftarrow$) Assume that $IJL \subseteq E$, where I, J are ideals in R and L is a submodule of B.Since B is a multiplication it follows that, $IDL = FDL \subseteq E$, so by hypothesis either $FL \subseteq E + soc(B)$ or $DL \subseteq E + soc(B)$, that is either $IL \subseteq E + soc(B)$ or $JL \subseteq E + soc(B)$. Hence by Proposition 2.4 E is an app-quasi-prime submodule of B.

Proposition 2.38 Let B be a faithful finitely generated multiplication R-module, and E be a proper submodule of B with $soc(B) \subseteq E$. then the following statements are equivalent.
1) E is an app-quasi-prime submodule of B.
2) $[E:_R B]$ is an app-quasi-prime ideal of R.
3) $E = IB$ for some app-quasi-prime ideal I of R.

Proof (1) $\Longrightarrow$(2) Follows by Proposition 2.12

(2) $\Longrightarrow$(1) Follows by Proposition 2.15

(2) $\Longrightarrow$(3) Since $[E:_R B]$ is an app-quasi-prime ideal of R, and $E = [E:_R B]B$, it is follows that $E = IB$ and $I = [E:_R B]$ an app-quasi-prime ideal of R.
(3) $\Longrightarrow$(2) Suppose that $E = IB$ for some app-quasi-prime ideal I of R. But B is a multiplication we have $E = [E:_R B]B = IB$. Thus since B is faithful finitely generated multiplication, then by Lemma 2.17 we have $I = [E:_R B]$, it follows that $[E:_R B]$ is an app-quasi-prime ideal of R.

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المقاسات الجزئية الاولية الظاهرية تقريباً ومفاهيم ذات علاقة

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المستخلص:

لتكن R حلقه ابداليه بمحايد و B مقاساً احادياً ايسراً. يدعى المقاس الجزئي الفعلي E من المقاس B مقاس جزئي اولي ظاهري اذا كان $rsb \in E$ حيث $r, s \in R$ ، $b \in B$ يؤدي الى اما $rb \in E$ او $sb \in E$ ، كأعمام لمفهوم المقاس الجزئي الاولي الظاهري تقريباً، حيث انه يدعى المقاس الجزئي الفعلي E من المقاس، مقاس جزئي اولي ظاهري تقريباً اذا كان $sb \in E$ حيث $r, s \in R$ ، $b \in B$ يؤدي الى اما $rb \in E + soc(B)$ او $sb \in E + soc(B)$ ، حيث ان $soc(B)$ هو تقاطع جميع المقاسات الجزئية الجوهرية من المقاس B. اعطينا العديد من الخصائص الاساسيه، المكافئات والامثلة لهذا المفهوم. بالاضافه الى ذالك درسنا سلوك المقاسات الجزئية الاولية الظاهرية تقريباً تحت تأثير التشاكلات. واخيراً قدمنا العديد من المكافئات للمقاسات الجزئية الاولية الظاهرية تقريباً في صنف المقاسات الضربية.