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Differential Subordination Results for Holomorphic Functions Related to Differential Operator

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Abstract:

In the present work, we introduce and study a certain class of holomorphic functions defined by differential operator in the open unit disk U. Also, we derive some important geometric properties for this class such as integral representation, inclusion relationship and argument estimate.

Key Words. Holomorphic functions, subordination, integral representation, differential operator**.**

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1. Introduction.

Let $\mathcal{A}$ stands for the family of all functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad (1.1)$$

which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Given two functions f and g which are holomorphic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function w which is holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ $\alpha, \gamma \geq 0, \mu, \lambda, \beta > 0$ and $\alpha \neq \lambda$, we consider the differential operator $A^{\eta}_{\mu,\lambda,\gamma}(\alpha, \beta) : \mathcal{A} \longrightarrow \mathcal{A}$, introduced by Amourah and Darus [2], where

$$A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta) f(z) = z + \sum_{n=2}^{\infty} \left[1 + \frac{(n-1)[(\lambda-\alpha)\beta + n\gamma]}{\mu+\lambda}\right]^{\eta} a_n z^n . \qquad (1.2)$$

It is readily verified from (1.2) that

$$z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta) f(z)\right)' = \frac{\mu+\lambda}{(\lambda-\alpha)\beta + n\gamma} A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta) f(z) - \left(1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta + n\gamma}\right) A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta) f(z). \quad (1.3)$$

Here, we would point out some of the special cases of the operator defined by (1.2) can be found in [1,3,7,9].

Let T stands for the family of mapping h of the form:

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

which are holomorphic and convex univalent in U and satisfy the condition:

$Re\{h(z)\} > 0,\ (z \in U)$.

Now, we need the following lemmas that will be used to prove our main results.

Lemma 1.1 [5]. Let $u, v \in \mathbb{C}$ and suppose that ψ is convex and univalent in U with $\psi(0) = 1$ and $Re\{u\psi(z) + v\} > 0,\ (z \in U)$. If q is holomorphic in U with $q(0) = 1$, then the subordination

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z),$$

which implies to $q(z) \prec \psi(z)$.

Lemma 1.2 [6]. Let h be convex univalent in U and $\mathcal{T}$ be holomorphic in U with $Re\{\mathcal{T}(z)\} \geq 0$, $(z \in U)$. If q is holomorphic in U and $q(0) = h(0)$, then the subordination

$$q(z) + \mathcal{T}(z) zq'(z) \prec h(z),$$

which implies to $q(z) \prec h(z)$.

Lemma 1.3 [4]. Let q be holomorphic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exists two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2} b_1 = arg\big(q(z_1)\big) < arg\big(q(z)\big) < arg\big(q(z_2)\big) = \frac{\pi}{2} b_2 ,$$

for some b_1 and b_2 $(b_1 > 0, b_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right) m$$

and

$$\frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right) m ,$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad and \quad \varepsilon = i \tan \frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

Such type of study was carried out for another classes in [10].

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2. Main Results

We begin this section with the function class $\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$ as follows:

Definition 2.1. A function $f\in\mathcal{A}$ is said to be in the class $\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$, if it satisfies the following differential subordination condition:

$$\frac{1}{1-\delta}\left(\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}-\delta\right)\prec h(z), \tag{2.1}$$

where $\eta\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\alpha,\gamma\geq 0,\mu,\lambda,\beta>0$, $\alpha\neq\lambda$ and $h\in T$.

In the following theorem, we find integral representation of the class $\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$.

Theorem 2.1. Let $f\in\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$. Then

$$A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)$$

$$=z.\exp\left[(1-\delta)\int_0^z\frac{h(w(s))-1}{s}ds\right],$$

where w is holomorphic in U with $w(0)=0$ and $|w(z)|<1$, $(z\in U)$.

Proof. Assume that $f\in\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$. It is easy to see that subordination condition (2.1) can be written as follows

$$\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}=(1-\delta)h(w(z))+\delta\,, \tag{2.2}$$

where w is holomorphic in U with $w(0)=0$ and $|w(z)|<1$, $(z\in U)$.

From (2.2), we find that

$$\frac{\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}-\frac{1}{z}=(1-\delta)\frac{h(w(z))-1}{z}, \tag{2.3}$$

After integrating both sides of (2.3), we have

$$\log\left(\frac{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}{z}\right)$$

$$=(1-\delta)\int_0^z\frac{h(w(s))-1}{s}ds \tag{2.4}$$

Therefore, from (2.4), we obtain the required result.

Next, we establish the inclusion relationship for the class $\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$.

Theorem 2.2. Let $Re\left\{(1-\delta)h(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right\}>0$. Then

$$\Psi(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)\subset\Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h).$$

Proof. Let $f\in\Psi(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$ and put

$$q(z)=\frac{1}{1-\delta}\left(\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}-\delta\right). \tag{2.5}$$

Then q is holomorphic in U with $q(0)=1$. Making use of the identity (1.3), we find from (2.5) that

$$\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\frac{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}$$

$$=(1-\delta)q(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}. \tag{2.6}$$

Differentiating both sides of (2.6) with respect to z and multiplying by z, we have

$$q(z)+\frac{zq'(z)}{(1-\delta)q(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}}$$
$$=\frac{1}{1-\delta}\left(\frac{z\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)}-\delta\right)\prec h(z). \tag{2.7}$$

Since $Re\left\{(1-\delta)h(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right\}>0$, then applying Lemma 1.1 to the subordination

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(2.7), yields $q(z) \prec h(z)$, which implies to $f \in \Psi(\eta,\mu,\lambda,\gamma,\alpha,\beta,\delta;h)$.

Theorem 2.3

Let $f \in \mathcal{A}$, $0 < a_1, a_2 \le 1$ and $0 \le \delta < 1$. If

$$-\frac{\pi}{2}a_1 < arg\left(\frac{z\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} - \delta\right) < \frac{\pi}{2}a_2,$$

for some $g \in \Psi\left(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;\frac{1+Az}{1+Bz}\right)$, $(-1 \le B < A \le 1)$, then

$$-\frac{\pi}{2}b_1 < arg\left(\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} - \delta\right) < \frac{\pi}{2}b_2,$$

where b_1 and b_2 $(0 < b_1, b_2 \le 1)$ are the solutions of the equations:

$$a_1 = \begin{cases} b_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1 \\ b_1, & B = -1 \end{cases} \tag{2.8}$$

and

$$a_2 = \begin{cases} b_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1 \\ b_2, & B = -1 \end{cases} \tag{2.9}$$

with

$$\varepsilon = i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right)$$

and

$$t = \frac{2}{\pi} \times$$

$$\times \sin^{-1}\left(\frac{(A-B)(1-\delta)}{\left(\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)(1-B^2)+(1-\delta)(1-AB)}\right). \tag{2.10}$$

Proof. Define the function G by

$$G(z) = \frac{1}{1-\tau}\left(\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} - \tau\right), \tag{2.11}$$

where $g \in \Psi\left(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;\frac{1+Az}{1+Bz}\right)$, $(-1 \le B < A \le 1)$ and $0 \le \tau < 1$.

Then G is holomorphic in U with $G(0) = 1$. Thus in view of (1.3) and (2.11), we observe that

$$\begin{aligned}&\left((1-\tau)G(z)+\tau\right)A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z) \\ &= \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z) \\ &-\left(1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z).\end{aligned} \tag{2.12}$$

So, it is required to differential with respect to z the relation (2.12), and then multiplying by z, we obtain

$$\begin{aligned}&\left((1-\tau)G(z)+\tau\right)z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)\right)' \\ &+(1-\tau)zG'(z)A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z) \\ &= \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}z\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)' \\ &-\left(1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'.\end{aligned} \tag{2.13}$$

Suppose that

$$H(z) = \frac{1}{1-\delta}\left(\frac{z\left(A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)\right)'}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} - \delta\right).$$

Using (1.3) again, we have

$$\begin{aligned}&\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\frac{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)}{A^{\eta}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} \\ &= (1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}.\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we easily get

$$\begin{aligned}&G(z)+\frac{zG'(z)}{(1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}} \\ &= \frac{1}{1-\tau}\left(\frac{z\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z)} - \tau\right).\end{aligned} \tag{2.15}$$

Notice that from Theorem 2.2, $g \in \Psi\left(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;\frac{1+Az}{1+Bz}\right)$ implies $g \in \Psi\left(\eta+1,\mu,\lambda,\gamma,\alpha,\beta,\delta;\frac{1+Az}{1+Bz}\right)$. Thus,

$$H(z) \prec \frac{1+AZ}{1+Bz} \quad (-1 \le B < A \le 1).$$

By applying the result of Silverman and Silvia [8], we have

$$\left|H(z)-\frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (B \ne -1,\ z \in U)\ (2.16)$$

and

$$Re\{H(z)\} > \frac{1-A}{2} \quad (B=-1,\ z \in U). \quad (2.17)$$

It follows from (2.16) and (2.17) that

$$\left|(1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma} - \frac{\left(\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)(1-B^2)+(1-\delta)(1-AB)}{1-B^2}\right|$$

$$< \frac{(A-B)(1-\delta)}{1-B^2}, \quad (B \ne -1,\ z \in U)$$

and

$$Re\left\{(1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right\}$$

$$> \frac{(1-A)(1-\delta)}{2}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma},$$

$$(B=-1,\ z \in U).$$

Putting

$$(1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma} = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(1-\delta)}{\left(\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)(1-B^2)+(1-\delta)(1-AB)} < \phi < \frac{(A-B)(1-\delta)}{\left(\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right)(1-B^2)+(1-\delta)(1-AB)},\ (B \ne -1)$$

and $-1 < \phi < 1,\ (B=-1)$,

then

$$\frac{(1-A)(1-\delta)}{1-B}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma} < \rho$$

$$< \frac{(1+A)(1-\delta)}{1+B}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma},$$

$(B \ne -1)$

and

$$\frac{(1-A)(1-\delta)}{1-B}+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma} < \rho < \infty,$$

$(B=-1)$.

An application of Lemma 1.2 with $\mathcal{T}(z) = \frac{1}{(1-\delta)H(z)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}}$, yields $G(z) \prec h(z)$.

If there exist two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2}b_1 = arg\big(G(z_1)\big) < arg\big(G(z)\big) < arg\big(G(z_2)\big) = \frac{\pi}{2}b_2\,,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)} = -\frac{mi}{2}(b_1+b_2)$$

and

$$\frac{z_2G'(z_2)}{G(z_2)} = \frac{mi}{2}(b_1+b_2)\,,$$

where

$$m \ge \frac{1-|\varepsilon|}{1+|\varepsilon|} \quad and \quad \varepsilon = i\tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B \ne -1$, we obtain

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_1\left(A_{\mu,\lambda,\gamma}^{\eta+1}(\alpha,\beta)f(z_1)\right)'}{A_{\mu,\lambda,\gamma}^{\eta+1}(\alpha,\beta)g(z_1)}-\tau\right)\right)$$

$$= arg\left(G(z_1) + \frac{z_1G'(z_1)}{(1-\delta)H(z_1)+\delta+1-\frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}}\right)$$

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$$= arg\left(G(z_1)\right) + arg\left(1 + \frac{z_1 G'(z_1)}{\left[(1-\delta)H(z_1) + \delta + 1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta + n\gamma}\right] G(z_1)}\right)$$

$$= -\frac{\pi}{2} b_1 + arg\left(1 - \frac{mi}{2\rho}(b_1+b_2)e^{-i\frac{\pi}{2}\phi}\right)$$

$$= -\frac{\pi}{2} b_1 + arg\left(1 - \frac{m}{2\rho}(b_1+b_2)\cos\frac{\pi}{2}(1-\phi) + \frac{mi}{2\rho}(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)\right)$$

$$\leq -\frac{\pi}{2} b_1 - \tan^{-1}\left(\frac{m(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho + m(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)}\right)$$

$$\leq -\frac{\pi}{2} b_1 - \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + 1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$

$$= -\frac{\pi}{2} a_1,$$

where a_1 and t are given by (2.8) and (2.10), respectively.

Also,

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z_2)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z_2)} - \tau\right)\right)$$

$$\geq \frac{\pi}{2} b_2 + \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B} + \delta + 1 - \frac{\mu+\lambda}{(\lambda-\alpha)\beta+n\gamma}\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$

$$= \frac{\pi}{2} a_2,$$

where a_2 and t are given by (2.9) and (2.10), respectively.

Similarly, for the case $B = -1$, we have

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_1\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z_1)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z_1)} - \tau\right)\right) \leq -\frac{\pi}{2} b_1$$

and

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)f(z_2)\right)'}{A^{\eta+1}_{\mu,\lambda,\gamma}(\alpha,\beta)g(z_2)} - \tau\right)\right) \geq \frac{\pi}{2} b_2.$$

The above two cases disagree the assumptions. Therefore, the proof is complete.

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نتائج التابعية التفاضلية للدوال التحليلية المرتبطة بالمؤثر التفاضلي

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المستخلص:

في العمل الحالي ، نقدم وندرس صنف مؤكد من الدوال التحليلية المعرفة بواسطة المؤثر التفاضلي في قرص الوحدة المفتوح U. كذلك نقدم بعض الخصائص الهندسية المهمة لهذا الصنف مثل تمثيل التكامل ، علاقة الاحتواء وتخمين السعة.