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EXISTENCE OF NONOSCILLATORY RELATIVELY BOUNDED SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper some sufficient conditions are obtained to insure the existence of positive solutions which is relatively bounded from one side for nonlinear neutral differential equations of second order. We used the Krasnoselskii's fixed point theorem and Lebesgue's dominated convergence theorem to obtain new sufficient conditions for the existence of a Nonoscillatory one side relatively bounded solutions. These conditions are more applicable than some known results in the references. Three examples included to illustrate the results obtained.
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Mathematics Subject Classification: Differential Equations.

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1. INTRODUTION

This paper is concerned with the existence of a positive relatively bounded solution of the neutral differential equations of the form

$$(a(t)(x(t)-q(t)x(\tau(t))')'$$
$$-p(t)f\big(t,x(t),x(\sigma(t)),x'(t),x'(\sigma(t))\big)=0. \qquad (1.1)$$

with respect to equation (1.1), throughout we shall assume the following:

(i) $p,q\in C([t_0,+\infty),\mathbb{R}^+), a\in C\big([t_0,+\infty),(0,\infty)\big), t\geq t_0>0, t_0\in\mathbb{R}$.
(ii) $\tau,\ \sigma\in C([t_0,+\infty),\mathbb{R})$, $\tau(t)\leq t$, $\sigma(t)\leq t, \lim_{n\to\infty}\tau(t)=\infty, \lim_{n\to\infty}\sigma(t)=\infty$
(ii)$f\in C([t_0,\infty)\times R^4,R), f$ is nondecreasing function, and $xf\big(t,x(t),x(\sigma(t)),x'(t),x'(\sigma(t))\big)>0,\ x\neq 0$. By a solution of Eq.(1.1) we mean a function $x\in C[(t_1-\rho(t_1),\infty),\mathbb{R})$, $\rho(t_1)=\min\{\tau(t),\sigma(t)\}$, forsome $t_1\geq t_0$, such that $a(t)\big(x(t)-q(t)x(t-\tau(t))\big)$ is continuously differentiable on $[t_1,\infty)$ and such that$x(t)$ satisfy Eq.(1.1) for $t\geq t_1$. A solution $x(t)$ is said to be nonoscillatory if it is either eventually positive or eventually negative that is there exists $t_*\geq t_0$, such that either $x(t)>0$ or $x(t)<0$ for all $t\geq t_*$, otherwise is said oscillatory [10].

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations. In 1999, S. Tanaka [12] study the first order differential equations:

$$\frac{d}{dt}[x(t)+h(t)x(\tau(t))]+\sigma f(t,x(g(t)))=0$$

and established some sufficient conditions to insure the existence of positive solution of previous equation. In 2002, Y. Zhou, B. Zhang [14], found some sufficient conditions for the existence of nonoscillatory solutions the following equation:

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)x(t-\sigma)$$
$$-Q(t)x(t-\delta)].$$

In 2005, Y. Yu, H. Wang [13] , studied the nonoscillatory solutions of a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients of the form:

$$\big(r(t)(x(t)+P(t)x(t-\tau)')'+Q_1(t)f\big(x(t-\sigma_1)\big)$$
$$-Q_2(t)g\big(x(t-\sigma_2)\big)=0$$

In 2009, B. Dorociakova and R. Olach [5] studied the first order delay differential equations:

$$x'(t)+p(t)x(t)+q(t)x\big(\tau(t)\big)=0.$$

In the same year I. Culkov, L. Hanutiakov, R. Olach [3] studied the second order nonlinearneutral differential equations

$$\frac{d^2}{dt^2}[x(t)-a(t)x(t-\tau)]=p(t)f\big(x(t-\sigma)\big).$$

In 2011,R. Olach etc. al [4], studied the first order neutral differential equations:

$$\frac{d}{dt}[x(t)-a(t)x(t-\tau)]=p(t)f\big(x(t-\sigma)\big)$$

In 2012, L. Lietc. al[9], studied the existence of a bounded nonoscillatory positive solution for the equation.

$$\frac{d}{dt}[x(t)+a(t)x(t-\tau)]+p(t)f\big(x(t-\alpha)\big)$$
$$+q(t)g\big(x(t-\beta)\big)=0$$

In 2013, T. Canadan [1],obtained sufficient conditions for first-order nonlinear neutral differential equations to have nonoscillatory solutions for different ranges of $p_1(t)$ and $p_2(t,\xi)$of the forms:

$$[[x(t)-p_1(t)x(t-\tau)]^{\gamma}]'+Q_1(t)G\big(x(t-\sigma)\big)=0$$

$$[[x(t)-p_1(t)x(t-\tau)]^{\gamma}]'+\int_c^d Q_2(t)G\big(x(t-\xi)\big)\,d\xi=0$$

and

$$[\left[x(t)+\int_a^b p_2(t,\xi)x(t-\xi)d\xi\right]^{\gamma}]'$$
$$+\int_c^d Q_2(t)G\big(x(t-\xi)\big)\,d\xi=0$$

$$\frac{d}{dx}[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)]$$
$$+Q_1(t)g_1\big(x(t-\sigma_1)\big)-Q_2(t)g_2\big(x(t+\sigma_2)\big)=0$$

In 2017, F. Kong [8], studied the Existence of nonoscillatory solutions of a kind of first-order neutral differential equation:

$$\frac{d}{dx}[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)]$$
$$+Q_1(t)g_1\big(x(t-\sigma_1)\big)-Q_2(t)g_2\big(x(t+\sigma_2)\big)-f(t)=0\,.$$

In 2018, B. Çına and M. Tamer Şenel[2], obtained some sufficient conditions for the existence of positive solutions of variable coefficient nonlinear second order neutral differential equation with distributed deviating arguments of the form:

$$\left(x(t)-\int_{a_1}^{b_1}P(t,\xi)x(t-\xi)d\xi\right)''+\int_{a_2}^{b_2}f(t,x(\sigma(t,\xi))d\xi$$
$$=0$$

In this paper we prove that the existence of solution of Eq.(1.1) is relatively bounded, and we show that the solution is bounded from one side by function from above and below by function and ratio function respectively. some sufficient conditions for this purpose are obtained.

Definition 1.1 A function $x(t)$ is said to be relatively bounded from below (above) if there is a function $y(t)$and constant $k$such that$y(t)\leq x(t)\leq k\big(k\leq x(t)\leq y(t)\big)$.

The following fixed point theorem and Lebesgue's dominated convergence theorem will be used to prove the main results in the next section.

Lemma 1.2[7] (Krasnoselskii's Fixed Point Theorem).

Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x+S_2y\in\Omega$ for every pair $x,y\in\Omega$. If S_1 is contractive and S_2 is completely continuous, then the equation $S_1x + S_2x = x$has a solution in Ω.

Theorem 1.3 [11] (The Lebesgue Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates$\{f_n\}$ on E in the sense that If $|f_n(x)| \le g(x)$ on E for all n. If $\{f_n\} \to \{f\}$ pointwise a.e. on E, then f is integrable over E and

$lim_{n\to\infty} \int_E f_n = \int_E f$.whereE is afinite measurable set.

2.EXISTENCE OF ONE SIDE RELATIVELY BOUNDED SOLUTIONS:

In this section we will establish several sufficient conditions to insure the existence of a nonoscillatory solutions which are one side relatively bounded by functions and a ratio of positive functions on $[t_1,\infty)$of Eq.(1.1), $t_1 \ge t_0$. Without loss of generality we will discuss the existence of eventually positive solution and the existence of eventually negative solution can be discussed in similar way.

The following conditions will be used in the next results:

H1. $0 < q(t) \le c < 1$

H2. $M_1 \le f(t,.) \le M_2$, $M_1, M_2 \ne 0$, are constants.

H3.$m_1x(t) \le f(t,.) \le m_2x(t)$, $m_1, m_2 \ne 0$, are constants

Theorem 2.1. Suppose that H1, H3 hold, and there exist bounded function $u \in C^1([t_0,\infty),[0,\infty))$, a constant $N^* > 0$, and $\rho(t_1) \ge t_0$ such that

$$u(t) \le \frac{q(t_1)u(\tau(t_1))}{q(t)} \qquad (2.1)$$

$$\frac{u(t) - q(t)u(\tau(t))}{m_1 \min_{t\ge t_1}\{u(t)\}} \le \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds \le \frac{1}{m_2}(1 - q(t)) \qquad (2.2)$$

Then Eq.(1.1) has a nonoscillatory relatively bounded from below.

Proof. Let $C([t_0,+\infty),\mathbb{R})$ be the set of all continuous bounded functions with the norm $\|x\| = sup_{t\ge t_0}|x(t)|$. Then $C([t_0,+\infty),\ \mathbb{R})$ is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0,+\infty),\mathbb{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0,+\infty),\mathbb{R}):\ u(t) \le x(t) \le N^*,\ N^* > 0,\ t \ge t_0\}. \qquad (2.3)$$

For simplicity let

$f(t,\mathbf{x}(t)) = f(t,x(t),x(\tau(t)),x'(t),x'(t-\sigma(t)))$.

Now we define two maps S_1 and $S_2 : \Omega \to C([t_0,+\infty),\mathbb{R})$as follows:

$$(S_1x)(t) = \begin{cases} q(t)x(\tau(t)), & t \ge t_1, \\ (S_1x)(t_1), & t_0 \le t \le t_1, \end{cases}$$

$$(S_2x)(t) = = \begin{cases} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{x}(\xi))\, d\xi ds & ,t \ge t_1, \\ u(t) - q(t_1)u(\tau(t_1)) & ,t_0 \le t \le t_1, \end{cases} \qquad (2.4)$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$.

From condition (2.2) it follows that for $t \ge t_1$

$$\int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds < \infty. \qquad (2.5)$$

For every $x, y \in \Omega$ and $t \ge t_1$, with regard (2.2) we obtain

$$(S_1x)(t) + (S_2y)(t)$$
$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{y}(\xi))\, d\xi ds$$
$$\le q(t)N^* + m_2N^* \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$
$$\le N^*(q(t) + m_2N^* \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds) \le N^*.$$

For $t \in [t_0, t_1]$, we have

$$(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + u(t) - q(t_1)u(\tau(t_1))$$
$$\le q(t_1)x(\tau(t_1)) + N^*(1 - q(t_1)).$$
$$\le q(t_1)N^* + N^*(1 - q(t_1)) = N^*.$$

Furthermore, for $t \ge t_1$, with regard (2.2) we obtain

$$(S_1x)(t) + (S_2y)(t) =$$
$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{y}(\xi))\, d\xi ds,$$
$$\ge q(t)u(\tau(t)) + m_1 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{y}(\xi))\, d\xi ds,$$
$$\ge q(t)u(\tau(t)) + m_1 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} \mathrm{y}(\xi)\, d\xi ds,$$
$$\ge q(t)u(\tau(t)) + m_1 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} u(\xi)\, d\xi ds,$$
$$\ge q(t)u(\tau(t)) + m_1 \min_{t\ge t_1}\{u(t)\} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$
$$\ge u(t).$$

Let $t \in [t_0, t_1]$, from Eq.(2.4), with regard to (2.1) we get

$$(S_1x)(t) + (S_2y)(t) =$$
$$= (S_1x)(t_1) + u(t) - q(t_1)u(\tau(t_1))$$
$$= q(t_1)x(\tau(t_1)) + u(t) - q(t_1)u(\tau(t_1)),$$
$$\geq q(t_1)u(\tau(t_1)) + u(t) - q(t_1)u(\tau(t_1))$$
$$= u(t).$$

Thus, we have proved that $S_1x + S_2y \in \Omega$, for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq t_1$ we have

$$\|S_1x - S_1y\| = \sup_{t\geq t_1}|(S_1x)(t) - (S_1y)(t)|$$
$$= \sup_{t\geq t_1}|q(t)x(\tau(t)) - q(t)y(\tau(t))|$$
$$\leq \sup_{t\geq t_1} q(t)|x(\tau(t)) - y(\tau(t))|$$
$$\leq c\|x - y\|$$

Also for $t \in [t_0, t_1]$.

$$\|S_1x - S_1y\| = \sup_{t_0\leq t\leq t_1}|(S_1x)(t) - (S_1y)(t)|$$
$$= |(S_1x)(t_1) - (S_1y)(t_1)|$$
$$= |q(t_1)x(\tau(t_1)) - q(t_1)y(\tau(t_1))|$$
$$= q(t_1)|x(\tau(t_1)) - y(\tau(t_1))|$$
$$\leq c \sup_{t_0\leq t\leq t_1}|x(\tau(t)) - y(\tau(t))|$$
$$= c\|x - y\|$$

Hence

$$\|S_1x - S_1y\| \leq c\|x - y\|.$$

Thus S_1 is a contraction mapping on Ω.

To show that S_2 is completely continuous. First we will show that S_2 is continuous. By (2.5) and H2 it follows:

$$\int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(s, \mathbf{x}(\xi)) d\xi ds$$
$$\leq m_2 N^* \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds < \infty. \qquad (2.8)$$

Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because of Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$\|(S_2x_k)(t) - (S_2x)(t)\| = \sup_{t\geq t_1}|(S_2x_k)(t) - (S_2x)(t)|$$
$$= \sup_{t\geq t_1}\left|\int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)}\left(f(\xi, \mathbf{x}_k(\xi)) - f(\xi, \mathbf{x}(\xi))\right) d\xi ds\right|$$
$$\leq \int_{t_1}^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)}\left|f(s, \mathbf{x}_k(\xi)) - f(s, \mathbf{x}(\xi))\right| d\xi ds.$$

Since

$$\left|f(s, \mathbf{x}_k(\xi)) - f(s, \mathbf{x}(\xi))\right| \to 0 \text{ as } k \to \infty,$$

by applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\|(S_2x_k)(t) - (S_2x)(t)\| = 0$$

This means that S_2 is continuous.

Now to prove $S_2\Omega$ is relatively compact, we have to show that $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty]$, according to Arzelã-Ascoli theorem [6]. It is clear that from (2.3) we get $\{S_2x : x \in \Omega\}$ is uniformly bounded. To show $\{S_2x : x \in \Omega\}$ is equicontinuouson $[t_0, \infty)$. Let $x \in \Omega$ and any $\varepsilon > 0$, with regard to (2.8), there exists $t_* \geq t_1$ large enough so that

$$\int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds < \frac{\varepsilon}{2}, t \geq t_* \qquad (2.9)$$

Then, for $x \in \Omega$, $T_2 > T_1 \geq t_*$, we have

$$\|(S_2x_k)(T_2) - (S_2x)(T_1)\| =$$
$$= \sup_{T_2>T_1\geq t_*}|(S_2x_k)(T_2) - (S_2x)(T_1)|$$
$$\leq |(S_2x_k)(T_2)| + |(S_2x)(T_1)|$$
$$\leq \int_{T_2}^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}_k(\xi)) d\xi ds$$
$$+ \int_{T_1}^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For $x \in \Omega$ and $t_1 \leq T_1 < T_2 \leq t_*$, we get

$$\|(S_2x)(T_2) - (S_2x)(T_1)\| =$$
$$= \sup_{t_1\leq T_1<T_2\leq t_*}|(S_2x)(T_2) - (S_2x)(T_1)|$$
$$\leq \sup_{t_1\leq T_1<T_2\leq t_*}\left|\int_{T_2}^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds - \int_{T_1}^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds\right|$$
$$\leq \int_{T_1}^{T_2} \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds$$
$$\leq m_2 \max_{T_1\leq t\leq T_2}\{\frac{p(t)}{a(t)}\}(T_2 - T_1).$$

Thus there exists $\delta_1 = \frac{\varepsilon}{m_2 \max_{T_1\leq t\leq T_2}\{\frac{p(t)}{a(t)}\}}$, such that

$$|(S_2x)(T_2) - (S_2x)(T_1)| < \varepsilon, \text{ if } 0 < T_2 - T_1 < \delta_1$$

Finally, for any $x \in \Omega$, $t_0 \le T_1 < T_2 \le t_1$, and for any $\varepsilon > 0$,

$|(S_2x)(T_2) - (S_2x)(T_1)| = 0 < \varepsilon$, if $0 < T_2 - T_1 < \delta_2$.

Hence, $S_2\Omega$ is relatively compact. By Lemma 1.2 then Eq.(1.1) has a nonoscillatory relatively bounded from below. The proof is complete.

The next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory relatively bounded from above by $v(t)$.

Theorem 2.2. Suppose that H1, H3 hold, and there exist bounded function $v \in C^1([t_0,\infty),[0,\infty)), \rho(t_1) \ge t_0$ such that

$$\frac{1}{m_1}[1-q(t)] \le \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds \le \frac{1}{m_2 \max\limits_{t \ge t_1}\{v(t)\}}[v(t) - q(t)v(\tau(t))], t \ge t_1. \quad (2.10)$$

Then Eq.(1.1) has a nonoscillatory relatively bounded from above.

Proof. Let $C([t_0,+\infty),\mathbb{R})$ be the set of all continuous bounded functions with the norm $\|x\| = sup_{t \ge t_0}|x(t)|$. Then $C([t_0,+\infty),\mathbb{R})$ is a Banach space. Let Ω be a closed bounded, and convex subset of $C([t_0,+\infty),\mathbb{R})$ defined as

$$\Omega = \{x = x(t) \in C([t_0,+\infty),\mathbb{R}): N_* \le x(t) \le v(t), N_* > 0, \quad t \ge t_0\}. \quad (2.11)$$

and the two maps S_1 and $S_2: \Omega \to C([t_0,+\infty),\mathbb{R})$ defined as

$$(S_1x)(t) = \begin{cases} q(t)x(\tau(t)), & t \ge t_1, \\ (S_1x)(t_1), & t_0 \le t \le t_1, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{x}(\xi)) d\xi ds & , t \ge t_1, \\ v(t) - q(t_1)v(\tau(t_1)) & , t_0 \le t \le t_1, \end{cases} \quad (2.12)$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$.

For every $x, y \in \Omega$ and $t \ge t_1$, we obtain

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{y}(\xi)) d\xi ds \\ &\le q(t)v(\tau(t)) + m_2 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds \\ &\le q(t)v(\tau(t)) + m_2 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} v(\xi) d\xi ds \\ &\le q(t)v(\tau(t)) + m_2 \max_{t \ge t_1}\{v(t)\} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds \\ &\le q(t)v(\tau(t)) + (v(t) - q(t)v(\tau(t))) = v(t)\end{aligned}$$

Let $t \in [t_0, t_1]$, using (2.12) we get

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + v(t) - q(t_1)v(\tau(t_1)) \\ &\le q(t_1)x(\tau(t_1)) + v(t) - q(t_1)v(\tau(t_1)) \\ &= v(t).\end{aligned}$$

Furthermore, for $t \ge t_1$, we get

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{y}(\xi)) d\xi ds \\ &\ge q(t)N_* + m_1 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} y(\xi) d\xi ds \\ &\ge q(t)N_* + \frac{m_1 N_*}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds \\ &\ge q(t)N_* + N_*(1 - q(t)) \ge N_*.\end{aligned}$$

Then for $t \in [t_0, t_1]$. From equation (2.11) and (2.12), we have

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + v(t) - q(t_1)v(\tau(t_1)) \\ &\ge q(t_1)x(\tau(t_1)) + v(t) - q(t_1)v(\tau(t_1)) \\ &\ge q(t_1)N_* + N_* - q(t_1)N_* = N_*.\end{aligned}$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. We can treat the rest of the proof in similar way as in the proof of theorem (2.1). By Lemma 1.2 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$is a positive solution of (2.2). The proof is complete.

In the next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory one side relatively bounded from below by ratio function $\frac{u(t)}{a(t)}$.

Theorem2.3. Suppose that H1, H2 hold, and there exist bounded function $u \in C^1([t_0,\infty),[0,\infty))$, $\rho(t_1) \ge t_0$ such that

$$\frac{u(t_1)}{a(t_1)} \ge \frac{u(t)}{a(t)}, \quad t_0 \le t \le t_1. \quad (2.13)$$

$$\frac{a(\tau(t))u(t) - q(t)a(t)u(\tau(t))}{M_1 a(t)a(\tau(t))} \le \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds \le \frac{1}{M_2}(1 - q(t)), t \ge t_1. \quad (2.14)$$

Then Eq.(1.1) has a nonoscillatory relatively bounded from below.

Proof. Let $C([t_0,+\infty),\mathbb{R})$be the set of all continuous bounded functions with the norm $\|x\| = sup_{t \ge t_0}|x(t)|$. Then $C([t_0,+\infty),\mathbb{R})$is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0,+\infty),\mathbb{R})$as follows:

$$\Omega = \left\{x = x(t) \in C([t_0,+\infty),R): \frac{u(t)}{a(t)} \le x(t) \le N^*, N^* > 0, t \ge t_0\right\}. \quad (2.15)$$

The two maps S_1 and $S_2: \Omega \to C([t_0,+\infty),\mathbb{R})$defined as

$$(S_1x)(t) = \begin{cases} q(t)x(\tau(t)) & , t \ge t_1 \\ (S_1x)(t_1) & , t_0 \le t \le t_1 \end{cases}$$

$$(S_2x)(t) = \begin{cases} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi,\mathbf{x}(\xi)) d\xi ds & , t \ge t_1 \\ (S_2x)(t_1) & , t_0 \le t \le t_1 \end{cases} \quad (2.16)$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$.

For every $x, y \in \Omega$ and $t \geq t_1$, we obtain

$$(S_1x)(t) + (S_2y)(t) =$$

$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{y}(\xi)) d\xi ds$$

$$\leq q(t)\mathrm{N}^* + \frac{M_2}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} \mathrm{y}(\xi) d\xi ds$$

$$\leq q(t)\mathrm{N}^* + \frac{M_2\mathrm{N}^*}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$

$$\leq q(t)\mathrm{N}^* + (\mathrm{N}^* - q(t)\mathrm{N}^*) = \mathrm{N}^*.$$

For $t \in [t_0, t_1]$, we have

$$(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + (S_2y)(t_1) \leq N^*.$$

Furthermore, for $t \geq t_1$, we get

$$(S_1x)(t) + (S_2y)(t) =$$

$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{y}(\xi)) d\xi ds$$

$$\geq \frac{q(t)}{a(\tau(t))} u(\tau(t)) + M_1 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$

$$\geq \frac{q(t)}{a(\tau(t))} u(\tau(t)) + \frac{a(\tau(t))u(t) - q(t)a(t)u(\tau(t))}{a(t)a(\tau(t))}$$

$$= \frac{u(t)}{a(t)}.$$

Let $t \in [t_0, t_1]$. Using (2.13), we get

$$(S_1x)(t) + (S_2y)(t) = (S_1x)(t_1) + (S_2y)(t_1)$$

$$\geq \frac{u(t_1)}{a(t_1)} \geq \frac{u(t)}{a(t)}.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.We can treat the rest of the proof in similar way as in the proof of theorem (2.1). By Lemma 1.2 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of Eq.(1.1). The proof is complete.

In the next theorem we will give another new sufficient conditions to prove that the Eq.(1.1) has a nonoscillatory relatively bounded from below by ratio function $\frac{v(t)}{a(t)}$.

Theorem2.4. Suppose that H1, H3 hold, and there exist bounded function $v \in C^1([t_0, \infty), [0, \infty))$,$\rho\ (t_1) \geq t_0$ such that

$$\frac{1}{m_1}(1 - q(t)) \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$

$$\leq \frac{a(\tau(t))v(t) - q(t)a(t)v(\tau(t))}{m_2 \max_{t \geq t_1}\{v(t)\}\, a(t)a(\tau(t))}, t \geq t_1. \quad (2.17)$$

Then Eq.(1.1) has a nonoscillatory relatively bounded from above.

Proof. Let $C\ ([t_0, +\infty), \mathbb{R})$ be the set of all continuous bounded functions with the norm $\|x\| = sup_{t \geq t_0}|x(t)|$. Then $C([t_0, +\infty), \mathbb{R})$is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0, +\infty), \mathbb{R})$as follows:

$$\Omega = \left\{x = x(t) \in C([t_0, +\infty), R): N_* \leq x(t) \leq \frac{v(t)}{a(t)}, t \geq t_0, N_* > 0\right\}. \quad (2.18)$$

We now define two maps S_1 and $S_2: \Omega \to C([t_0, +\infty), \mathbb{R})$as follows:

$$(S_1x)(t) = \begin{cases} q(t)x(\tau(t)) & , t \geq t_1 \\ (S_1x)(t_1) & , t_0 \leq t \leq t_1 \end{cases}$$

$$(S_2x)(t) = \begin{cases} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{x}(\xi)) d\xi ds, & t \geq t_1 \\ \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))} & , t_0 \leq t \leq t_1 \end{cases} \quad (2.19)$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$.

For every $x, y \in \Omega$ and $t \geq t_1$, we obtain

$$(S_1x)(t) + (S_2y)(t) =$$

$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{y}(\xi)) d\xi ds$$

$$\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + m_2 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} \mathrm{y}(\xi) d\xi ds$$

$$\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + m_2 \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} v(\xi) d\xi ds$$

$$\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + m_2 \max_{t \geq t_1}\{v(t)\} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$

$$\leq \frac{q(t)}{a(\tau(t))} v(\tau(t)) + \frac{a(\tau(t))v(t) - q(t)a(t)v(\tau(t))}{a(t)a(\tau((t))}$$

$$= \frac{v(t)}{a(t)}.$$

For $t \in [t_0, t_1]$, we have

$$(S_1x)(t) + (S_2y)(t)$$

$$= (S_1x)(t_1) + \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))}$$

$$\leq q(t_1)x(\tau(t_1)) + \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))}$$

$$\leq q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))} + \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))} = \frac{v(t)}{a(t)}.$$

Furthermore, for $t \geq t_1$, we get

$$(S_1x)(t) + (S_2y)(t) =$$

$$= q(t)x(\tau(t)) + \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} f(\xi, \mathbf{y}(\xi)) d\xi ds$$

$$\geq q(t)N_* + \frac{M_1}{a(t)} \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$

$$\geq q(t)N_* + N_*(1 - q(t)) = N_*.$$

Let $t \in [t_0, t_1]$. From Eqs. (3.21) and (3.22), we get

$$(S_1x)(t) + (S_2y)(t) =$$

$$= (S_1x)(t_1) + \frac{v(t)}{a(t)} - q(t_1)\frac{v(t_1 - \tau(t_1))}{a(t_1 - \tau(t_1))}$$

$$= q(t_1)x(\tau(t_1)) + \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))}$$

$$\geq q(t_1)N_* + \frac{v(t)}{a(t)} - q(t_1)\frac{v(\tau(t_1))}{a(\tau(t_1))}$$
$$\geq q(t_1)N_* + \frac{v(t)}{a(t)} - q(t_1)N_* = N_*.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. We can treat the rest of the proof in similar way as in the proof of theorem 2.1. By Lemma 1.2 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$is a positive solution of Eq.(1.1). The proof is complete.

Example2.5. Consider the following nonlinear Neutral differential equation

$$\frac{d}{dx}\left(\frac{d}{dx}(a(t)(x(t) - q(t)x(t-2))\right)$$
$$- p(t)\left(\frac{2}{t} + 1\right) = 0. \qquad (2.20)$$

Where $a(t) = e^{-1.5t}$, $p(t) = \frac{1}{8}e^{-2t}$, and $q(t) = 0.3$, Let $u(t) = e^{-0.5t}$,$1 \leq f(t, \mathbf{x}(t)) = \frac{2}{t} + 1 \leq 3, t \geq t_1 = 1$.

Solution: It is clear that condition (2.1) holds, since

$$e^{-0.5t} \leq e^{-0.5(t_1-2)}, t_0 \leq t \leq t_1$$

To show condition (2.2) oftheorem (2.1) verified:

$$\frac{1}{M_1}\left(u(t) - q(t)u(\tau(t))\right) \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$
$$\leq \frac{1}{M_2}(1 - q(t)), \qquad t \geq 1,$$

Let $R_1(t) = \frac{1}{M_1}(u(t) - q(t)u(\tau(t)))$, $R_2(t) = 0.5e^{-0.5t}$, and

$$R_3(t) = \frac{1}{M_2}(1 - q(t)).$$

Then $R_1(t) \leq R_2(t) \leq R_3(t)$, for $t \geq 1$ so all conditions of Theorem 2.1 hold, by Theorem there exists a positive solution of Eq.(2.20).

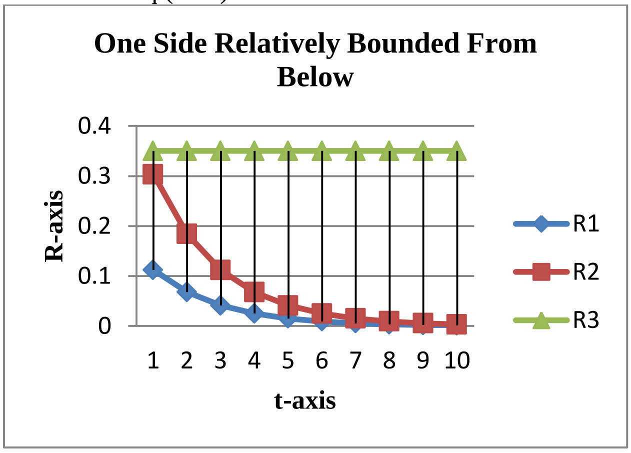

Figure 2.1: Thegraph of $R_1(t), R_2(t)$, and$R_3(t)$, of theorem (2.1)

Example 2.6. Consider the following nonlinear Neutral differential equation

$$\frac{d}{dt}\left(\frac{d}{dt}\left(a(t)(x(t) - q(t)x(t-1))\right)\right) -$$
$$p(t)\left(\frac{1}{2}\sin t + 1.5\right) = 0, \quad t \geq 0, \quad (2.21)$$

where $a(t) = \frac{-(10+5t)}{3}$, $p(t) = \frac{2}{(2+t)^2}$, $q(t) = 0.5$, and $v(t) = 2 - e^{-t}$,$1 \leq f(t, \mathbf{x}(t)) = \frac{1}{2}\sin t + 1.5 \leq 2$, $t_1 = 2$.

Solution. To show condition (2.10) of theorem (2.2) verified:

$$\frac{1}{m_1}[1 - q(t)] \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$
$$\leq \frac{[v(t) - q(t)v(\tau(t))]}{m_2 \max_{t\geq t_1}\{v(t)\}}, t \geq t_1.$$

Let $R_1(t) = \frac{1}{m_1}[1 - q(t)]$, $R_2(t) = \frac{1+0.8t}{2+t}$ and

$$R_3(t) = \frac{[v(t) - q(t)v(\tau(t))]}{m_2 \max_{t\geq t_1}\{v(t)\}}.$$

Then $R_1(t) \leq R_2(t) \leq R_3(t)$, for $t \geq 1$ so all conditions of Theorem 2.2 hold, by Theorem there exists a positive solution of Eq.(2.21).

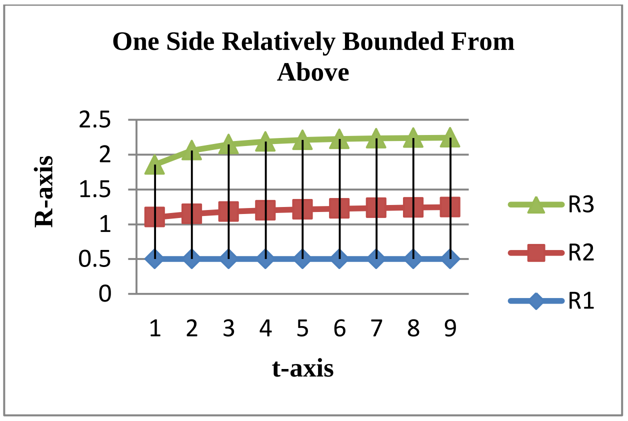

Figure 2.2: Thegraph of $R_1(t), R_2(t)$, and $R_3(t)$, of theorem (2.2)

Example 2.7. Consider the following nonlinear Neutral differential equation

$$\frac{d}{dt}\left(\frac{d}{dt}\left(a(t)(x(t) - q(t)x(t-1))\right)\right)$$
$$- p(t)(\cos t + 1) = 0, \quad t \geq 0, \qquad (2.22)$$

Where$a(t) = 2^{-1.5t}$, $p(t) = 2^{-3t-3}$, and $q(t) = 0.5$, and $u(t) = 2^{-0.5t}$,$1 \leq f(t, \mathbf{x}(t)) = \cos t + 1 \leq 2$, $t_1 = 2$.

Solution: It is clear thatcondition (2.13)holds, since

$$\frac{u(t_1)}{a(t_1)} - \frac{u(t)}{a(t)} = \frac{2^{-0.5t_1}}{2^{-1.5t_1}} - \frac{2^{-0.5t}}{2^{-1.5t}} \geq 0 , 0 \leq t \leq 2.$$

To show condition (2.14) of theorem (2.3) verified:

$$\frac{a(\tau(t))u(t) - q(t)a(t)u(\tau(t))}{M_1 a(t)a(t-\tau)} \leq \int_t^\infty \int_s^\infty \frac{p(\xi)}{a(\xi)} d\xi ds$$
$$\leq \frac{1}{M_2}(1 - q(t)), t \geq t_1.$$

$$\text{Let } R_1(t) = \frac{a(\tau(t))u(t) - q(t)a(t)u(\tau(t))}{M_1 a(t)a(\tau(t))},$$

$$R_2(\mathrm{t}) = \frac{2^{-0.5t-3}}{\ln(2)}$$

and

$$R_3(\mathrm{t}) = \frac{1}{M_2}\big(1 - q(t)\big).$$

Then $R_1(\mathrm{t}) \le R_2(\mathrm{t}) \le R_3(\mathrm{t})$, for $t \ge 1$ so all conditions of Theorem 2.3 hold, by Theorem there exists a positive solution of Eq.(2.22).

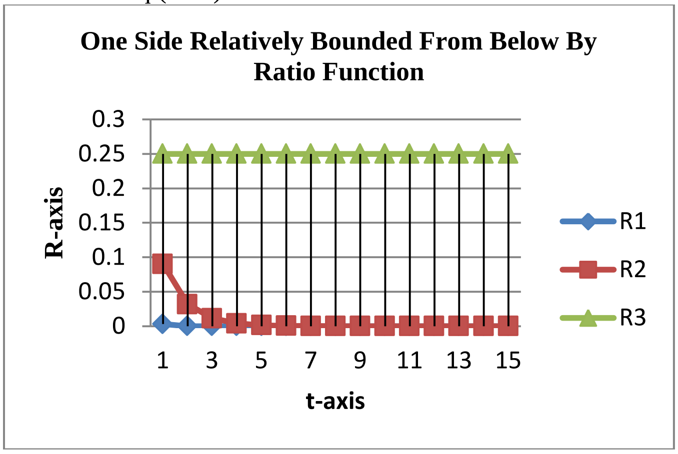

Figure 2.3: Thegraph of $R_1(\mathrm{t}), R_2(\mathrm{t})$, and $R_3(\mathrm{t})$, of theorem (2.3).

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وجود الحل المقيد النسبي الغير متذبذب للمعادلات التفاضلية المحايدة من الرتبة الثانية

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المستخلص:
في هذا البحث حصلنا على الشروط الكافية لوجود الحل الموجب المقيد النسبي من جهة واحدة للمعادلات التفاضلية المحايدة من الرتبة الثانية.
استخدمنا مبرهنة النقطة الثابته ل (Krasnoselskii)
ومبرهنة التقارب المهيمنه لليبيك للحصول على شروط كافية لوجود الحلول النسبية المقيدة من جهة واحدة. هذه الشروط اكثر تطبيق من النتائج المعروفة في المصادر. ثلاثة امثلة لبرهنة النتائج التي حصلنا عليها.