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On Sandwich Theorems for Certain Univalent Functions Defined by a New operator

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Abstract:

In this paper, we study some differential subordination and superordination results for certain univalent functions in the open unit disc U by using a new operator $\mathfrak{f}_{s,\alpha,\mu}^{\lambda}$. Also, we derive some sandwich theorems.

Keywords: Analytic function, Differential Subordination, Hadamard Product, Univalentfunction.

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1 . Introduction

Denote by $\mathcal{H} = \mathcal{H}(U)$ the class of analytic functions in the open unit disk $U = \{z: z \in \mathbb{C}$ and$|z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}[a,n]$ be the subclass of the function $f \in \mathcal{H}$ of the from:

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}, n \in \mathcal{N} = \{1,2,3,\ldots\})$.(1.1)

Also ,Let T be the subclass of $\mathcal{H}$consisting of functions of the form:

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (1.2)

If $f \in T$ is given by (1.2) and $g \in T$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_n z^n.$$

The Hadamard product (or the convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).$$

If f and g are analytic functions in U, We say that f is subordination to g.

Let l , $h \in \mathcal{H}$, and $\phi(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$.

If $l$and $\phi(l(z), zl'(z), z^2 l''(z); z)$are univalent functions in U and if l satisfies the second- order superordination:

$$h(z) \prec \phi(l(z), zl'(z), z^2 l''(z); z), (z \in U)$$

then l is called a solution of the differential superordination(1.2), (if f subordinate to g, then g is superordinate to f).

An analytic function q is called subordinate of the differential superordination if $q \prec k$ for all l satisfying (1.3). A univalent subordinate $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinats q of (1.3)is said to be the best subordinat. Recently, Miller and Mocanu [9] obtained sufficient conditions on the functions(h,k) and ϕ for which the following implication holds:

$h(z) \prec \phi(l(z), zl'(z), z^2 l''(z); z) \Longrightarrow q(z) \prec l(z)$, $(z \in U)$. (1.4)

Using the results, Bulboacă [5] considered certain classes of first order differential superordinations as well as superordinationpreserving integral operator [6]. Ali et al. [1], have used the results of Bulboacă[5] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [13] obtained a sufficient conditions for starlikeness of f in terms of the quantity

$$\frac{f''(z)f(z)}{(f'(z))^2}.$$

Recently, Shanmugam et al. [11,12] and Goyal et al. [7],Atshan and Hiress[2] ,Atshan and Kazim[4] , Atshan and Jawad [3], Wanas and Majeed [14] , also obtained sandwich results for certain classes of analytic functions.

Komatu [8] introduced and investigated a family of integral operator

$\mathfrak{I}_\mu^\lambda: T \to T$

that is obtain as follows:

$$\mathfrak{I}_\mu^\lambda \ f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\mu}{\mu + n - 1}\right)^n a_n z^n,$$

$(z \in U^*, \mu > 1, \lambda \geq 0)$. (1.5)

The Hurwitiz - Lerch zeta function

$\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(1+a)^s}$, $r \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $0 < |z| < 1$.

In terms of (Hadamard) product (or convolution) where $G_{s,a(z)}$ is given by

$$G_{s,a(z)} = (1+a)^s [\Phi(z,s,a) - a^{-s}], (z \in U).$$

Definition $(2.1.1)$:Let $f \in T, z \in U^*, a \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ and $\lambda > 1$,we define a new operator $\mathfrak{f}_{s,a,\mu}^\lambda f(z): T \longrightarrow T$, where

$$\mathfrak{f}_{s,a,\mu}^\lambda f(z) = G_{s,a}(z) * \mathfrak{I}_\mu^\lambda f(z)$$

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$$= z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s \left(\frac{\mu}{\mu+n-1}\right)^\lambda a_n z^n \quad (1.6)$$

We note from (1.6) that

$$z\left(\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)\right)' = \mu \mathfrak{f}_{s,a,\mu}^{\lambda} f(z) - (\mu - 1)\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) \quad (1.7)$$

The specific aim of this document is to find sufficient-conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho + \xi)z}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

and

$$q_1(z) \prec \left(\frac{\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)}{z}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with$q_1(0) = q_2(0) = 1$.

2 . Preliminaries

In order to establish our subordination and superordination results, that require the following lemmas and definitions.

Definition $(2.1)[6]$: Denote by Q the class of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $\overline{U} = U \cup \{z \in \partial U\}$, and $E(q) = \{\zeta \in \partial U : lim_{z\to\zeta} f(z) = \infty\}$and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further,let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) = Q_0$and $Q(1) = Q_1 = \{q \in Q : q(0) = 1\}$.

Lemma $(2.1)[1]$: Let q(z) be convex univalentfunction in U, let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\text{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\text{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If $l(z)$ is analytic in U and

$$\alpha l(z) + \beta z l'(z) \prec \alpha q(z) + \beta z q'(z),$$

then $l(z) \prec q(z)$ and q is the best dominant.

Lemma $(2.2)[3]$: Let q be univalent in U and let $\emptyset$ and θ be analytic in the domain D containing $q(U)$ with $\emptyset(w) \neq 0$, when $w \in q(U)$. Set

$$Q(z) = zq'(z)\emptyset(q(z)) \text{and} h(z) = \theta(q(z)) + Q(z),$$

suppose that

1) Q is starlike univalent in U,

$$\text{2) Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0, z \in U.$$

If l is analytic in U with $l(0) = q(0)$, $l(U) \subseteq D$ and

$$\emptyset(l(z)) + zl'(z)\emptyset(l(z)) \prec \emptyset(q(z)) + zq'(z)\emptyset(q(z)),$$

then$l(z) \prec q(z)$,and q is the best dominant.

Lemma $(2.3)[6]$:Let $q(z)$ be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

$$\text{1) Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0 \text{ for } z \in U,$$

2) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in $z \in U$.

If $l \in \mathcal{H}[q(0), 1] \cap Q$, with $l(U) \subseteq D$, and $\theta(l(z)) + zl'(z)\phi(l(z))$is univalent in U, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(l(z)) + zl'(z)\phi(l(z)),$$

then $q(z) \prec l(z)$, and q is the best subordinant.

Lemma $(2.4)[6]$: Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\text{Re}\{\beta\} > 0$. If $l(z) \in \mathcal{H}[q(0), 1] \cap Q$and $l(z) + \beta z l'(z)$is univalent in U, then

$$q(z) + \beta z q'(z) \prec l(z) + \beta z l'(z),$$

which implies that $q(z) \prec l(z)$ and $q(z)$ is the best subordinant .

3 . Subordination Results

Theorem(3.1):Let $q(z)$ be convex univalent in $U$with $q(0) = 1, 0 < \delta < 1, \eta, \in \mathbb{C}\backslash\{0\}$. Suppose that

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$$\text{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\text{Re}\left(\frac{1}{\delta\eta}\right)\right\}. \quad (3.1)$$

If $f \in T$ is satisfies the subordination

$$l(z) \prec q(z) + \delta\eta\, zq'(z), \quad (3.2)$$

where

$$l(z) = \left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}}\left(1 + \eta\left(\frac{\rho\left[\mu\mathfrak{f}^{\lambda}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z)\right] + \xi\left[\mu\mathfrak{f}^{\lambda-1}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda}_{s,a,m}f(z)\right]}{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}\right)\right) \quad (3.3)$$

then

$$\left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

Proof :consider a function $l(z)$ by

$$l(z) = \left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \times \left(1 + \eta\left(\frac{\rho\left[\mu\mathfrak{f}^{\lambda}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z)\right] + \xi\left[\mu\mathfrak{f}^{\lambda-1}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)\right]}{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}\right)\right), \quad (3.5)$$

then the function $q(z)$ is analytic in U and $q(0)$=1, therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.7)in the resulting equation,

$$l(z) = \left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \times \left(1 + \eta\left(\frac{\rho\left[\mu\mathfrak{f}^{\lambda}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z)\right] + \xi\left[\mu\mathfrak{f}^{\lambda-1}_{s,a,\mu}f(z) - (\mu-1)\mathfrak{f}^{\lambda}_{s,a,m}f(z)\right]}{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}\right)\right).$$

Thus the subordination (3.2) is equivalent to

$$l(z) + \delta\eta\, zl'(z) \prec q(z) + \delta\eta\, zq'(z)$$

An application of Lemma (2.1)with $\beta = \delta\eta$ and $\alpha = 1$, we obtain(3.4).

Taking$q(z) = \frac{1+Az}{1+Bz}(-1 \le B < A \le 1)$, in Theorem(3.1),we obtain the following corollary.

Corollary (3.2):Let $0 < \delta < 1$, $\eta \in \mathbb{C}\backslash\{0\}$ and ($-1 \le B < A \le 1$). Suppose that

$$\text{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\text{Re}\left(\frac{1}{\delta\eta}\right)\right\}.$$

If $f \in T$ is satisfy the following subordination condition:

$$l(z) \prec \frac{1+Az}{1+Bz} + \delta\eta\frac{(A-B)z}{(1+Bz)^2},$$

where$l(z)$given by (3.3), then

$$\left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz},$$

and$\frac{1+Az}{1+Bz}$is best dominant.

Taking $A = 1$ and $B = -1$ in corollary (3.2), we get following result.

Corollary (3.3): Let $0 < \delta < 1, \eta \in \mathbb{C}\backslash\{0\}$ and suppose that

$$\text{Re}\left(\frac{1+z}{1-z}\right) > \max\{0, -\text{Re}(\delta\eta)\}.$$

If $f \in T$ is satisfy the following subordination

$$l(z) \prec \frac{1+z}{1-z} + \delta\eta\frac{2z}{(1-z)^2},$$

where$l(z)$ given by (3.3), then

$$\left(\frac{\rho\mathfrak{f}^{\lambda+1}_{s,a,\mu}f(z) + \xi\mathfrak{f}^{\lambda}_{s,a,\mu}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is best dominant.

Theorem (3.4):Let $q(z)$ be convex univalent in unit disk $U$with $q(0) = 1$, let $0 < \delta < 1$, $\eta \in$

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$\mathbb{C}\backslash\{0\}, u, \nu, \varepsilon, \alpha \in \mathbb{C}, f \in T$ and suppose that f and q satisfy the following conditions

$$\operatorname{Re}\left\{\frac{\nu}{\eta}q(z) + \frac{2\varepsilon}{\eta}[q(z)]^2 + 1 + \frac{3\alpha}{\eta}[q(z)]^3 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right\} > 0, \qquad (3.6)$$

and

$$\left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \neq 0. \qquad (3.7)$$

If

$$r(z) \prec u + \nu q(z) + \varepsilon[q(z)]^2 + \alpha[q(z)]^3 + \eta\frac{zq'(z)}{q(z)}, \qquad (3.8)$$

where

$$r(z) = u + \nu\left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} + \varepsilon\left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{2\delta}} + \alpha\left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{3\delta}} + \frac{1}{\eta\delta}\left[\frac{\rho\left(\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)\right)'+\xi\left(\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)\right)'}{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)} - 1\right] \qquad (3.9)$$

then

$$\left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec q(z),$$ and $q(z)$ is best dominant.

Proof :consider a function $l(z)$ by

$$l(z) = \left(\frac{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}}. \qquad (3.10)$$

Then the function p is analytic in U and $l(0) = 1$, differentiating (3.10) logarithmically with respect to z, we get

$$\frac{zl'(z)}{l(z)} = \frac{1}{\delta}\left[\frac{\rho\left(\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)\right)'+\xi\left(\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)\right)'}{\rho\mathfrak{I}_{s,a,\mu}^{\lambda+1}f(z)+\xi\mathfrak{I}_{s,a,\mu}^{\lambda}f(z)} - 1\right]. (3.11)$$

By setting $\theta(w) = u + \nu w + \varepsilon w^2 + \alpha w^3$ and $\phi(w) = \frac{\eta}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$is analytic in$\mathbb{C}\backslash\{0\}$and that $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \eta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z)$$

$$= u + \nu q(z) + \varepsilon[q(z)]^2 + \alpha[q(z)]^3 + \eta\frac{zq'(z)}{q(z)},$$

It is observe that $Q(z)$ is starlike univalent in U, we have

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\nu}{\eta}q(z) + \frac{2\varepsilon}{\eta}[q(z)]^2 + \frac{3\alpha}{\eta}[q(z)]^3 + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

By Making use of (2.2),we obtain

$$\nu l(z) + \varepsilon[l(z)]^2 + \alpha[l(z)]^3 \prec \nu q(z) + \varepsilon[q(z)]^2 + \alpha[q(z)]^3 + \eta\frac{zq'(z)}{q(z)}$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that

$l(z) \prec q(z)$and the function $q(z)$is the best dominant.

Taking the function$q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, in Theorem (3.4) for every $\eta \in \mathbb{C}\backslash\{0\}$ the condition(3.6)becomes

$$\operatorname{Re}\left(\frac{\nu}{\eta}\frac{1+Az}{1+Bz} + \frac{2\varepsilon}{\eta}\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{2\alpha}{\eta}\left(\frac{1+Az}{1-Bz}\right)^3 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz}\right) > 0, (3.12)$$

hence, we have the following corollary.

Corollary(3.5):Let$(-1 \leq B < A \leq 1)$, $0 < \delta < 1$, $\eta \in \mathbb{C}\backslash\{0\}$, $u, \nu, \varepsilon, \alpha \in \mathbb{C}$.
Assume that (3.12) holds.

If $f \in T$ and

$$r(z) \prec u + \nu\frac{1+Az}{1+Bz} + \varepsilon\left(\frac{1+Az}{1+Bz}\right)^2 + \alpha\left(\frac{1+Az}{1+Bz}\right)^3 + \eta\frac{(A-B)z}{(1+Bz)(1+Az)},$$

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where $r(z)$ is defined in (3.9), then

$\left(\frac{\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)}{z}\right)^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is best dominant.

Taking the function $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$

$(0 < \rho \leq 1)$, in Theorem(3.4), the condition

(2.12)becomes

$$\operatorname{Re}\left\{\frac{v}{\eta}\left(\frac{1+z}{1-z}\right)^{\rho} + \frac{2\varepsilon}{\eta}\left(\frac{1+z}{1-z}\right)^{2\rho} + \frac{\alpha}{\eta}\left(\frac{1+z}{1-z}\right)^{3\rho} + \frac{2z^2}{1-z^2}\right\} > 0 \quad (\eta \in \mathbb{C}\backslash\{0\}), \qquad (3.13)$$

hence, we have the following corollary.

Corollary (3.6):Let $0 < \rho \leq 1, 0 < \delta < 1, \eta \in \mathbb{C}\backslash\{0\}, u,, v, \varepsilon, \alpha \in \mathbb{C}$. Assume that (3.13) holds.If $f \in T$ and

$$r(z) \prec u + v\left(\frac{1+z}{1-z}\right)^{\rho} + \varepsilon\left(\frac{1+z}{1-z}\right)^{2\rho} + \alpha\left(\frac{1+z}{1-z}\right)^{3\rho} + \eta\frac{2\rho z}{1-z^2}, r(z) =$$

where $r(z)$ is defined in (3.9), then

$\left(\frac{\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)}{z}\right)^{\frac{1}{\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$, and $\left(\frac{1+z}{1-z}\right)^{\rho}$ is best dominant.

4. Superordination Results

Theorem (4.1):Let $q(z)$ be convex univalent in U with $q(z) = 1$, $0 < \delta < 1, \eta \in \mathbb{C}$ with $\operatorname{Re}(\eta) > 0$, if $f \in T$, such that

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \neq 0$$

and suppose that f satisfies the condition:

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \in \mathcal{H}[q(0), 1] \cap Q. \quad (4.1)$$

If the function $l(z)$ given by (3.3) is univalent and the following superordination condition:

$$q(z) + \delta\eta\, zq'(z) \prec l(z), \quad (4.2)$$

holds, then

$$q(z) \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \quad (4.3)$$

and $q(z)$ is the best subordinant.

Proof : Consider the analytic function $l(z)$ by

$$l(z) = \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}}. \quad (4.4)$$

Differentiate Euquation(4.4) with the respect to z logarithmically, we get

$$\frac{zl'(z)}{l(z)} = \frac{1}{\delta}\left(\frac{\rho z\left(\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)\right)' + \xi z\left(\mathfrak{f}_{s,a,\mu}^{\lambda} f(z)\right)'}{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda}(z)}\right) \quad (4.5)$$

A simple computation and using (1.6), from (4.5), we get

$$l(z) = \left(\frac{\rho \mathfrak{f}_{a,\mu}^{s,\lambda+1} f(z) + \xi \mathfrak{f}_{a,\mu}^{s,\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \times \left(1 + \eta\left(\frac{\rho\left[\mu \mathfrak{f}_{a,\mu}^{s,\lambda+1} f(z) - (\mu-1)\mathfrak{f}_{a,\mu}^{s,\lambda} f(z)\right] + \xi\left[\mu \mathfrak{f}_{a,\mu}^{s,\lambda} f(z) - (\mu-1)\mathfrak{f}_{a,m}^{s,\lambda-}\right.}{\rho \mathfrak{f}_{a,\mu}^{s,\lambda+1} f(z) + \xi \mathfrak{f}_{a,\mu}^{s,\lambda} f(z)}\right.\right.$$

$$= l(z) + \delta\eta\, zl'(z),$$

now, by using Lemma (2.4), we get the desired result.

Taking $q(z) = \frac{1+Az}{1+Bz}(-1 \leq B < A \leq 1)$, in Theorem (4.1), we get thefollowing corollary.

Corollary (4.2):Let $\operatorname{Re}\{\eta\} > 0, 0 < \delta < 1$ and $-1 \leq B < A \leq 1$, such that

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \in \mathcal{H}[q(0), 1] \cap Q.$$

If the function $l(z)$ given by (3.3) is univalent in U and $f \in T$ satisfies the following superordination condition:

$$\frac{1+Az}{1+Bz} + \delta\eta\frac{(A-B)z}{(1+Bz)^2} \prec l(z), \text{ then}$$

$$\frac{1+Az}{1+Bz} \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}},$$

and the function $\frac{1+Az}{1+Bz}$is the bestsubordinant.

Theorem $(\mathbf{4.3})$:Let $q(z)$ be convex univalent in unit disk U, let $\eta \in \mathbb{C}\backslash\{0\}, 0<\delta<1, u, v, \varepsilon \in \mathbb{C}, q(z) \neq 0$, and $f \in T$. Suppose that

$$\operatorname{Re}\left\{(v+2\varepsilon q(z)+3\alpha q(z)) \frac{q(z) q'(z)}{\eta}\right\}>0,$$

Let $f(z) \in T$ and suppose that satisfies the next condition:

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \in \mathcal{H}[q(0),1] \cap Q, (4.6)$$

and

$$\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z} \neq 0.$$

If the function $r(z)$is given by (3.9) is univalent in U, and

$$u+v q(z)+\varepsilon[q(z)]^2+\alpha[q(z)]^3+\eta \frac{z q'(z)}{q(z)} \prec r(z), \quad (4.7)$$

implies

$$q(z) \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}}$$

and $q(z)$ is the best subordinant.

Proof : Let the function $A(z)$ defined on U by (3.14). Then a computation show that

$$\frac{z l'(z)}{l(z)}=\frac{1}{\delta}\left[\frac{\rho\left(\mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)\right)'+\xi\left(\mathfrak{f}_{s,a,\mu}^{\lambda} f(z)\right)'}{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}-1\right] (4.8)$$

By setting $\theta(w)=u+v w+\varepsilon w^2+\alpha w^3$ and $\phi(w)=\frac{\eta}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$is analytic in$\mathbb{C}\backslash\{0\}$and that $\phi(w) \neq 0\ (w \in \mathbb{C}\backslash\{0\})$. Also, we get $Q(z)=z q'(z) \phi(q(z))=\eta \frac{z q'(z)}{q(z)}$, it observed that$Q(z)$is starlike univalent in U.

Since $q(z)$ is convex, it follows that

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=\operatorname{Re}\left\{\frac{q(z)}{\eta}(2\varepsilon q(z)+3\alpha[q(z)]^2+v)\right\} q'(z)>0.$$

By making use of (4.8)the hypothesis (4.7) can be equivalently written as

$$\theta(q(z)+z q'(z) \phi(q(z))) = \theta(l(z)+z l'(z) \phi(A(z))),$$

thus, by applying Lemma (2.3), the proof is completed.

5. Sandwich Results

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich Theorem :

Theorem $(\mathbf{5.1})$:Let q_1 and $q_2$be convex univalent in $U$with $q_1(0)=q_2(0)=1$ and q_2 satisfies (3.1). Suppose that Re$\{\eta\}>0, 0<\delta<1, \eta \in \mathbb{C}\backslash\{0\}$.

If $f \in T$, such that

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \in \mathcal{H}[q(0),1] \cap Q,$$

and the function $l(z)$defined by (3.3) is univalent and satisfies

$$q_1(z)+\delta \eta z q_1'(z) \prec l(z) \prec q_2(z)+\delta \eta z q_2'(z), (5.1)$$

implies that

$$q_1(z) \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z)+\xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

where $q_1$andq_2 are, respectively , the best subordinant and the best dominant of (5.1).

Combining Theorem (3.4) with Theorem (4.3), we obtain the following sandwich Theorem.

Theorem $(\mathbf{5.2})$:Let q_i be two convex univalent functions in U, such that $q_i(0)=1$, $q_i(0) \neq 0 (i=1,2)$. Suppose that $q_1$andq_2 satisfies (4.8)and(3.8), respectively.

If $f \in T$ and suppose that f satisfies the next conditions:

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \neq 0$$

and

$$\left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \in \mathcal{H}[q(1),1] \cap Q,$$

and $r(z)$is univalent in U, then

$$\mathrm{u} + \nu\, q_1(z) + \varepsilon\, [q_1(z)]^2 + \alpha[q_1(z)]^3 + \eta \frac{zq_1'(z)}{q_1(z)} \prec r(z) \prec \mathrm{u} + \nu\, q_2(z) + \alpha[q_2(z)]^3 + \eta \frac{zq_2'(z)}{q_2(z)} \quad (5.2)$$

implies

$$q_2(z) \prec \left(\frac{\rho \mathfrak{f}_{s,a,\mu}^{\lambda+1} f(z) + \xi \mathfrak{f}_{s,a,\mu}^{\lambda} f(z)}{(\rho+\xi)z}\right)^{\frac{1}{\delta}} \prec q_2(z),$$

and $q_1$and$q_2$are the best subordinant and the best dominant respectively and $r(z)$ is given by (3.9).

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حول مبرهنات الساندوج للدوال الاحادية التكافؤ الاكيدة والمعرفة بواسطة مؤثر جديد

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المستخلص:

في هذا البحث درسنا بعض نتائج التبعية التفاضلية العليا للدوال احادية التكافؤ الاكيدةفي قرص الوحدة المفتوح باستخدام مؤثر جديد $\mathfrak{I}_{s,a,\mu}^{\lambda}$ اشتقينا ايضا بعض مبرهنات الساندوج.