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Using inverse triangular and hyperbolic functions of Al-Tememe acceleration methods of first kind for improving the numerical integration results

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the inverse sine triangular acceleration rule for Al-Tememe of the first kind. We refer to it by $(A^F{}_{\sin^{-1}})$. the inverse cosine triangular acceleration rule for Al-Tememe of the first kind, We refer to it by $(A^F{}_{\cos^{-1}})$.

ABSTRACT

The main aim of this work is to introduce the acceleration methods which are called the inverse triangular acceleration methods and inverse hyperbolic acceleration methods, which are considered a series of numerated methods. In general, these methods are named as AL-Tememe's acceleration methods of first kind discovered by (Ali Hassan Mohammed). They are very beneficial to acceleration the numerical results for definite integrations with continuous integrands which are of 2nd order main error, with respect to the accuracy and the number of the used subintervals and the speed of obtaining results. Especially, for accelerating the results which are obviously obtained by trapezoidal and midpoint methods. Moreover, these methods could be enhancing the results of numerical of the ordinary differential equations, where the main errors are of 2nd order.

MSC.

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1 . Introduction

There are several numerical methods used for evaluating definite single integrals such as:

1) Trapezoidal Rule
2) Midpoint Rule
3) Simpson's Rule

which are called "Newton-cotes formulas".

In this paper, we introduce two methods which are trapezoidal and midpoint methods for finding approximate values of single Integrals which their integrands are continuous on the interval of integration using triangular acceleration methods which are part of a series of AL-Tememe's acceleration methods of first kind. We will make a comparison between these methods regarding to the accuracy and the speed of reaching to the approximate values, which are closed to the real values (analytic) for those integrals.

Consider the integral J defined as:

$$J=\int_{x_0}^{x_m} f(x)dx \qquad \dots (1)$$

such that, f(x) is a continuous function defined on $[x_0,x_m]$. We need to calculate the integral J approximately. In general we can write Newton-cotes formula as:

$$J=\int_{x_0}^{x_m} f(x)dx = f(x)dx = G(h) + E_{G(h)} + R_G \qquad \dots (2)$$

Here, G(h) is the Lagrangian approximation to the value of the integral J, (the letter G symbolizes the rule type), E_G is the remainder and related to amputation after the use of certain limits of $E_G(h)$ and $h=\frac{x_m-x_0}{m}$; m is the number of sub interval used and the general form of G(h) where

$$G(h) = h(w_0f_0 + w_1f_1 + w_2f_2 + \dots + w_2f_{m-2} + w_1f_{m-1} + w_0f_m) \qquad \dots (3)$$

where $f_r = f(x_r)$ and $x_r = x_0 + rh$; r = 0,1,2,…,m and the weight coefficients w_r take the sequence $(w_0, w_1, w_2, \dots, w_2, w_1, w_0)$. To simplify the formula (3), we write weights by w_0 such that $w_1=2(1-w_0)$, $w_2=2w_0$, we note that when $w_0 = \frac{1}{2}$ we get the trapezoidal rule and refer for G (h) by T(h) where $T(h) = \frac{h}{2}(f_0+2f_1+2f_2+\dots+2f_{m-1}+f_m)$. When $w_0=0$, we get the midpoint rule and we refer to it by M(h) where $M(h)=h(f_1+f_3+\dots+f_{2i-1})$; i=1,2,…,m. The general formula of $E_G(h)$ is the following:

1- For Trapezoidal rule:

$$E_T(h) = \frac{-1}{12}h^2(f_m'-f_0') + \frac{1}{720}h^4\left(f_m^{(3)} - f_0^{(3)}\right) - \frac{1}{30240}h^6\left(f_m^{(5)} - f_0^{(5)}\right) + \cdots \qquad \dots (4)$$

2- Midpoint rule:

$$E_M(h) = \frac{1}{6}h^2(f_m' - f_0') - \frac{7}{360}h^4\left(f_m^{(3)} - f_0^{(3)}\right) + \frac{31}{15120}h^6\left(f_m^{(5)} - f_0^{(5)}\right) - \dots \qquad \dots (5)$$

In these methods when the integrands is a continuous function and their derivatives are defined at each point of integration points interval $[x_0, x_m]$ the error formula can be written as:

$$\left.\begin{array}{l} J - T(h) = A_1h^2 + A_2h^4 + A_3h^6 + \cdots \\ J - M(h) = B_1h^2 + B_2h^4 + B_3h^6 + \cdots \end{array}\right] \qquad \dots (6)$$

where $A_1, A_2, A_3, \dots$ and $B_1, B_2, B_3, \dots$ are constants that do not depend on h, but they depend on the values of their derivatives at the end of integration interval.

2. Derivation of Al-Tememe's acceleration of inverse triangular and inverse hyperbolic Functions of the first kind:

A series of acceleration methods of Al-Tememe's are introduced and we will call it the inverse triangular and the inverse hyperbolic accelerations. Due to the similarity in of both trapezoidal and midpoint methods regarding h basics, we will deal with the error for trapezoidal method to derive our acceleration methods following the same way to derive these methods as for the midpoint method.

$$J=\int_{x_0}^{x_m} f(x)dx = h\left[\frac{1}{2}f_0+f_1+f_2+\cdots\ldots.+f_{m-1}+\frac{1}{2}f_m\right]+E(h) \qquad \ldots(7)$$

where

$E(h) = A_1h^2+A_2h^4+\ldots$; $A_1, A_2, \ldots$ are constants ...(8)

$=h(A_1h+A_2h^3+A_3h^5+\ldots) \cong h\sin^{-1}h$ ...(9)

Since, $\sin^{-1}h = h+\frac{h^3}{6}+\frac{3h^5}{40}+\cdots$ [3]

So ,we can write

$$J\cong \frac{h}{2}[f_0+2f_1+\cdots+2f_{m-1}+f_m]+h\sin^{-1}h \qquad \ldots(10)$$

Therefore,

$$E=J-T(h)\cong h\sin^{-1}h \qquad \ldots(11)$$

We assumed that $T(h_1)$ represents the value of the above mentioned numerical integration numerically when $h=h_1$, also, $T(h_2)$ represents the value of numerical integration when $h=h_2$, So;

$$J-T(h_1)\cong h_1\sin^{-1}h_1 \qquad \ldots(12)$$

$$J-T(h_2)\cong h_2\sin^{-1}h_2 \qquad \ldots(13)$$

From equations (12) and (13) we get:

$$A^F_{\sin^{-1}}\cong \frac{h_2\sin^{-1}h_2T(h_1)-h_1\sin^{-1}h_1T(h_2)}{h_2\sin^{-1}h_2-h_1\sin^{-1}h_1} \qquad \ldots(14)$$

The formula (14) is called the inverse sine triangular acceleration rule for Al-Tememe of the first kind. We refer to it by ($A^F_{\sin^{-1}}$). Also it is possible to write the equation (11) by using the formula:

$E= J-T(h)\cong h(\frac{\pi}{2}-\sin^{-1}h)$; $\cos^{-1}h=\frac{\pi}{2}-h-\frac{h^3}{6}-\frac{3h^5}{40}-\ldots$ [3]

By following the same steps as above method, we get;

$$A^F_{\cos^{-1}}\cong \frac{h_2(\frac{\pi}{2}-\sin^{-1}h_2)\,T(h_1)-h_1(\frac{\pi}{2}-\sin^{-1}h_1)\,T(h_2)}{h_2(\frac{\pi}{2}-\sin^{-1}h_2)-h_1(\frac{\pi}{2}-\sin^{-1}h_1)} \qquad \ldots(15)$$

The formula (15) is called the inverse cosine triangular acceleration rule for Al-Tememe of the first kind, We refer to it by ($A^F_{\cos^{-1}}$). If we used the rule in the formula (15), we get the correct value for 7 decimal when n=100, but we expect [4] the inverse cosine triangular acceleration rule, take the following formula:

$$A^F_{\cos^{-1}}\cong \frac{h_2{}^2\cos^{-1}h_2T(h_1)-h_1{}^2\cos^{-1}h_1T(h_2)}{h_2{}^2\cos^{-1}h_2-h_1{}^2\cos^{-1}h_1} \qquad \ldots(16)$$

In the same way, we can conclude the third triangular acceleration inverse rule that we call inverse tangent triangular acceleration rule, which is referred to as $(A^F{}_{\tan^{-1}})$:

$$A^F{}_{\tan^{-1}} \cong \frac{h_2 \tan^{-1} h_2 T(h_1) - h_1 \tan^{-1} h_1 T(h_2)}{h_2 \tan^{-1} h_2 - h_1 \tan^{-1} h_1} \quad ...(17)$$

So that, it is possible to write the equation (11) in the following formula:

$J\text{-}T(h)=h \tan^{-1}h$; $\tan^{-1}h= h-\frac{h^3}{3}+\frac{h^5}{5}- ...$ [3]

Also, we can write the equation (11) in the following formula:

$E= J\text{-}T(h) \cong h(\frac{\pi}{2} - \tan^{-1}h)$; $\cot^{-1}h= \frac{\pi}{2} - h + \frac{h^3}{3} - \frac{h^5}{5} + ...$; $h \neq 1$ [3]

By following the same steps as above method, we get;

$$A^F{}_{\cot^{-1}} \cong \frac{h_2(\frac{\pi}{2}\text{-}\tan^{-1} h_2)T(h_1) - h_1(\frac{\pi}{2}\text{-}\tan^{-1} h_1)T(h_2)}{h_2(\frac{\pi}{2}\text{-}\tan^{-1} h_2) - h_1(\frac{\pi}{2}\text{-}\tan^{-1} h_1)}; \quad h_1, h_2 \neq 1 \quad ...(18)$$

The formula (18) is called the inverse cotangent triangular acceleration rule for Al-Tememe of the first kind, We refer to it by $(A^F{}_{\cot^{-1}})$. If we used the rule in the formula (18), we get the correct value for 7 decimal when n=100, but we expect [4] the inverse cotangent triangular acceleration rule, take the following formula:

$$A^F{}_{\cot^{-1}} \cong \frac{h_2^2 \cot^{-1} h_2 T(h_1) - h_1^2 \cot^{-1} h_1 T(h_2)}{h_2^2 \cot^{-1} h_2 - h_1^2 \cot^{-1} h_1} \quad ...(19)$$

Based the on the same method that is followed in finding inverse triangular acceleration rules, we can find the inverse hyperbolic acceleration rules:

$$A^F{}_{\sinh^{-1}} \cong \frac{h_2 \sinh^{-1} h_2 T(h_1) - h_1 \sinh^{-1} h_1 T(h_2)}{h_2 \sinh^{-1} h_2 - h_1 \sinh^{-1} h_1} \quad ...(20)$$

$$A^F{}_{\tanh^{-1}} \cong \frac{h_2 \tanh^{-1} h_2 T(h_1) - h_1 \tanh^{-1} h_1 T(h_2)}{h_2 \tanh^{-1} h_2 - h_1 \tanh^{-1} h_1}; \quad h_1, h_2 \neq 1 \quad ...(21)$$

Since $\sinh^{-1}h= h-\frac{h^3}{6}+\frac{3h^5}{40}-\frac{5h^7}{112}+\cdots$

and $\tanh^{-1} h= h+\frac{h^3}{3}+\frac{h^5}{5}+\frac{h^7}{7}+\cdots$ [3]

3. Examples:

We will review some integrals that have continuous integrands on the interval of integration using inverse triangular and inverse hyperbolic acceleration methods of Al-Tememe to improve the numerical results :

3.1: $\int_3^4 \frac{\sqrt{x^2+1}}{x} dx$ and its exact value, 1.04081165413928 is rounded to 14 decimal .

3.2: $\int_0^{0.5} \sin^{-1}(x)\, dx$ and its exact value, 0.12782479158359 is rounded to 14 decimal.

3.3: $\int_{-1}^{-0.5} e^x\, dx$ and its exact value, 0.23865121854119 is rounded to 14 decimal .

We will compare the values of the acceleration methods with values of trapezoidal rule and midpoint rule. The priority of the acceleration methods can be calculated based on n values, n=1,2,3,..., and the results we adopted in Matlab codes throughout putting Eps=10^{-10} that represents (the absolute error of the subsequent value minus the previous value).

4 The numerical results:

Clearly, the integrand of integration $\int_3^4 \frac{\sqrt{x^2+1}}{x}dx$ is continuous in the integration interval [3,4] and the formula of the correction terms for the (trapezoidal and midpoint rules) identical for the formula in the equation (8).

n	Values of trapezoidal Rule	$A^F{}_{sin^{-1}}$	$A^F{}_{cos^{-1}}$	$A^F{}_{tan^{-1}}$	$A^F{}_{cot^{-1}}$	$A^F{}_{sinh^{-1}}$	$A^F{}_{tanh^{-1}}$
1	1.04243447989694						
2	1.04122508918279	1.04082197941370	1.04081893330603	1.04082191801540	1.04081893411905	1.04082193848020	1.04082199988240
3	1.04099608150483	1.04081287768785	1.04081226033088	1.04081287071194	1.04081226038052	1.04081287303720	1.04081288001324
4	1.04091553008167	1.04081196455041	1.04081174369773	1.04081196279775	1.04081174370637	1.04081196338196	1.04081196513464
5	1.04087817512477	1.04081176652324	1.04081166344163	1.04081176589108	1.04081166344403	1.04081176610180	1.04081176673396
6	1.04085786454414	1.04081170422737	1.04081164800936	1.04081170394614	1.04081164801022	1.04081170403988	1.04081170432111
7	1.04084561141878	1.04081167973487	1.04081164576417	1.04081167959132	1.04081164576454	1.04081167963917	1.04081167978273
8	1.04083765605283	1.04081166855096	1.04081164647466	1.04081166847019	1.04081164647484	1.04081166849711	1.04081166857789
9	1.04083220068861	1.04081166286315	1.04081164771567	1.04081166281428	1.04081164771577	1.04081166283057	1.04081166287945
10	1.04082829790357	1.04081165972511	1.04081164888455	1.04081165969382	1.04081164888460	1.04081165970425	1.04081165973553
11	1.04082540996434	1.04081165787979	1.04081164985617	1.04081165785884	1.04081164985620	1.04081165786583	1.04081165788677
12	1.04082321326761	1.04081165673750	1.04081165063338	1.04081165672295	1.04081165063340	1.04081165672780	1.04081165674235
13	1.04082150361211	1.04081165599990	1.04081165124863	1.04081165598948	1.04081165124865	1.04081165599295	1.04081165600337
14	1.04082014698496	1.04081165550647	1.04081165173601	1.04081165549881	1.04081165173602	1.04081165550136	1.04081165550902
15	1.04081905248365	1.04081165516631	1.04081165212423	1.04081165516056	1.04081165212423	1.04081165516248	1.04081165516823
16	1.04081815668543	1.04081165492568	1.04081165243579	1.04081165492128	1.04081165243580	1.04081165492275	1.04081165492715
17	1.04081741424995	1.04081165475157	1.04081165268792	1.04081165474815	1.04081165268792	1.04081165474929	1.04081165475272
18	1.04081679206794	1.04081165462310	1.04081165289368	1.04081165462039	1.04081165289368	1.04081165462129	1.04081165462400
19	1.04081626550546	1.04081165452662	1.04081165306298	1.04081165452446	1.04081165306298	1.04081165452518	1.04081165452735
20	1.04081581592779		1.04081165320339		1.04081165320339		
21	1.04081542902798		1.04081165332070		1.04081165332070		
22	1.04081509367446		1.04081165341942		1.04081165341942		

Table (1) calculating the integral $\int_3^4 \frac{\sqrt{x^2+1}}{x}dx = 1.04081165413928$ by using trapezoidal rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of first kind

n	Values of midpoint rule	$A^F_{sin^{-1}}$	$A^F_{cos^{-1}}$	$A^F_{tan^{-1}}$	$A^F_{cot^{-1}}$	$A^F_{sinh^{-1}}$	$A^F_{tanh^{-1}}$
1	1.04001569846865						
2	1.04060597098054	1.04080271849414	1.04080420522097	1.04080274846107	1.04080420482416	1.04080273847273	1.04080270850389
3	1.04071964758346	1.04081058771150	1.04081089415992	1.04081059117425	1.04081089413528	1.04081059002002	1.04081058655720
4	1.04075978202399	1.04081138315644	1.04081149319544	1.04081138402969	1.04081149319114	1.04081138373861	1.04081138286534
5	1.04077842068236	1.04081155596987	1.04081160740355	1.04081155628530	1.04081160740236	1.04081155618016	1.04081155586473
6	1.04078856199109	1.04081161037320	1.04081163844351	1.04081161051362	1.04081163844307	1.04081161046682	1.04081161032640
7	1.04079468255114	1.04081163177048	1.04081164873919	1.04081163184218	1.04081164873900	1.04081163181828	1.04081163174657
8	1.04079865731804	1.04081164154311	1.04081165257317	1.04081164158347	1.04081165257308	1.04081164157002	1.04081164152966
9	1.04080138344726	1.04081164651385	1.04081165408329	1.04081164653828	1.04081165408324	1.04081164653014	1.04081164650571
10	1.04080333395202	1.04081164925654	1.04081165467435	1.04081164927217	1.04081165467432	1.04081164926696	1.04081164925133
11	1.04080477738458	1.04081165086947	1.04081165487978	1.04081165087993	1.04081165487977	1.04081165087645	1.04081165086598
12	1.04080587539272	1.04081165186795	1.04081165491906	1.04081165187522	1.04081165491905	1.04081165187280	1.04081165186553
13	1.04080672999653	1.04081165251272		1.04081165251792		1.04081165251619	1.04081165251098
14	1.04080740815784	1.04081165294405		1.04081165294788		1.04081165294660	1.04081165294278
15	1.04080795530207	1.04081165324141		1.04081165324429		1.04081165324333	1.04081165324046
16	1.04080840312498	1.04081165345177		1.04081165345398		1.04081165345324	1.04081165345104
17	1.04080877428700	1.04081165360398		1.04081165360569		1.04081165360512	1.04081165360341
18	1.04080908533651	1.04081165371629		1.04081165371764		1.04081165371719	1.04081165371584
19	1.04080934858634	1.04081165380063		1.04081165380172		1.04081165380136	1.04081165380027

Table (2) calculating the integral $\int_3^4 \frac{\sqrt{x^2+1}}{x}dx = 1.04081165413928$ by using midpoint rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of first kind

To find the value of the integral $\int_0^{0.5} \sin^{-1}(x)dx$ numerically, we note that the integrand is continuous in the integration interval [0,0.5] and the formula of the correction terms for both (trapezoidal, midpoint) rules, respectively, is as in the formula (8).

n	Values of trapezoidal Rule	$A^F_{sin^{-1}}$	$A^F_{cos^{-1}}$	$A^F_{tan^{-1}}$	$A^F_{cot^{-1}}$	$A^F_{sinh^{-1}}$	$A^F_{tanh^{-1}}$
1	0.13089969389958						
2	0.12861991073531	0.12785999265945	0.12785715004214	0.12785996372377	0.12785715023168	0.12785997336884	0.12786000230498
3	0.12818072941216	0.12782938546850	0.12782879670697	0.12782938212395	0.12782879671880	0.12782938323880	0.12782938658336
4	0.12802553107562	0.12782599063861	0.12782577868433	0.12782598979440	0.12782577868641	0.12782599007580	0.12782599092002

5	0.12795342298339	0.12782523092113	0.12782513171896	0.12782523061606	0.12782513171954	0.12782523071775	0.12782523102282
6	0.12791417912752	0.12782498859126	0.12782493440807	0.12782498845541	0.12782493440828	0.12782498850069	0.12782498863654
7	0.12789049086250	0.12782489261329	0.12782485984212	0.12782489254391	0.12782485984221	0.12782489256704	0.12782489263642
8	0.12787510595310	0.12782484859541	0.12782482728561	0.12782484855636	0.12782482728566	0.12782484856937	0.12782484860843
9	0.12786455339939	0.12782482614622	0.12782481151833	0.12782482612259	0.12782481151835	0.12782482613047	0.12782482615410
10	0.12785700286256	0.12782481373694	0.12782480326487	0.12782481372180	0.12782480326488	0.12782481372685	0.12782481374198
11	0.12785141505132	0.12782480642975	0.12782479867695	0.12782480641962	0.12782479867696	0.12782480642299	0.12782480643312
12	0.12784716433958	0.12782480190189	0.12782479600265	0.12782480189485	0.12782479600266	0.12782480189720	0.12782480190423
13	0.12784385585358	0.12782479897592	0.12782479438340	0.12782479897088	0.12782479438341	0.12782479897256	0.12782479897760
14	0.12784123040149	0.12782479701739	0.12782479337244	0.12782479701368	0.12782479337244	0.12782479701492	0.12782479701862
15	0.12783911214665	0.12782479566660	0.12782479272546	0.12782479566382	0.12782479272546	0.12782479566474	0.12782479566752
16	0.12783737839455	0.12782479471065	0.12782479230316	0.12782479470852	0.12782479230316	0.12782479470923	0.12782479471136
17	0.12783594142424	0.12782479401880	0.12782479202328	0.12782479401714	0.12782479202328	0.12782479401770	0.12782479401935
18	0.12783473717400	0.12782479350813	0.12782479183570	0.12782479350682	0.12782479183570	0.12782479350726	0.12782479350857
19	0.12783371797776	0.12782479312458	0.12782479170908	0.12782479312353	0.12782479170908	0.12782479312388	0.12782479312493
20	0.12783284777602	0.12782479283199	0.12782479162337	0.12782479283114	0.12782479162337	0.12782479283142	0.12782479283227
21	0.12783209888261	0.12782479260566	0.12782479156550	0.12782479260497	0.12782479156550	0.12782479260520	0.12782479260589
22	0.12783144975546	0.12782479242838	0.12782479152676	0.12782479242780	0.12782479152676	0.12782479242799	0.12782479242857
23	0.12783088343023	0.12782479228792	0.12782479150129	0.12782479228744	0.12782479150129	0.12782479228760	0.12782479228808
24	0.12783038640075	0.12782479217547	0.12782479148509	0.12782479217507	0.12782479148509	0.12782479217520	0.12782479217561
25	0.12782994780636	0.12782479208460	0.12782479147537	0.12782479208426	0.12782479147537	0.12782479208437	0.12782479208471
26	0.12782955883359	0.12782479201051		0.12782479201022		0.12782479201032	0.12782479201061
27	0.12782921227000	0.12782479194963		0.12782479194938		0.12782479194947	0.12782479194971
28	0.12782890216745	0.12782479189922		0.12782479189901		0.12782479189908	0.12782479189930
29	0.12782862358519	0.12782479185720		0.12782479185702		0.12782479185708	0.12782479185727
30	0.12782837239181	0.12782479182195		0.12782479182179		0.12782479182184	0.12782479182200
31	0.12782814511129	0.12782479179219		0.12782479179205		0.12782479179210	0.12782479179224
32	0.12782793880202	0.12782479176694		0.12782479176681		0.12782479176686	0.12782479176698
33	0.12782775096117	0.12782479174539		0.12782479174528		0.12782479174532	0.12782479174543
34	0.12782757944846	0.12782479172692		0.12782479172682		0.12782479172686	0.12782479172695
35	0.12782742242488	0.12782479171101		0.12782479171092		0.12782479171095	0.12782479171104
36	0.12782727830322	0.12782479169725		0.12782479169717		0.12782479169720	0.12782479169728
37	0.12782714570782	0.12782479168530		0.12782479168523		0.12782479168525	0.12782479168532

Table (3) calculating the integral $\int_0^{0.5} \sin^{-1}(x)dx = 0.12782479158359$ by using trapezoidal rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of the first kind

n	Values of midpoint rule	$A^F_{\sin^{-1}}$	$A^F_{\cos^{-1}}$	$A^F_{\tan^{-1}}$	$A^F_{\cot^{-1}}$	$A^F_{\sinh^{-1}}$	$A^F_{\tanh^{-1}}$
1	0.12634012757104						
2	0.12743115141593	0.12779482141486	0.12779618179129	0.12779483526246	0.12779618170058	0.12779483064667	0.12779481679885
3	0.12764762884287	0.12782081023490	0.12782110044210	0.12782081188346	0.12782110043627	0.12782081133395	0.12782080968537
4	0.12772468083059	0.12782374753223	0.12782385276208	0.12782374795136	0.12782385276105	0.12782374781165	0.12782374739252
5	0.12776058274173	0.12782440831092	0.12782445770270	0.12782440846281	0.12782445770242	0.12782440841218	0.12782440826029
6	0.12778014955165	0.12782461955159	0.12782464656709	0.12782461961933	0.12782464656698	0.12782461959675	0.12782461952902
7	0.12779196994047	0.12782470331338	0.12782471966611	0.12782470334800	0.12782471966607	0.12782470333646	0.12782470330183
8	0.12779965083599	0.12782474175486	0.12782475239375	0.12782474177436	0.12782475239373	0.12782474176786	0.12782474174836
9	0.12780492094861	0.12782476136865	0.12782476867405	0.12782476138046	0.12782476867404	0.12782476137652	0.12782476136472
10	0.12780869268948	0.12782477221383	0.12782477744497	0.12782477222139	0.12782477744496	0.12782477221887	0.12782477221131
11	0.12781148445960	0.12782477860134	0.12782478247477	0.12782477860640	0.12782478247477	0.12782477860471	0.12782477859965
12	0.12781360846192	0.12782478255994	0.12782478550768	0.12782478256345	0.12782478550767	0.12782478256228	0.12782478255876
13	0.12781526181359	0.12782478511834	0.12782478741336	0.12782478512086	0.12782478741336	0.12782478512002	0.12782478511750
14	0.12781657393342	0.12782478683100	0.12782478865263	0.12782478683285	0.12782478865263	0.12782478683223	0.12782478683038
15	0.12781763263698	0.12782478801229	0.12782478948228	0.12782478801368	0.12782478948228	0.12782478801322	0.12782478801183
16	0.12781849920949	0.12782478884834	0.12782479005166	0.12782478884940	0.12782479005166	0.12782478884905	0.12782478884798
17	0.12781921747267	0.12782478945344	0.12782479045089	0.12782478945427	0.12782479045089	0.12782478945399	0.12782478945316
18	0.12781981943245	0.12782478990010	0.12782479073608	0.12782478990075	0.12782479073608	0.12782478990053	0.12782478989988
19	0.12782032890536	0.12782479023558	0.12782479094316	0.12782479023611	0.12782479094316	0.12782479023593	0.12782479023541
20	0.12782076391003	0.12782479049151	0.12782479109569	0.12782479049194	0.12782479109569	0.12782479049179	0.12782479049137
21	0.12782113828182	0.12782479068949	0.12782479120946	0.12782479068983	0.12782479120946	0.12782479068972	0.12782479068937
22	0.12782146278637	0.12782479084456	0.12782479129529	0.12782479084485	0.12782479129529	0.12782479084476	0.12782479084447
23	0.12782174590197	0.12782479096743	0.12782479136068	0.12782479096767	0.12782479136068	0.12782479096759	0.12782479096735
24	0.12782199437889	0.12782479106579	0.12782479141093	0.12782479106600	0.12782479141093	0.12782479106593	0.12782479106573
25	0.12782221364538	0.12782479114529	0.12782479144986	0.12782479114546	0.12782479144986	0.12782479114540	0.12782479114523
26	0.12782240810662	0.12782479121010	0.12782479148022	0.12782479121024	0.12782479148022	0.12782479121020	0.12782479121005
27	0.12782258136768	0.12782479126336	0.12782479150403	0.12782479126348	0.12782479150403	0.12782479126344	0.12782479126332
28	0.12782273640172	0.12782479130746	0.12782479152281	0.12782479130756	0.12782479152281	0.12782479130753	0.12782479130742
29	0.12782287567845	0.12782479134422	0.12782479153768	0.12782479134431	0.12782479153768	0.12782479134428	0.12782479134419
30	0.12782300126300	0.12782479137506	0.12782479154951	0.12782479137514	0.12782479154951	0.12782479137511	0.12782479137503
31	0.12782311489300	0.12782479140109		0.12782479140116		0.12782479140114	0.12782479140107
32	0.12782321803890	0.12782479142318		0.12782479142324		0.12782479142322	0.12782479142316
33	0.12782331195185	0.12782479144203		0.12782479144209		0.12782479144207	0.12782479144201

34	0.12782339770179	0.12782479145819		0.12782479145824		0.12782479145822	0.12782479145817
35	0.12782347620803	0.12782479147211		0.12782479147215		0.12782479147214	0.12782479147209
36	0.12782354826406	0.12782479148415		0.12782479148419		0.12782479148417	0.12782479148414

Table (4) calculating the integral by using midpoint rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of the first kind

To find the value of the integral $\int_{-1}^{-0.5} e^{x}\,dx$ numerically, where that the integrand is continuous in the integration interval [-1,-0.5] and the formula of the correction terms for both (trapezoidal, midpoint) rules, respectively, is as in the formula (8).

n	Values of trapezoidal Rule	$A^{F}{}_{sin^{-1}}$	$A^{F}{}_{cos^{-1}}$	$A^{F}{}_{tan^{-1}}$	$A^{F}{}_{cot^{-1}}$	$A^{F}{}_{sinh^{-1}}$	$A^{F}{}_{tanh^{-1}}$
1	0.24360252522102						
2	0.23989290079576	0.23865637501577	0.23865174955719	0.23865632793214	0.23865174986562	0.23865634362643	0.23865639071081
3	0.23920339633029	0.23865179450821	0.23865087016629	0.23865178925734	0.23865087018487	0.23865179100762	0.23865179625851
4	0.23896188142257	0.23865136269342	0.23865103285665	0.23865136137968	0.23865103285988	0.23865136181759	0.23865136313134
5	0.23885006141853	0.23865127045793	0.23865111662240	0.23865126998485	0.23865111662329	0.23865127014254	0.23865127061563
6	0.23878931090314	0.23865124162006	0.23865115774307	0.23865124140977	0.23865115774339	0.23865124147987	0.23865124169016
7	0.23875267726889	0.23865123031750	0.23865117963726	0.23865123021020	0.23865117963740	0.23865123024596	0.23865123035326
8	0.23872889942752	0.23865122516584	0.23865119223091	0.23865122510548	0.23865119223098	0.23865122512560	0.23865122518596
9	0.23871259687019	0.23865122254889	0.23865119995037	0.23865122251237	0.23865119995040	0.23865122252455	0.23865122256106
10	0.23870093547355	0.23865122110620	0.23865120493265	0.23865122108283	0.23865120493268	0.23865122109062	0.23865122111399
11	0.23869230721215	0.23865122025831	0.23865120828704	0.23865122024266	0.23865120828705	0.23865122024788	0.23865122026352
12	0.23868574462820	0.23865121973365	0.23865121062595	0.23865121972279	0.23865121062595	0.23865121972641	0.23865121973728
13	0.23868063734480	0.23865121939498	0.23865121230555	0.23865121938720	0.23865121230556	0.23865121938979	0.23865121939757
14	0.23867658484065	0.23865121916847	0.23865121354233	0.23865121916276	0.23865121354233	0.23865121916466	0.23865121917038
15	0.23867331546704	0.23865121901235	0.23865121447291	0.23865121900806	0.23865121447291	0.23865121900949	0.23865121901378
16	0.23867063971098	0.23865121890193	0.23865121518637	0.23865121889864	0.23865121518637	0.23865121889974	0.23865121890302
17	0.23866842210073	0.23865121882204	0.23865121574245	0.23865121881948	0.23865121574245	0.23865121882034	0.23865121882289
18	0.23866656371542	0.23865121876310	0.23865121618221	0.23865121876108	0.23865121618221	0.23865121876175	0.23865121876377
19	0.23866499095948	0.23865121871884	0.23865121653453	0.23865121871722	0.23865121653453	0.23865121871776	0.23865121871938
20	0.23866364816268	0.23865121868509	0.23865121682008	0.23865121868378	0.23865121682009	0.23865121868421	0.23865121868553
21	0.23866249258521	0.23865121865898	0.23865121705396	0.23865121865791	0.23865121705396	0.23865121865827	0.23865121865934
22	0.23866149097422	0.23865121863854	0.23865121724733	0.23865121863766	0.23865121724733	0.23865121863795	0.23865121863884
23	0.23866061714464		0.23865121740860		0.23865121740860		
24	0.23865985025026		0.23865121754415		0.23865121754415		

25	0.23865917352878		0.23865121765892		0.23865121765892		
26	0.23865857337809		0.23865121775673		0.23865121775673		
27	0.23865803866735		0.23865121784060		0.23865121784060		
28	0.23865756021716		0.23865121791294		0.23865121791294		
29	0.23865713040297		0.23865121797566		0.23865121797566		
30	0.23865674284938		0.23865121803030		0.23865121803029		
31	0.23865639219242		0.23865121807812		0.23865121807812		
32	0.23865607389291		0.23865121812015		0.23865121812015		
33	0.23865578408882		0.23865121815726		0.23865121815726		
34	0.23865551947758		0.23865121819012		0.23865121819012		
35	0.23865527722158		0.23865121821934		0.23865121821934		
36	0.23865505487175		0.23865121824541		0.23865121824541		
37	0.23865485030544		0.23865121826874		0.23865121826874		

Table (5) calculating the integral $\int_{-1}^{-0.5} e^x\,dx = 0.23865121854119$ by using trapezoidal rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of the first kind

n	Values of midpoint rule	$A^F{}_{\sin^{-1}}$	$A^F{}_{\cos^{-1}}$	$A^F{}_{\tan^{-1}}$	$A^F{}_{\cot^{-1}}$	$A^F{}_{\sinh^{-1}}$	$A^F{}_{\tanh^{-1}}$
1	0.23618327637051						
2	0.23803086204938	0.23864671612536	0.23864901984402	0.23864673957546	0.23864901969041	0.23864673175888	0.23864670830841
3	0.23837522547599	0.23865071534311	0.23865117699284	0.23865071796558	0.23865117698356	0.23865071709143	0.23865071446894
4	0.23849591743247	0.23865109258625	0.23865125741518	0.23865109324276	0.23865125741357	0.23865109302393	0.23865109236741
5	0.23855180952858	0.23865117317616	0.23865125006930	0.23865117341263	0.23865125006885	0.23865117333381	0.23865117309734
6	0.23858217835327	0.23865119837434	0.23865124030396	0.23865119847947	0.23865124030380	0.23865119844443	0.23865119833930
7	0.23860049241242	0.23865120825065	0.23865123358696	0.23865120830430	0.23865123358689	0.23865120828642	0.23865120823277
8	0.23861237999444	0.23865121275230	0.23865122921791	0.23865121278248	0.23865122921788	0.23865121277242	0.23865121274224
9	0.23862053056064	0.23865121503908	0.23865122633736	0.23865121505734	0.23865122633734	0.23865121505125	0.23865121503300
10	0.23862636085181	0.23865121629976	0.23865122438597	0.23865121631144	0.23865122438596	0.23865121630755	0.23865121629587
11	0.23863067473629	0.23865121704069	0.23865122302598	0.23865121704851	0.23865122302597	0.23865121704590	0.23865121703808
12	0.23863395587231	0.23865121749915	0.23865122205279	0.23865121750459	0.23865122205279	0.23865121750278	0.23865121749734
13	0.23863650941138	0.23865121779511	0.23865122133968	0.23865121779900	0.23865122133968	0.23865121779770	0.23865121779381
14	0.23863853559368	0.23865121799304	0.23865122080601	0.23865121799590	0.23865122080601	0.23865121799494	0.23865121799209
15	0.23864017023171	0.23865121812947	0.23865122039912	0.23865121813161	0.23865122039912	0.23865121813090	0.23865121812875
16	0.23864150807483	0.23865121822596	0.23865122008369	0.23865121822760	0.23865122008369	0.23865121822706	0.23865121822541
17	0.23864261685444	0.23865121829577	0.23865121983553	0.23865121829705	0.23865121983553	0.23865121829662	0.23865121829534

18	0.23864354602808	0.23865121834728	0.23865121963769	0.23865121834829	0.23865121963769	0.23865121834795	0.23865121834694
19	0.23864433239166	0.23865121838595	0.23865121947809	0.23865121838676	0.23865121947809	0.23865121838649	0.23865121838568
20	0.23864500377900	0.23865121841545	0.23865121934793	0.23865121841610	0.23865121934793	0.23865121841588	0.23865121841523
21	0.23864558155913	0.23865121843826	0.23865121924076	0.23865121843879	0.23865121924076	0.23865121843861	0.23865121843808
22	0.23864608235784		0.23865121915172		0.23865121915172		
23	0.23864651926723		0.23865121907714		0.23865121907714		
24	0.23864690271007		0.23865121901421		0.23865121901421		
25	0.23864724106729		0.23865121896075		0.23865121896075		
26	0.23864754113974		0.23865121891503		0.23865121891503		
27	0.23864780849273		0.23865121887572		0.23865121887572		
28	0.23864804771584		0.23865121884172		0.23865121884172		
29	0.23864826262129		0.23865121881217		0.23865121881217		
30	0.23864845639669		0.23865121878638		0.23865121878637		
31	0.23864863172399		0.23865121876375		0.23865121876375		

Table (6) calculating the integral $\int_{-1}^{-0.5} e^x\,dx = 0.23865121854119$ by using midpoint rule with the inverse triangular and inverse hyperbolic acceleration methods of AL-Tememe of the first kind

5. Discussion:

We note that,

- In table (1) that the integration values are correct for 9 decimal with all inverse hyperbolic and triangular acceleration methods with slight differences in n values. While the value in trapezoidal method and without acceleration is correct for 5 decimal only when n=22.
- In table (2), it is clear that the integration values are correct for 9 decimal with all inverse hyperbolic and triangular acceleration methods with slight difference in n values. While the value in midpoint method without acceleration is correct for 5 decimal only when n=19.
- From table (3) that the integration values are correct for 9 decimal with all inverse hyperbolic and triangular acceleration methods with slight difference in n values. but they are correct for 5 decimal in trapezoidal method without acceleration when n=37.
- In table (4), the integration values are correct for 9 decimal with all inverse hyperbolic and triangular acceleration methods with slight difference in n values., but they are correct for 5 decimal only in midpoint method without acceleration when n=36.
- In table (5), the values of both acceleration methods :$A^F{}_{\cos^{-1}}$ and $A^F{}_{\cot^{-1}}$ are correct for 9 decimal when n=35,36,37, but they are correct for 5 decimal only in trapezoidal method without acceleration when n=37, We get 10 decimal correct values with other acceleration methods when n=22.
- The integration values in table (6) are correct for 9 decimal with all inverse and hyperbolic triangular acceleration methods with slight difference in n value. While the value in midpoint method without acceleration is correct for 5 decimal only when n=31.

6. Conclusion:

We can say that acceleration methods of Al-Tememe have the same efficiency to improve the results of integrals, which are reviewed regarding accuracy and partial periods used in addition to the speed of calculating their values.

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