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On Quadratic Unbiased Estimator for Variance Components of One-Way Repeated Measurements Model

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ABSTRACT

In this paper, we investigate the estimator of variance components of one-way repeated measurements model (RMM) using MINQUE-principle (Rao 1971a and Rao 1971b) and method of MINQUE (1) which using priori values for components of variance.

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1. Introduction

A models of repeated measurements (RMM) are widespread in statistical studies (life, health, social, agricultural and others). And since the study of estimating the components of variance is of great importance in statistical studies, there are many statistical methods for estimating these components. The interclass correlation model is a special case of repeated measurements model introduced by Wilks (1946). Vonesh and Chinchlli (1997) introduce univariate repeated measurements Model (called One-Way Repeated Measurement Model). AL-Mouel (2004) studied the multivariate repeated measurements models and comparison of estimators. Al-Mouel A. H. S. eand others (2017) studied Bayesian One- Way Repeated Measurements Model Based on Bayes Quadratic Unbiased Estimator. Al-Isawi JA. M. A. and Al-Mouel A. H. S. (2018) studied Best Quadratic Unbiased Estimator for Variance Component of One-Way Repeated Measurements Model, in this article we study the quadratic unbiased estimator for variance components of one-way repeated measurements model. Now we introduce some definitions and remarks which used in this article.

Definition 1 [7]: For given matrix Λ of size $n \times m$ we called a matrix Λ^+ of size $m \times n$ is *Moore-Penrose* generalized inverse (*MP-inverse*) of Λ if satisfy the following conditions

(a) $\Lambda\Lambda^+\Lambda = \Lambda$, (b) $\Lambda^+\Lambda\Lambda^+ = \Lambda^+$,

(c) matrix $\Lambda\Lambda^+$ is symmetric (d) matrix $\Lambda^+\Lambda$ is symmetric.

Definition 2 [7]: The *Kronecker product* ($\otimes$) of an $n \times m$ and $p \times q$ matrix A and B, is denoted by $A \otimes B$. This is an $np \times mq$ matrix with the (i,j) block $A_{ij}B$, where $i = 1, \dots, n$ and $j = 1, \dots, m$.

Kronecker product have the following properties:

1- $(A \otimes B)(D \otimes C) = AD \otimes BC$
2- $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$ and $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
3- $(A \otimes B)^+ = A^+ \otimes B^+$ and $(A \otimes B)' = A' \otimes B'$
4- $(c_1A) \otimes (c_2B) = (c_1c_2)(A \otimes B)$
5- $A \otimes [B_1 \vdots B_2] = [A \otimes B_1 \vdots A \otimes B_2]$
6- $tr(A \otimes B) = tr(A)\, tr(B)$.

Remark 1: If j_p denotes to $p \times 1$ vector of $1's$, J_p denotes to the $p \times p$ matrix of $1's$ and I_p denotes to $p \times p$ identity matrix then

1- $j_p^+ = \frac{1}{p} j_p'$.
2- $J_p^+ = \frac{1}{p^2} J_p = (\frac{1}{p} j_p)(\frac{1}{p} j_p')$.
3- $(I_p + J_p)^+ = I_p - \frac{1}{p+1} J_p$.
4- $(I_p - \frac{1}{p} J_p)^+ = (I_p - \frac{1}{p} J_p)$.

Remark 2: If A is any square matrix of size $q \times q$, then

$$(I_q \otimes j_p)A(I_q \otimes j_p') = A \otimes J_p.$$

Remark 3: If H_p denotes to $p \times p$ Idempotent matrix and A any matrix, then

1- $(A \otimes H_p)^+ = A^+ \otimes H_p$.
2- $(H_p \otimes A)^+ = H_p \otimes A^+$
3- $H_p^+ = H_p$
4- $Y^+ = \frac{Y'}{Y'Y}$ for any $n \times 1$ vector Y.

2. The one-way repeated measurements model

Consider the following linear model and parameterization for the one-way repeated measurement model with one between−units factor incorporating univariate random effects.

$$Y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + \varepsilon_{ijk} \quad (1)$$

where

$i = 1, \dots, n$ is an index for an experimental unit within group j,

$j = 1, \dots, q$ is an index for levels of the between−units factor (group),

$k = 1, \dots, p$ is an index for levels of the within−units factor (time),

Y_{ijk} is the response measurements at time k for unit i within group j,

μ is the overall mean,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect due to experimental unit i within treatment group j,

γ_k is the added effect for time k,

$(\tau\gamma)_{jk}$ is the added effect for the group $j \times$ time k interaction, and

ε_{ijk} is the random error on time k for unit i within group j.

For the parameterization to be of full rank, we impose the following set of conditions:

$$\left.\begin{array}{l} \sum_{j=1}^{q} \tau_j = 0 \;\;,\;\; \sum_{k=1}^{p} \gamma_k = 0 \,, \\ \sum_{j=1}^{q} (\tau\gamma)_{jk} = 0 \;\; \forall\, k = 1, \dots, p \,,\;\; \sum_{k=1}^{p} (\tau\gamma)_{jk} = 0 \;\; \forall\, j = 1, \dots, q, \end{array}\right\} \quad (2)$$

We assume that the ε_{ijk}'s and the $\delta_{i(j)}$'s are independent with

$$\varepsilon_{ijk} \sim^{i.i.d} N(0, \sigma_\varepsilon^2) \quad \text{and} \quad \delta_{i(j)} \sim^{i.i.d} N(0, \sigma_\delta^2) \quad (3)$$

We can write model (1) as follows

$$Y = X\beta + Z\delta + \varepsilon \,, \quad (4)$$

Where Y is nqp-dimensional response vector,

Z : is a $nqp \times nq$ design matrix,

β: is a $(q+1)(p+1)$-dimensional vector of fixed effects parameters,

δ : is a nq-dimensional vector of random effects,

ε: is error term has length nqp with $\varepsilon \sim N_{nqp}\left(0_{nqp}, \sigma_\varepsilon^2 I_{nqp}\right)$.

And design matrix X of size $nqp \times (q+1)(p+1)$ is

$$\left.\begin{aligned} &X = [x1 \vdots x2 \vdots x3 \vdots x4]_{nqp\times(qp+q+p+1)} \\ &where \\ &x1 = j_{nqp}\,,\;\; x2 = j_n \otimes I_q \otimes j_p\,,\;\; x3 = j_{nq}\otimes I_p\,,\;\; x4 = j_n \otimes I_{qp} \end{aligned}\right\} \quad (5)$$

Then from (3)

$$\left.\begin{aligned} &\varepsilon \sim N_{nqp}\left(0_{nqp}, \sigma_\varepsilon^2 I_{nqp}\right) \;, \delta \sim N_{nq}\left(0_{nq}, \sigma_\delta^2 I_{nq}\right) \;\; and \;\; cov(\varepsilon\,,\delta)) = 0 \\ &Y \sim N_{nqp}(X\beta, \Sigma) \;\; where,\; \Sigma = \sigma_\delta^2 Z\,Z' + \sigma_\varepsilon^2 I_{nqp} \end{aligned}\right\} \quad (6)$$

*($\Sigma = var(Y)$ is variance-covariance matrix).

Lemma 1 [13]: Let Λ^+ be MP-inverse of Λ and put $X_{i.j} = \left(I_n - E_j\right)X_i$, where

X_i is a given $n \times m_i$ matrix and $E_j = X_j\, X_j^+$. Then with $X = [X_1 \vdots X_2]$

and $E_{i.j} = X_{i.j}\, X_{i.j}^+$ we have $X\,X^+ = E_1 + E_{2.1} = E_2 + E_{1.2}$.

Proposition 1: For model (4) and using Lemma 1; If $X = [X_1 \vdots X_2]$, where X_1 and X_2

are a matrix of size $nqp \times (q+1)(p+1) - m$ and $nqp \times m$, $1 \le m < (q+1)(p+1)$ then

$$E_2 = \begin{cases} (J_n/n)\otimes I_{qp} & ;\; m \ge qp \\ (J_n/n)\otimes H_{qp} & ;\; m < qp \;\;, H \text{ is Idempotent matrix} \end{cases}$$

$$\text{and} \;\; E_{1.2} = \begin{cases} 0 & ;\; m \ge qp \\ (J_n/n)\otimes(I_{qp} - H_{qp}) & ;\; m < qp,\;\; H = H^2 \end{cases}$$

$$\rightarrow \qquad X\,X^+ = \frac{J_n \otimes I_{qp}}{n} \qquad (7)$$

Proof:

$$E_2 = X_2\, X_2^+$$

$$= \;(j_n \otimes \chi)(j_n^+ \otimes \chi^+) \;\; \text{where } \chi \text{ is a matrix of size } qp \times m$$

$$= j_n j_n^+ \otimes \chi\,\chi^+ = \begin{cases} (J_n/n)\otimes I_{qp} & ;\; m \ge qp \\ (J_n/n)\otimes H_{qp} & ;\; m < qp \;\;, H \text{ is Idempotent matrix} \end{cases}$$

$$\text{Similarly} \;\; E_1 = X_1\, X_1^+ = \begin{cases} (J_n/n)\otimes I_{qp} & ;\; m < qp \\ (J_n/n)\otimes H_{qp} & ;\; m \ge qp \;\;, H \text{ is Idempotent matrix} \end{cases}$$

$$\text{And} \;\; E_{1.2} = X_{1.2}\, X_{1.2}^+ = \left[\left(I_{nqp} - E_2\right)X_1\right]\left[\left(I_{nqp} - E_2\right)X_1\right]^+$$

$$= \left(I_{nqp} - E_2\right)X_1\, X_1^+\left(I_{nqp} - E_2\right)$$

$$= \begin{cases} (I_n - (J_n/n))\otimes I_{qp})((J_n/n)\otimes H_{qp})(I_n - (J_n/n))\otimes I_{qp}) & ;\; m \ge qp \\ \left(I_{nqp} - (J_n/n)\otimes H_{qp}\right)((J_n/n)\otimes I_{qp})\left(I_{nqp} - (J_n/n)\otimes H_{qp}\right) & ;\; m < qp \end{cases}$$

$$= \begin{cases} (I_n - (J_n/n))(J_n/n)(I_n - (J_n/n))\otimes H_{qp} & ;\; m \ge qp \\ (J_n/n)\otimes(I_{qp} - H_{qp}) & ;\; m < qp \end{cases}$$

$$= \begin{cases} ((J_n/n) - (J_n/n))\otimes H_{qp} = 0 & ;\; m \ge qp \\ (J_n/n)\otimes I_{qp} - (J_n/n)\otimes H_{qp} & ;\; m < qp \end{cases}$$

$$\rightarrow \;E_{1.2} = \begin{cases} 0 & ;\; m \ge qp \\ (J_n/n)\otimes(I_{qp} - H_{qp}); & m < qp, \qquad \mathrm{H} = \mathrm{H}^2 \end{cases}$$

$$\rightarrow X\,X^+ = E_2 + E_{1.2} = \frac{J_n \otimes I_{qp}}{n}$$

It is clear that

$$X = [j_{nqp} \vdots j_n \otimes I_q \otimes j_p \vdots j_{nq} \otimes I_p \vdots j_n \otimes I_{qp}]$$

$$= j_n \otimes [j_{qp} \vdots I_q \otimes j_p \vdots j_q \otimes I_p \vdots I_{qp}]$$

$$= (j_n \otimes \chi) \text{ where } \chi = [j_{qp} \vdots I_q \otimes j_p \vdots j_q \otimes I_p \vdots I_{qp}]_{qp \times (q+1)(p+1)}$$

$$\rightarrow X X^+ = (j_n \otimes \chi)(j_n^+ \otimes \chi^+)$$

$$= j_n j_n^+ \otimes \chi \chi^+ \quad (j_n^+ = \frac{j_n'}{n} \quad and \ \chi \chi^+ = I_{qp})$$

$$= \frac{J_n \otimes I_{qp}}{n}$$

Complete proof.

Proposition 2: For model (4) and using Lemma 1, with $U = [U_1 \vdots U_2]$

where $U_1 = X \vdots U_2 = Z$, then we have $E_1 = U_1 U_1^+ = X X^+ = \frac{J_n \otimes I_{qp}}{n}$

and $E_{2.1} = U_{2.1} U_{2.1}^+ = \frac{I_{nq} \otimes J_p}{p} - \frac{J_n \otimes I_q \otimes J_p}{np} = \frac{(I_n - (J_n/n)) \otimes I_q \otimes J_p}{p}$

Implies to

$$U(U'U)^+ U' = UU^+ = \frac{J_n \otimes I_{qp}}{n} + \frac{(I_n - (J_n/n)) \otimes I_q \otimes J_p}{p} \qquad (8)$$

Proof:

Since $U_{2.1} = (I_{nqp} - E_1)Z, \ Z = I_{nq} \otimes j_p \ and \ E_1 = \frac{J_n \otimes I_{qp}}{n}$ (proved in Proposition 1).

$$\therefore \ U_{2.1} = (I_{nqp} - (J_n/n)) \otimes I_{qp})Z$$

$$U_{2.1} = ((I_n - (J_n/n)) \otimes I_{qp})Z$$

$$\rightarrow E_{2.1} = U_{2.1} U_{2.1}^+ = [((I_n - (J_n/n)) \otimes I_{qp})Z \ ][((I_n - (J_n/n)) \otimes I_{qp})Z \ ]^+$$

$$= ((I_n - (J_n/n)) \otimes I_{qp})(ZZ^+)((I_n - (J_n/n)) \otimes I_{qp})^+$$

$$= ((I_n - (J_n/n)) \otimes I_{qp})((I_{nq} \otimes j_p)(I_{nq} \otimes j_p)^+)((I_n - (J_n/n)) \otimes I_{qp})^+$$

$$= ((I_n - (J_n/n)) \otimes I_{qp})(I_{nq} \otimes J_p/p)((I_n - (J_n/n)) \otimes I_{qp})$$

$$= ((I_n - (J_n/n)) \otimes I_{qp})(I_n \otimes I_q \otimes (J_p/p))((I_n - (J_n/n)) \otimes I_{qp})$$

$$= ((I_n - (J_n/n)) \otimes I_q \otimes I_p)(I_n \otimes I_q \otimes (J_p/p))((I_n - (J_n/n)) \otimes I_{qp})$$

$$= ((I_n - (J_n/n)) \otimes I_q \otimes (J_p/p))((I_n - (J_n/n)) \otimes I_q \otimes I_p)$$

$$= ((I_n - (J_n/n)) \otimes I_q \otimes (J_p/p)) \qquad (9)$$

$$\rightarrow \ UU^+ = E_1 + E_{2.1} = \frac{J_n \otimes I_{qp}}{n} + \frac{(I_n - (J_n/n)) \otimes I_q \otimes J_p}{p} \ . \qquad (10)$$

Complete proof.

3.1- MINQUE for σ_ε^2

Let A be an $nqp \times nqp$ matrix; then a quadratic estimator for σ_ε^2 is defined as

$$\hat{\sigma}_\varepsilon^2 = Y'AY \qquad (11)$$

Note: For matrix A without loss of generality, we can assume that A is a symmetric matrix and nonnegative definite, to make $\hat{\sigma}_\varepsilon^2$ nonnegative for all Y.

When a ratio $\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}$ is known which equal to θ , or σ_δ^2 and σ_ε^2 are equal ($\theta = 1$).

We can write model (4) as follows

$$Y = X\beta + \boldsymbol{\varepsilon}, \qquad (12)$$

Where $\boldsymbol{\varepsilon} = Z\delta + \varepsilon \rightarrow E(\boldsymbol{\varepsilon}) = 0, var(\boldsymbol{\varepsilon}) = \sigma_\varepsilon^2(I + \theta ZZ')$

Since $E(\boldsymbol{\varepsilon}'A\boldsymbol{\varepsilon}) = tr\big(A\, var(\boldsymbol{\varepsilon})\big)$ and $E(\boldsymbol{\varepsilon}') = 0$, , then

$$\begin{aligned} E(\hat{\sigma}_\varepsilon^2) &= E(Y'AY) = E[(X\beta + \boldsymbol{\varepsilon})'A(X\beta + \boldsymbol{\varepsilon})] \\ &= \beta'X'AX\beta + E(\boldsymbol{\varepsilon}'A\boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon}'AX\beta) \\ &= \beta'X'AX\beta + tr\big(A\, var(\boldsymbol{\varepsilon})\big) \\ &= \beta'X'AX\beta + \sigma_\varepsilon^2 tr(A(I + \theta ZZ')) \end{aligned}$$

$$\rightarrow \quad E(\hat{\sigma}_\varepsilon^2) = \beta'X'AX\beta + \sigma_\delta^2 tr(Z'A\,Z) + \sigma_\varepsilon^2\, tr(A)$$

To make $\hat{\sigma}_\varepsilon^2$ unbiased that minimizes the norm of matrix A must be have

$$\left.\begin{aligned} &\|A\|^2 = tr(AA') = \min \\ \text{and} \quad &X'AX = 0\,, Z'A\,Z = 0, \text{and } tr(A) = 1\,. \end{aligned}\right\} \qquad (13)$$

To solve this problem let we assume that

$$U = [X \vdots Z] \rightarrow U'AU = \begin{bmatrix} X'AX & X'AZ \\ Z'AX & Z'AZ \end{bmatrix} \qquad (14)$$

Since $X'AX = 0$, $Z'AZ = 0$ and A is a symmetric and nonnegative matrix then

$$AX = 0 \text{ and } AZ = 0 \rightarrow X'AZ = 0 \text{ and } Z'AX = 0 \qquad (15)$$

From (14) and (15) we have $U'AU = 0$ which implies that problem (13) becomes

$$tr(AA') = min, \qquad (16)$$

under restrictions

$$U'AU = 0 \text{ and } tr(A) = 1 \qquad (17)$$

To solve problems (16-17) using a Lagrange function for multiplier matrix (Lagrange multipliers technique), the Lagrange function can be defined as

$$f(A, L, \lambda) = \frac{1}{2}\, tr(AA') + tr(U'AU\,L') + (1 - tr(A))\lambda$$

where L is $m \times m$ Lagrange multiplier matrix and λ is scalar .

17 ng this formula, for any matrices A, B, C of appropriate size.

$$\frac{\partial tr(A)}{\partial A} = I, \qquad \frac{\partial tr(AA')}{\partial A} = 2A, \qquad \frac{\partial tr(BAC)}{\partial A} = B'C'$$

differentiate function f with respect to A,

For optimization (16) under restrictions (17) is that the derivative $\partial f/\partial A$ must be equal to zero,

$$\frac{\partial f}{\partial A} = A + ULU' - \lambda I = 0 \qquad (18)$$

From equation (18) we have

$$A = \lambda I - ULU' \qquad (19)$$

To find L and λ in (19) , we using the conditions in (17) , we have

$$U'\lambda IU - U'ULU'U = 0 \rightarrow L = \lambda(U'U)^{+} \qquad (20)$$

substituting the value of L (20) in (19) , we have

$$A = \lambda\,(I - U(U'U)^{+}U') = \lambda\,(I - UU^{+}) \qquad (21)$$

and $\lambda = \dfrac{1}{rank(I - UU^{+})}$

The matrix $[I - UU^{+}]$ is *idempotent* (proved by [6] Graybill(1983)) then

$$rank[I - UU^{+}] = tr(I - UU^{+}) = nqp - rank(U)$$

$$\therefore \quad \lambda = \frac{1}{nqp - rank(U)}, \qquad rank(U) = q(n + p - 1)$$

$$\rightarrow \quad A = \frac{(I - U(U'U)^{+}U')}{q(np - n - p + 1)} = \frac{(I - UU^{+})}{q(n-1)(p-1)} \qquad (22)$$

From (11) and (22), we have

$$\hat{\sigma}_{\varepsilon}^{2} = Y'\frac{(I - UU^{+})}{q(n-1)(p-1)}Y. \;\square \qquad (23)$$

And

$$\hat{\sigma}_{\delta}^{2} = Y'\frac{\theta(I - UU^{+})}{q(n-1)(p-1)}Y. \;\square \qquad (24)$$

From Proposition 2,

Since $U(U'U)^{+}U' = E_1 + E_{2.1}$

we have $U(U'U)^{+}U' = UU^{+} = \dfrac{J_n \otimes I_{qp}}{n} + \dfrac{\big(I_n - (J_n/n)\big)\otimes I_q \otimes J_p}{p}$ (25)

Substituting (25) in(22) $A = \dfrac{(I_{nqp} - (J_n/n)\otimes I_{qp} + \big(I_n - (J_n/n)\big)\otimes I_q \otimes (J_p/p))}{q(n-1)(p-1)}$

$$\rightarrow \qquad A = \frac{\big(I_n - (J_n/n)\big)\otimes I_q \otimes \big(I_p - (J_p/p)\big)}{q(n-1)(p-1)} \qquad (26)$$

From (23), (24) and (26) we have, the quadratic estimator for σ_{ε}^{2} and σ_{δ}^{2} *are*

$$\hat{\sigma}_{\varepsilon}^{2} = Y'\frac{\big(I_n - (J_n/n)\big)\otimes I_q \otimes \big(I_p - (J_p/p)\big)}{q(n-1)(p-1)}Y \qquad (27)$$

And

$$\hat{\sigma}_{\delta}^{2} = Y'\frac{\theta\big(I_n - (J_n/n)\big)\otimes I_q \otimes \big(I_p - (J_p/p)\big)}{q(n-1)(p-1)}Y \qquad (28)$$

3.2- MINQUE for $\boldsymbol{\sigma} = [\sigma_\delta^2 , \sigma_\varepsilon^2]'$

The derivation of the law is based on minimizing the Euclidean norm.
If the mixed linear model is expressed as a matrix

For model (4) we have that $\boldsymbol{Y} \sim N\left(X\beta, \boldsymbol{\Sigma} = \sigma_0^2\, I_{nqp} + \sigma_1^2 Z_1 Z_1'\right)$,

Such that $\sigma_0^2 = \sigma_\varepsilon^2,\ \sigma_1^2 = \sigma_\delta^2$ and $Z_0 = I_{nqp},\ Z_1 = Z$ in model (4)

We can express the model (4) as

$$\boldsymbol{Y} = X\beta + \boldsymbol{Z\delta} \tag{29}$$

where $\boldsymbol{Z} = [\,Z_0 \vdots Z_1]$ and $\boldsymbol{\delta}' = [\,\varepsilon' \vdots \delta']$. The model (29) is called a mixed linear model. Thus generally we have $E(Y) = X\beta$ and $\boldsymbol{\Sigma} = \sigma_0^2 V_0 + \sigma_1^2 V_1$, where $V_r = Z_r Z_r'\,, r = 0, 1$. $\boldsymbol{\Sigma}$ is called the covariance matrix and the parameters σ_0^2, σ_1^2 are the unknown components of variance whose values should be estimated.

We can write a linear combination for the components of variance $\sigma_r^2 s$, by a quadratic form $\boldsymbol{Y}'A\boldsymbol{Y}$, where A is a symmetric matrix chosen subject to the conditions which, guarantee the estimator's unbiasedness and invariance we have

$$E(\boldsymbol{Y}'A\boldsymbol{Y}) = c_0\sigma_0^2 + c_1\sigma_1^2 = \boldsymbol{C}'\boldsymbol{\sigma} \tag{30}$$

and
$$\boldsymbol{Y}'A\boldsymbol{Y} = (X\beta + \boldsymbol{Z\delta})'A(X\beta + \boldsymbol{Z\delta})$$
$$= \beta'X'AX\beta + 2\beta'X'A\boldsymbol{Z\delta} + \boldsymbol{\delta}'\boldsymbol{Z}'A\boldsymbol{Z\delta}$$

Under unbiasedness and invariance, the estimator reduces to

$$\boldsymbol{Y}'A\boldsymbol{Y} = \boldsymbol{\delta}'\boldsymbol{Z}'A\boldsymbol{Z\delta} \tag{31}$$

Where $\boldsymbol{C} = [c_0 , c_1]'$ and A is chosen to satisfy the restrictions

$$AX = 0 \quad and \quad c_r = tr(Z_r' A Z_r)\,, r = 0, 1. \tag{32}$$

Clear that: $\boldsymbol{\delta}$ has a normal distribution (since $\varepsilon \sim iid.N(0, \sigma_0^2 I)$ and $\delta \sim iid.N(0, \sigma_1^2 I)$). The components of variance are a linear function of the natural estimated, so it should be $\boldsymbol{\delta}'D\boldsymbol{\delta}$ where D is known diagonal matrix.

The difference between the proposed estimator (30) and the natural unbiased estimator ($\boldsymbol{\delta}'D\boldsymbol{\delta}$) is

$$\boldsymbol{\delta}'(\boldsymbol{Z}'A\boldsymbol{Z} - D)\boldsymbol{\delta} \tag{33}$$

Remark 4: $\|\boldsymbol{Z}'A\boldsymbol{Z} - D\|^2 = tr[(\boldsymbol{Z}'A\boldsymbol{Z} - D)^2]$=min. (Rao1971a deduced)[9].

The MINQUE method tries to find minimize the difference in (33) with the restrictions in (32).

19 nimize the square of Euclidean norm ($\|\ \|$) using (Remark 4) inasmuch

$$\|\boldsymbol{Z}'A\boldsymbol{Z} - D\|^2 = tr[(\boldsymbol{Z}'A\boldsymbol{Z} - D)^2] = tr[(A\boldsymbol{V})^2] - tr[D^2]$$

Where $\boldsymbol{V} = V_0 + V_1$

Inasmuch as $tr[D^2]$ does not involve A, the problem of MINQUE reduces to minimizing $tr[(A\boldsymbol{V})^2]$ with the conditions in (32) attained at, according to Rao [11]

$$A = a_0 Q V_0 Q + a_1 Q V_1 Q \tag{34}$$

Where

$$Q = \boldsymbol{V}^{-1}(I - X(X'\boldsymbol{V}^{-1}X)^{+}X'\boldsymbol{V}^{-1})$$

And $\boldsymbol{a} = [a_0 , a_1]'$ is determined from the equations $\boldsymbol{a} = \boldsymbol{S}^{+}\boldsymbol{C}$, with

$$\boldsymbol{S} = (S_{r,s}) = tr(Q\,V_r Q V_s\,), r = 0,1 \text{ and } s = 0,1.$$

where X is the matrix in the model in (29) and $\boldsymbol{V}$ is a positive definite matrix.

For the problem of MINQUE, choosing $\boldsymbol{V} = V_0 + V_1$ or $\Sigma_{(t)} = t_0V_0 + t_1V_1$,

where $t = [t_0, t_1]'$ are a priori ratios of unknown components of variance.

On using (34), we have the MINQUE of $\boldsymbol{C'\sigma}$ is given by

$$\boldsymbol{C'\hat{\sigma}} = \boldsymbol{Y}'A\boldsymbol{Y} = \boldsymbol{Y}'(a_0QV_0Q + a_1QV_1Q)\boldsymbol{Y} \qquad (35)$$

the estimator (35) can be written as

$$\boldsymbol{C'\hat{\sigma}} = \boldsymbol{a}'\boldsymbol{\gamma}\,, \ \text{ where } \boldsymbol{\gamma} = \begin{bmatrix}\boldsymbol{Y}'QV_0Q\boldsymbol{Y}\\ \boldsymbol{Y}'QV_1Q\boldsymbol{Y}\end{bmatrix} \qquad (36)$$

On substituting $\boldsymbol{a} = \boldsymbol{S}^{+}\boldsymbol{C}$ in (36), we have

$$\boldsymbol{C'\hat{\sigma}} = \boldsymbol{C}'\boldsymbol{S}^{+}\,\boldsymbol{\gamma} \;\rightarrow\; \boldsymbol{\hat{\sigma}} = \boldsymbol{S}^{+}\,\boldsymbol{\gamma}. \qquad (37)$$

The solution vector (37) is unique if and only if the individual components are unbiased.

Now since σ_0^2 and σ_1^2 not equal

$$\text{Let} \quad \alpha_0 = \frac{\varepsilon}{\sigma_0} \quad \text{and} \quad \alpha_1 = \frac{\delta}{\sigma_1} \qquad (38)$$

Then the difference in (33) is given by

$$\boldsymbol{\alpha}'\Psi^{1/2}(\boldsymbol{Z}'A\boldsymbol{Z} - D)\Psi^{1/2}\boldsymbol{\alpha} \qquad (39)$$

Where $\boldsymbol{\alpha}' = (\alpha_0' \,\vdots\, \alpha_1')$ *and* $\Psi = \begin{bmatrix}\sigma_0^2 I & 0\\ 0 & \sigma_1^2 I\end{bmatrix}$

Now, the minimization of (39) using (Remark 4) is equivalent to minimizing $tr[(A\boldsymbol{\Sigma})^2]$ under the restrictions in (32),

Where $\boldsymbol{\Sigma}$ defined in (29) as.

$$\boldsymbol{\Sigma} = \sigma_0^2V_0 + \sigma_1^2V_1 = \sigma_0^2(V_0 + \tfrac{\sigma_1^2}{\sigma_0^2}V_1) \qquad (40)$$

The matrix $\boldsymbol{\Sigma}$ in (40) have two unknown variance $(\sigma_r^2, r = 0,1)$.

Then according to Rao [10], we have two amendments to this problem:

1. If we have a priori knowledge of the approximate ratio $\frac{\sigma_1^2}{\sigma_0^2}$, we can substitute them in (40) and use the $\boldsymbol{\Sigma}$ thus computed like as estimator in (section **3.1**).
2. We can use a priori estimates in (40) and obtain MINQUEs of $\sigma_r^2, r = 0,1$.

These estimates then may be substituted in (40) many times. The procedure is called iterative MINQUE or I-MINQUE (Rao and Kleffé,1988) [8]. In this procedure, the MINQUE estimator of the variance components can be obtained by solving the system of equations (37)

$$\begin{bmatrix} tr\left(Q_{(t)}V_0Q_{(t)}V_0\right) & tr\left(Q_{(t)}V_0Q_{(t)}V_1\right) \\ tr\left(Q_{(t)}V_1Q_{(t)}V_0\right) & tr\left(Q_{(t)}V_1Q_{(t)}V_1\right) \end{bmatrix} \begin{bmatrix} \hat{\sigma}_0^2 \\ \hat{\sigma}_1^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}'Q_{(t)}V_0Q_{(t)}\boldsymbol{Y} \\ \boldsymbol{Y}'Q_{(t)}V_1Q_{(t)}\boldsymbol{Y} \end{bmatrix} \quad (41)$$

Where

$$\left.\begin{aligned} Q_{(t)} &= \Sigma_{(t)}^{-1}\left(I - X\left(X'\Sigma_{(t)}^{-1}X\right)^{+}X'\Sigma_{(t)}^{-1}\right) \\ \Sigma_{(t)} &= t_0V_0 + t_1V_1 \quad ; t = [t_0, t_1]', \end{aligned}\right\} \quad (42)$$

Although the estimate of the variance component depends on a priori value of the human choice t_r, as long as these a priori values do not depend on the experimental data, the MINQUE estimator is still unbiased. Choose any a priori t_r, can be obtained the variance component estimate $[\sigma_r^2]$. New estimates can be obtained if the estimates are replaced with priori estimates for reevaluation value. This process is repeated until the new estimate is very close to the old estimate. This iterative estimation method is like to relative maximum likelihood estimator (REML) method, which is a result of the maximum likelihood estimate (EML). In other word, REML estimates and MINQUE estimates are relatively close. For more see [12].

3.3- MINQUE (1) for σ_ε^2 and σ_δ^2

The choice of a priori t_r in (42) can based on experience or even on past analysis. The easier way is to take it all a priori values are 1 ($t_r = 1$). This method is called the MINQUE (1) method, and the variance component obtained is estimated metering is a MINQUE (1) estimate.

The unbiased estimator of the MINQUE (1) {MINQUE on $t = j_2$} is,

Assume $t = j_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, substituting t_r in (42) we have

$$\left.\begin{aligned} \Sigma_{(1)} &= V_0 + V_1 = \boldsymbol{V} = Z_0Z_0' + Z_1Z_1' = I_{nq}\otimes(I_p + J_p), \\ Q_{(1)} &= \Sigma_{(1)}^{-1}(I - X\left(X'\Sigma_{(1)}^{-1}X\right)^{+}X'\Sigma_{(1)}^{-1}) \end{aligned}\right\} \quad (43)$$

Let $U = (\Sigma_{(1)}^{-1})^{1/2}X$ and $P_U = U(U'U)^{+}U'$ is projection matrix.
Then

$$\begin{aligned} Q_{(1)} &= (\Sigma_{(1)}^{-1})^{\frac{1}{2}}(I - P_U)(\Sigma_{(1)}^{-1})^{\frac{1}{2}} \\ Q_{(1)}X &= (\Sigma_{(1)}^{-1})^{\frac{1}{2}}(I - P_U)(\Sigma_{(1)}^{-1})^{\frac{1}{2}}X \\ &= (\Sigma_{(1)}^{-1})^{\frac{1}{2}}(I - P_U)U \\ &= (\Sigma_{(1)}^{-1})^{\frac{1}{2}}(U - U) = 0 \end{aligned} \quad (44)$$

Therefor $r = 0{,}1$ *and* $s = 0{,}1$ *we have*

$$\begin{aligned} E[\boldsymbol{Y}'Q_{(1)}V_rQ_{(1)}\boldsymbol{Y}] &= tr\left(Q_{(1)}V_rQ_{(1)}\boldsymbol{\Sigma}\right) + (X\beta)'Q_{(1)}V_rQ_{(1)}X\beta \\ &= tr\left(Q_{(1)}V_rQ_{(1)}(\sigma_0^2V_0 + \sigma_1^2V_1)\right) \\ &= tr\left(\sigma_0^2Q_{(1)}V_rQ_{(1)}V_0 + \sigma_1^2Q_{(1)}V_rQ_{(1)}V_1\right) \end{aligned}$$

Implies to

$$E\left[\begin{bmatrix} \boldsymbol{Y}'Q_{(t)}V_0Q_{(t)}\boldsymbol{Y} \\ \boldsymbol{Y}'Q_{(t)}V_1Q_{(t)}\boldsymbol{Y} \end{bmatrix}\right] = \begin{bmatrix} tr\left(Q_{(1)}V_0Q_{(1)}V_0\right) & tr\left(Q_{(1)}V_0Q_{(1)}V_1\right) \\ tr\left(Q_{(1)}V_1Q_{(1)}V_0\right) & tr\left(Q_{(1)}V_1Q_{(1)}V_1\right) \end{bmatrix} \begin{bmatrix} \sigma_0^2 \\ \sigma_1^2 \end{bmatrix}$$

Where

$$\boldsymbol{S} = \begin{bmatrix} tr\left(Q_{(1)}V_0Q_{(1)}V_0\right) & tr\left(Q_{(1)}V_0Q_{(1)}V_1\right) \\ tr\left(Q_{(1)}V_1Q_{(1)}V_0\right) & tr\left(Q_{(1)}V_1Q_{(1)}V_1\right) \end{bmatrix}_{2\times 2} = \begin{bmatrix} \lambda_{00} & \lambda_{01} \\ \lambda_{10} & \lambda_{11} \end{bmatrix} \quad (45)$$

and $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_0^2 \\ \sigma_1^2 \end{bmatrix}$, $\lambda_{r,s} = sumation\ all\ eigenvalues\ of\ Q_{(1)}V_rQ_{(1)}V_s$.

we have $\boldsymbol{S}\,\widehat{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{Y}'Q_{(1)}V_0Q_{(1)}\boldsymbol{Y} \\ \boldsymbol{Y}'Q_{(1)}V_1Q_{(1)}\boldsymbol{Y} \end{bmatrix}_{2\times 1}$

$\hat{\sigma}^2 = \boldsymbol{S}^{+}\begin{bmatrix} \boldsymbol{Y}'Q_{(1)}V_0Q_{(1)}\boldsymbol{Y} \\ \boldsymbol{Y}'Q_{(1)}V_1Q_{(1)}\boldsymbol{Y} \end{bmatrix}$, $\boldsymbol{S}^{+} = \frac{1}{|S|}\begin{bmatrix} \lambda_{1,1} & -\lambda_{0,1} \\ -\lambda_{1,0} & \lambda_{0,0} \end{bmatrix}$, $|\boldsymbol{S}|$ is determain of $\boldsymbol{S}$.

We can write as $\left[tr\left(Q_{(t)}Z_rZ_r'Q_{(t)}Z_sZ_s'\right)\right][\hat{\sigma}_r^2] = [\,\boldsymbol{Y}'Q_{(t)}Z_rZ_r'Q_{(t)}\boldsymbol{Y}]$ (46)

Relationships (43-46) are proof of (41).□

Then the MINQUE for σ_ε^2 and σ_δ^2 are

$$\hat{\sigma}_\varepsilon^2 = \hat{\sigma}_0^2 = \boldsymbol{Y}'(s_{0,0}^{+}Q_{(1)}Z_0Z_0'Q_{(1)} + s_{0,1}^{+}Q_{(1)}Z_1Z_1'Q_{(1)})\boldsymbol{Y}. \quad (47)$$

$$= \boldsymbol{Y}'\left[\frac{1}{|\boldsymbol{S}|}\left(\lambda_{1,1}Q_{(1)}Q_{(1)} - \lambda_{0,1}Q_{(1)}(I_{nq}\otimes J_p)Q_{(1)}\right)\right]\boldsymbol{Y}. \quad (48)$$

and

$$\hat{\sigma}_\delta^2 = \hat{\sigma}_1^2 = \boldsymbol{Y}'(s_{1,0}^{+}Q_{(1)}Z_0Z_0'Q_{(1)} + s_{1,1}^{+}Q_{(1)}Z_1Z_1'Q_{(1)})\boldsymbol{Y}. \quad (49)$$

$$= \boldsymbol{Y}'\left[\frac{1}{|\boldsymbol{S}|}\left(-\lambda_{1,0}Q_{(1)}Q_{(1)} + \lambda_{0,0}Q_{(1)}(I_{nq}\otimes J_p)Q_{(1)}\right)\right]\boldsymbol{Y}. \quad (50)$$

4. Conclusions

The conclusions obtained throughout this work are as follows:

1. The MINQUE for σ_ε^2 and σ_δ^2 *are*

$$\hat{\sigma}_\varepsilon^2 = Y'\left[\frac{\left(I_n - (J_n/n)\right)\otimes I_q\otimes\left(I_p - (J_p/p)\right)}{q(n-1)(p-1)}\right]Y$$

And

$$\hat{\sigma}_\delta^2 = Y'\left[\frac{\theta\left(I_n - (J_n/n)\right)\otimes I_q\otimes\left(I_p - (J_p/p)\right)}{q(n-1)(p-1)}\right]Y$$

2. The MINQUE(1) for σ_ε^2 and σ_δ^2 *are*

$$\hat{\sigma}_\varepsilon^2 = \boldsymbol{Y}'\left[\frac{1}{|\boldsymbol{S}|}\left(\lambda_{1,1}Q_{(1)}Q_{(1)} - \lambda_{0,1}Q_{(1)}(I_{nq}\otimes J_p)Q_{(1)}\right)\right]\boldsymbol{Y}.$$

And

$$\hat{\sigma}_\delta^2 = \boldsymbol{Y}'\left[\frac{1}{|\boldsymbol{S}|}\left(-\lambda_{1,0}Q_{(1)}Q_{(1)} + \lambda_{0,0}Q_{(1)}(I_{nq}\otimes J_p)Q_{(1)}\right)\right]\boldsymbol{Y}.$$

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