JQCM
University Of AL-Qadisiyah
Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On a Subclass of Meromorphic Univalent Functions Involving Hypergeometric Function

Mazin Sh.Mahmoud[a], Abdul Rahman S.Juma[b] ,Raheam A. Mansor Al-Saphory[c]

[a] *Department of Mathematics , Tikrit University Tikrit , Iraq.Email:*maz2004in@yahoo.com

[b] *Department of Mathematics, University of Anbar ,Ramadi , Iraq.Email: :* dr-juma@hotmail.com

[c] *Department of Mathematics , Tikrit University ,Tikrit , Iraq, Email:* saphory@hotmail.com

ARTICLE INFO

Article history:
Received: 01 /04/2019
Rrevised form: 24 /04/2019
Accepted : 29 /04/2019
Available online: 26 /05/2019

Keywords:

Hypergeometric function , meromorphic univalent function , Hadamard product" , convex function."starlike function ,

ABSTRACT

The main object of the present paper is to, introduce the. class of meromorphic univalent functions Involving! hypergeomatrc function .We obtain~ some interesting geometric properties according to coefficient inequality , growth and distortion theorems , radii of starlikeness and convexity for the" functions in our subclass.

MSC.

1 . Introduction

Let $\sum$ denoted be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \qquad (1)$$

which are analytic andmeromorphic univalent in punctured unit disk$\text{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

-*A function* $f \in \sum$ is meromorphic starlike of order α, $(0 \le \alpha < 1)$ if $\mathrm{R}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, $(z \in \text{U}^*)$.

The class of all such function is denoted by $\sum^*(\alpha)$. A function$f\in \sum$ is meromorphic convex of order α , $(0\le \alpha < 1)$ - $\mathrm{R}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha$, $(z \in \text{U}^*)$.Let $\sum_q$ be the class of function $\boldsymbol{f}\in \sum$,if

---

Corresponding author Mazin Sh.Mahmoud

Email addresses: maz2004in@yahoo.com

Communicated by Qusuay Hatim Egaar

with $a_k \geq 0$. The subclass of $\sum_q$ consisting of starlike functions of order α is denoted by $\sum_q^*(\alpha)$, and convex functions of order α by $\sum_q^k(\alpha)$. Various subclasses of $\sum$ have been defined and studied by varions authors see [1 , 2 , 3 , 4 , 5, 6 , 7 ,10 , 11,12].

For function *f(z)* given by (1) and g (z)= $\frac{1}{z}+\sum_{k=1}^{\infty} b_k z^k$,we define theHadamard product or(convolution) *f* and g by

$$f * g = \frac{1}{z}+\sum_{k=1}^{\infty} a_k b_k Z^k .$$

For positive real parameters($\alpha_1, A_1, \ldots, \alpha_\ell, A_\ell, \beta_1, B_1, \ldots, \beta_p, B_p$)

(ℓ, p ∈ N = {1 ,2,...}) such that

$1+\sum_{n=1}^{p} B_n-\sum_{n=1}^{\ell} A_n \geq 0 ,(z \in Ụ^*)$ The Wright generalized hypergeometric function

$$={}_\ell\Psi_p[(\alpha_t, A_t)_{1,\ell},(\beta_t, B_t)_{1,p};Z],\Psi_p[(\alpha_1, A_1), \ldots,(\alpha_\ell, A_\ell);(\beta_1, B_1), \ldots,(\beta_p, B_p);Z]_\ell$$

assigned by

$${}_\ell\Psi_p[(\alpha_t, A_t)_{1,\ell},(\beta_t, B_t)_{1,p};Z]=\sum_{n=0}^{\infty}\left\{\prod_{t=0}^{\ell} \Gamma(\alpha_1+nA_t)\right\}\left\{\prod_{t=0}^{p} \Gamma(\beta_t+n\ B_t)\right\}^{-1} \frac{z^n}{n!}.$$

If A_t= 1,(t= 1 ,2,3, ...ℓ) and B_t=1 ,(t = 1 ,2 ,3,....,P), then

$$\Omega_\ell\Psi_p[(\alpha_t,A_t)_{1,\ell},(\beta_t, B_t)_{1,p};z] \equiv F_p(\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_m ; z)$$

$$=\sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots(\alpha_1)_n z^n}{(\beta_1)_n \cdots(\beta_p)_n n!}$$

$(\ell \leq p+1 ; \ell, p \in \mathbb{N}_0=\mathbb{N}=\{0,1,2,3, \ldots\} ; Z \in Ụ)$.

That is the generalized hypergeometric function (see[8]). Here (α_k) is the Pochammer symbol and 'Ω= $\left(\prod_{t=0}^{\ell} \Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{p} \Gamma(\beta_t)\right)$.

When assign the generalized hypergeometric function , we take a Linear operator

$$\boldsymbol{W}[(\alpha_1, A_t)_{1,\ell},(\beta_t, B_t)_{1,P}]: \textstyle\sum_q \longrightarrow \sum_q .$$

$$\boldsymbol{W}[(\alpha_t, A_t)_1, \ell,(\beta_t, B_t)_{1,P}]\, f(z)=z^{-1} \Omega_l \Psi_P\left[(\alpha_t, A_t)_{1,\ell}\ (\beta_t, B_t)_{1,P}\right] ; z] * f(z) \qquad (2)$$

for convenience ,we denote $\boldsymbol{W}[(\alpha_t, A_t)_{1,\ell},(\beta_t, B_t)_{1,p}]$ by $\boldsymbol{W}[\alpha_1]$.

If f has the from (1) then we obtain

$$W[\alpha_1]\ f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} \sigma_k(\alpha_1) a_k z^k , \qquad (3)$$

where

$$\sigma_k(\alpha_1)=\frac{\Omega \Gamma(\alpha_1+A_1(k+1)) \ldots \Gamma(\alpha_\ell+A_\ell(k+1))}{(k+1)!\, r(\beta_1+B_1(k+1)) \ldots \Gamma(\beta_\ell+B_\ell(k+1))} . \qquad (4)$$

Definition1.1: A subclass of $\boldsymbol{\Sigma_q}$ by utilizing operator $W\,[\alpha_1]$ we let $V(\alpha,\eta)$ denote a subclass of Σ_q consisting of function in (1) satisfying the condition

$$\left|\frac{\frac{z\,(W(\alpha_1)f(z)\ )''}{(W(\alpha_1)f(z))'}+2}{\frac{z\,(W(\alpha_1)f(z)\ )''}{(W(\alpha_1)f(z))'}+2\alpha}\right|<\eta. \qquad (5)$$

$0<\alpha<1\,,0<\eta\leq 1$ and A_t=1 (t=1,2,3,…), B_t=1 (t=1,2,3,…) Where
Now we must prove the Coefficient Inequality

2.Coeffcient Inequality

Theorem 2.1: f is a function defined by (1) in the class V(α,η), if and only if

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\big[k\big(1+\eta\big)+\big(1+\eta(2\alpha-1)\big)\big]a_k\leq 2\eta(1-\alpha). \qquad (6)$$

The result is sharp"

Proof : Let the inequity (6) holds true and let |z|=1 by (5). Then we get

$$\left|\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'}+2\right|-\eta\left|\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'}+2\alpha\right|<0\,,$$

$$|z(W(\alpha_1)f(z))''+2(W(\alpha_1)f(z))'|-\eta|z(W(\alpha_1)f(z))''+2\alpha(W(\alpha_1)f(z))'|,$$

and by utilizing (3) we have

$$(W(\alpha_1)f(z))'=\frac{-1}{z^2}+\sum_{k=1}^{\infty}n\,\sigma_k(\alpha_1)a_k z^{k-1}\quad ,$$

$$(W(\alpha_1)f(z))''=\frac{+2}{z^3}+\sum_{k=1}^{\infty}k(k-1)\,\sigma_k(\alpha_1)a_k z^{k-2},$$

$$z(w(\alpha_1)f(z))''=\frac{2}{z^2}+\sum_{k=1}^{\infty}k(k-1)\sigma_k(\alpha_1)a_k z^{k-1},$$

$$=\left|\tfrac{2}{z^2}+\textstyle\sum_{k=1}^{\infty}k(k-1)\sigma_k(\alpha_1)a_k z^{k-1}-\tfrac{2}{z^2}+2\sum_{k=1}^{\infty}k\sigma_k(\alpha_1)a_k z^{k-1}\right|$$
$$-\eta\left|\tfrac{2}{z^2}+\textstyle\sum_{k=1}^{\infty}k(k-1)\sigma_k(\alpha_1)a_k z^{k-1}-\tfrac{2\alpha}{z^2}+2\alpha\sum_{k=1}^{\infty}k\sigma_k(\alpha_1)a_k z^{k-1}\right|$$

$$=|\textstyle\sum_{k=1}^{\infty}k\sigma_k(\alpha_1)a_k z^{k-1}(k-1+2)|-\eta\left|\tfrac{2}{z^2}(1-\alpha)+\sum_{k=1}^{\infty}k\sigma_k(\alpha_1)a_k z^{k-1})(k-1+2\alpha)\right|$$

$$\leq\sum_{k=1}^{\infty}k\,\sigma_k(\alpha_1)a_k|z|^{k-1}(k+1)\frac{-2\eta}{|z|^2}(1-\alpha)+\eta\sum_{k=1}^{\infty}k\,\sigma_k(\alpha_1)a_k|z|^{k-1}(k-1+2\alpha)$$

$$\leq\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|[k(1+\eta)+(1+\eta(2\alpha-1)]a_k-2\eta(1-\alpha)\leq 0.$$

Therefore, by the maximum modules theorem we have $f\in V(\alpha,\eta)$,
Conversely, suppose $f\in V(\alpha,\eta)$,then

$$\left|\frac{\frac{z\,(W(\alpha_1)f(z)\ )''}{W(\alpha_1)f(z))'}+2}{\frac{z\,(W(\alpha_1)f(z)\,)''}{(W(\alpha_1)f(z))'}+2\alpha}\right|<\eta,$$

$$\left|\frac{\sum_{k=1}^{\infty}(k+1)|\sigma_k(\alpha_1)|a_k z^{k-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{k=1}^{\infty}(k-1+2\alpha)|\sigma_k(\alpha_1)|a_k z^{k-1}}\right|<\eta.$$

Since $|\mathrm{Re}(z)|\leq |z|$ for all z , we get

$$Re\left\{\frac{\sum_{k=1}^{\infty}(k+1)|\sigma_k(\alpha_1)|a_k z^{k-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{k=1}^{\infty}(k-1+2\alpha)|\sigma_k(\alpha_1)|a_k z^{k-1}}\right\}<\eta \qquad (7)$$

on the real axis when choosing the value of z thus the value of

$$\frac{z\,(W(\alpha_1)f(z)\ )''}{(W(\alpha_1)f(z))'}\ ,$$

is real , therefore clearing the denominator of (7) and when $z \to 1^-$ through real axis the result is sharp for the function

$$f_k(z)=\frac{1}{2}+|\sigma_k\ (\alpha_1)|^{-1}\times\frac{2\eta(1-\alpha)}{[k(1+\eta)+(1+\eta(2\alpha-1))}z^k,\ k\geq 1 \quad (8)$$

Corollary 2.1 : When $f\in V(\alpha,\eta)$, then

$$a_k\leq|\sigma_k(\alpha_1)|^{-1}\times\frac{2\eta(1-\alpha)}{[k(1+\eta)+(1+\eta(2\alpha-1))}\,,$$

where $0<\alpha<1$, $0<\eta\leq 1$.

3. Growth and Distortion Theorems

Distortion and growth Theorems property for the function $f\in V(\alpha,\eta)$, is given as follows :

Theorem 3.1: Let f be a function defined by (1) is in the class $V(\alpha,\eta)$.
Then for $0<|z|=r<1$ we get

$$\frac{1}{r}-r|\sigma_1(\alpha_1)|^{-1}\times\frac{\eta(1-\alpha)}{(1+\alpha\eta)}\leq|f(z)|\leq\frac{1}{r}+r\frac{\eta(1-\alpha)}{(1+\alpha\eta)}|\sigma_1(\alpha_1)|^{-1},$$

equivalences for

$$f\ (z)\ =\ \frac{1}{z}+|\sigma_1(\alpha_1)|^{-1}\times\frac{\eta(1-\alpha)}{(1+\alpha\eta)}z.$$

Proof: Since $f\in V(\alpha,\eta)$, then we get by theorem 2.1,then the inequality

$$\sum_{k=1}^{\infty}\sigma_k(\alpha_1)[k(\eta+1)+(1+\eta(2\alpha-1))]a_k\leq 2\eta(1-\alpha).$$

Then

$$|f(z)|\leq\left|\frac{1}{z}\right|+\sum_{k=1}^{\infty}a_k|z|^k,$$

for $0<|z|=r<1$, we get

$$|f(z)|<\frac{1}{r}+r\sum_{k=1}^{\infty}a_k\leq\frac{1}{r}-|\sigma_1(\alpha_1)|^{-1}\times\frac{\eta(1-\alpha)}{(1+\alpha\eta)}r.$$

In addition to

$$|f(z)|\geq\left|\frac{1}{z}\right|-\sum_{k=1}^{\infty}a_k|z|^k\geq\frac{1}{r}-|\sigma_1(\alpha_1)|^{-1}\times\frac{\eta(1-\alpha)}{(1+\alpha\eta)}r,\ \ |z|=r.$$

Theorem 3.2:Let A function f defined by (1) in the class $f\in V(\alpha,\eta)$. *Then*

for $0 < |z| = r < 1$ we get

$$\frac{1}{r^2} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} \le |f'(z)| \le \frac{1}{r^2} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)}.$$

Equivalences for

$$f(z) = \frac{1}{z} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} z\,.$$

proof: Form Theorem 2.1, we get

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)|[k(1+\eta) + (1+\eta(2\alpha-1))]a_k \le 2\eta(1-\alpha).$$

Thus

$$|f'(z)| \le |\frac{-1}{z^2}| + \sum_{k=1}^{\infty} k a_k |z|^{k-1},$$

for $0 < r = |z| < 1$ we get

$$|f'(z)| \le |\frac{-1}{r^2}| + \sum_{k=1}^{\infty} k a_k$$
$$\le \frac{1}{r^2} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} \quad .$$

And

$$|f'(z) \ge \frac{-1}{z^2}| - \sum_{k=1}^{\infty} k a_k |z|^{k-1},$$
$$\ge |\frac{1}{r^2}| - \sum_{k=1}^{\infty} k a_k$$
$$\ge \frac{1}{r^2} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)}.$$

4.Hadamard product

Theorem 4.1: If the function $g, f \in V(\alpha,\eta)$. Then $(f * g) \in V(\alpha,\eta)$, for

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$
$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k,$$

and

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k,$$

where

$$\delta = \frac{2\eta^2\ (1-\alpha)(k+1)}{2\eta^2(1-\alpha)(k+2\alpha-1) - |\sigma_n(\alpha_1)|[k(1+\eta) + (1+\eta(2\alpha-1))]^2}$$

Proof: Since f, g $\in V(\alpha,\eta)$, then by Theorem 2.1, we have

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} a_k \le 1,$$

and

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[n(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} b_k \le 1,$$

we must find the largest δ *such that*

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}\sqrt{a_k b_k}\leq 1. \qquad (9)$$

To prove the theorem it is over to show that

$$|\sigma_k(\alpha_1)|\frac{\left[k(1+\delta)+\left(1+\delta(2\alpha-1)\right)\right]}{2\delta(1-\alpha)}a_k b_k$$
$$\leq |\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}\sqrt{a_k b_k} \quad ,$$

which is equivalent to

$$\sqrt{a_k b_k}\leq\frac{\delta\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{\beta\left[k(1+\delta)+\left(1+\delta(2\alpha-1)\right)\right]} \quad .$$

From (9) we get

$$\sqrt{a_k b_k}\leq |\sigma_r(\alpha_1)|\frac{2\eta(1-\alpha)}{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]} \quad .$$

We must proof that

$$|\sigma_k(\alpha_1)|\frac{2\eta(1-\alpha)}{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}\leq\frac{\delta\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{\eta\left[k(1+\delta)+\left(1+\delta(2\alpha-1)\right)\right]},$$

which gives

$$\delta\leq\frac{2\eta^2(\alpha-1)(k+1)}{2\eta^2(1-\alpha)(k+2\alpha^{-1})-|\sigma_k(\alpha_1)|[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)]^2}.$$

Theorem 4.2 : If the function f_i (i=1 ,2) defined by

$$f_{\mathrm{i}}(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_{k,\mathrm{i}}z^k \,,(a_{k,\mathrm{i}}\geq 0\,,\mathrm{i}=1,2)$$

be in the class *V(α,η)*, then the function defined

$$\mathrm{g(z)}=\frac{1}{z}+\sum_{k=1}^{\infty}\left(a^2{}_{k,1}+a^2{}_{k,2}\right)z^k \;,$$

is in the class *V(α,η)*, where

$$\beta=\frac{4\eta^2(\alpha-1)(k+1)}{4\eta^2(\alpha-1)(k+2\alpha-1)-|\sigma_k(\alpha_1)|[k(1+\eta)+(1+\eta(2\alpha-1))]^2}$$

proof: Since$f_i \in V(\alpha,\eta)$,(i= 1 ,2), then by Theorem 2.1, we get

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}a_{k,i}\leq 1,(i=1,2)\,.$$

Hence

$$\sum_{k=1}^{\infty}(|\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)})^2 a_{k,i}{}^2$$
$$\leq(\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}a_{k,i})^2\leq 1,(i=1,2).$$

Thus

$$\sum_{k=1}^{\infty}\frac{1}{2}|\sigma_k(\alpha_1)|(\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)})^2({a_{k,1}}^2+{a_{k,2}}^2)\leq 1,$$

to prove the theorem we must find the largestβ such that

$$\frac{\left[k(\beta+1)+\left(1+\beta(2\alpha-1)\right)\right]}{\beta}\leq\frac{|\sigma_k(\alpha_1)|\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]^{\ 2}}{4\eta^{\ 2}\ (1-\alpha)},k\geq 1,$$

so that

$$\beta\leq\frac{4\eta^2(\alpha-1)(k+1)}{4\eta^2(\alpha-1)(k+2\alpha-1)-|\sigma_k(\alpha_1)|\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]^{\ 2}}.$$

Theorem 4.3: If *f(z)*= $\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k\in V(\alpha,\eta)$, and

$g\ (z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_kz^k$with $|b_k|\leq 1$ is in the class $V(\alpha,\eta$
then $f(z)*g(z)\in V(\alpha,\eta)$.

Proof : By Theorem 2.1, we get

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]a_k\leq 2\eta(1-\alpha).$$

Since

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|(\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}|a_kb_k|,$$

$$=\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|(\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]}{2\eta(1-\alpha)}a_k|b_k|,$$

$$\leq\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right]a_k\leq 1.$$

Thus$f(z)*g(z)\in V(\alpha,\eta)$.

Corollary 4.1: If $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k\in V(\alpha,\eta)$, and$g\ (z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_kz^k$with $0\leq b_k\leq\ 1$ *is in the* $V(\alpha,\eta)$, *then* $f(z)*\ g(z)\in\ V(\alpha,\eta)$.

5. Radil of starlikness and convexity

Theorem 5.1: Let $f\ (z)$ be the function defined by (1) be in the subclass V(α,η). Then f is meromorphicallystarlike of orderδ($0\leq\delta<1$) in the disk $|z|<r_1(\alpha,\eta,\delta)$, where

$$r_1(\alpha,\eta,\delta)=\inf_k\{|\sigma_k(\alpha_1)|\frac{\left[k(1+\eta)+\left(1+\eta(2\alpha-1)\right)\right](1-\delta)}{2\eta(k+2-\delta)(1-\delta\alpha)}\}^{\frac{1}{k+1}}$$

the result is sharp for the function given by (8).

Proof: We show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\delta,$$

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{k=1}^{\infty}(k+1)a_k z^k}{z^{-1}+\sum_{k=1}^{\infty} a_k z^k}\right| \leq \frac{\sum_{k=1}^{\infty}(k+1)a_k |z|^{k+1}}{1-\sum_{k=1}^{\infty} a_k |z|^{k+1}}.$$

This will be bounded by $1-\delta$,

$$\frac{\sum_{k=1}^{\infty}(k+1)a_k |z|^{k+1}}{1-\sum_{k=1}^{\infty} a_k |z|^{k+1}} \leq 1-\delta\,,$$

$$\sum_{k=1}^{\infty}(2+k-\delta)a_k\,|z|^{k+1} \leq 1-\delta\,,$$

from Theorem 2.1, we get

$$\sum_{k=1}^{\infty}|\sigma_k(\alpha_1)|(\frac{[k(1+\eta)+(1+\eta(2\alpha-1))]}{2\eta(1-\alpha)}a_k \leq 1.$$

Hence

$$|z|^{k+1} \leq |\sigma_k(\alpha_1)|\frac{[k(1+\eta)+(1+\eta(2\alpha-1))](1-\delta)}{2\eta(k+2-\delta)(1-\alpha)}\,,$$

$$|z|\leq\left\{|\sigma_k(\alpha_1)|\frac{[k(1+\eta)+(1+\eta\ \ (2\alpha-1))](1-\delta)}{2\eta(k+2-\delta)(1-\alpha)}\}^{\frac{1}{k+1}}\right\}^{\frac{1}{k+1}} \quad .$$

Theorem 5.2: Let the function $f(z)$ defined by (1)be in the subclass $V(\alpha,\eta)$.
Then f is meromorphically convex of order $\curlyvee$ ($0\leq\curlyvee<1$)in the disk
$|z|<r_2(\eta\ ,\alpha,\curlyvee)$, where

$$r_2(\eta\ ,\alpha,\curlyvee)=\inf\left\{|\sigma_k(\alpha_1)|\frac{[k(1+\eta)+(1+\eta(2\alpha-1))](1-\curlyvee)}{2\eta(k+2-\curlyvee)(1-\alpha)}\}^{\frac{1}{k+1}}\right\}.$$

The result is sharp for the function given by (7).
Proof: By utilizing the same way in the proof of theorem 5.1 we can get this

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\curlyvee,\left((0\leq\curlyvee<1)\right).$$

For $|z|< r_2$ depending on the help of the Theorem 2.1,lead to confirmed of theorem 5.2.□

References

[1] M.K.Aoufand H. E.Darwish,Certainmeromorphicallystarlike functions with positive and fixed second coefficient ,*Turkish J.Math* ,21(3 (1997) ,311-316.

[2] W. G. Atshan and S. R. Kulkarni,Subclass of Meromorphic Function with Positive Coefficients defined by Ruscheweyh Derivative I, J. Rajasthan Academy of Physical Scieces,6(2)(20),129-140.

[3] N. E. Cho ,S. H. Lee and S .Owa,Aclass of meromorphicunivalent functions with positive coefficients, *Kobe J.Math.*,4(1)(1987),43-50.

[4] J. Clunie ,On meromorphicschlicht functions ,*J.London Math .Soc.* ,34(1959) ,215-216 .

[5] J. Dziok ,G. Murugusundaramoorthyand J. Sokol, On certain class of meromorphic functions with positive coefficients, *ActaMath.Sci.Ser .BEngl. Ed., 32(4)(2012),1376-1390.*

[6] F. Ghanimand M. Darus, On acertain subclass of meromorphicunivalent functions with fixed second positive coefficients ,*Surv .Math. Appl .*,5(2010) ,49-60.

[7] F. Ghanimand M. Darus, On class of HypergeometricMeromorphic with fixed second positive coefficients ,*GenMath .* 1 7(4)(2009), 1328.

[8] F. Ghanim and M. Darus, On acertain subclass of meromorphicunivalent functions with fixed second positive coefficients,*Surv.Math .Appl.*,5 (2010) , 49 - 60 .

[9] A. R. S. Juma and H. I. Dhayea,On a subclass of Meromorphic univalent function with Fixed Second Coeffcient Involving Fox-Wright's HyPergeometricfunction,Gen.Math,vol 28 (2015),PP.30-41.

[10] A. R. S. Juma and H . Zirar,On a Class of Meromorphic univalent function defined by Hypergetric Function, Gen.Math,vol 17 (2013),pp.63-73.

[11] S. Kavitha ,S. Sivasubramanianand K. Muthunagai, Anew subclass of Meromorphicfunction with positive coefficients, *Bull. Math. Anal .Appl.*, 2 (3) (2010) , 109-121.

[12] J. Ma, Extreme points and minimal outerareaproblem forMeromorphicunivalent functions, *J. Math.Anal.Appl.*,220(2)(1998),769-773.

[13] N. Magesh, N. B. Gattiand S. Mayilvaganan, On certain subclasses of meromorphic functions with positive and fixed second coefficients involving the Liu-Srivastava linear operator, *ISRNMath. Anal.*, ArtileID 698307 (2012),11pages.

[14] S. Owaand N. Pascu, Coefficient inequalities for certain classes ofMeromorphicallystarlike and meromorphicallyconvex functions,*JIPAM. J. Inequal. Pure Appl.Math.*,4(1)(2003),6.