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Weakly Secondary Submodules

Ghaleb Ahmed [a] , **Zainab Saadi** [b]

[a] Department of Mathematics, University of Baghdad, Baghdad, Iraq. Email: ghaleb.a.h@ihcoedu.uobaghdad.edu.iq

[b] Department of Mathematics, University of Baghdad, Baghdad, Iraq. Email: Zoozaih89@gmail.com

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ABSTRACT

Let M be a non-zero right module over a ring R with an identity. The weakly secondary submodule is introduced in this paper. A non-zero submodule N of M is weakly secondary submodule when $Nab \subseteq K$, where $a, b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t. Some relationships between this class of modules and other related modules are discussed and number of conclusions and characterizations are obtained.

MSC.

1. Introduction

R is indicated a ring has an identity and M is viewed as a non-zero left S-right R-bimodule where $S = \mathrm{End}_R(M)$ the endomorphism ring of M.We use the notation " $\subseteq$ " to denote inclusion. $0 \neq N$ is a second submodule of M if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or zero (that is either $Imf_a = Na = N$ or $Imf_a = Na = 0$) [1]. Equivalently $0 \neq N$ is a second submodule of M if $NI = N$ or $NI = 0$ for every ideal I of R [1]. In that situation, $ann_R(N)$ is a prime ideal of R[1]. $0 \neq M$ is second (or coprime) if M is a second submodule of itself [1]. As a new type of second submodules, the concept of weakly second submodule was presented and studied in [2]. $0 \neq N$ is a weakly second submodule of M whenever $Nab \subseteq K$ where $a, b \in R$ and K a submodule of M implies either $Na \subseteq K$ or $Nb \subseteq K$ [2]. $0 \neq M$ is a weakly second module if M is a weakly second submodule of itself [2]. In fact this idea as a dual notion of the concept weakly prime (sometimes is called classical prime) submodule. A proper submodule N of M is wekly prime whenever $Kab \subseteq N$ where $a, b \in R$ and K a submodule of M implies either $Ka \subseteq N$ or $Kb \subseteq N$ [3]. In our work, we supplied the idea of weakly secondary as a generalization of weakly second concept and the same time it is a new class of secondary submodules and a dual notion of classical primary submodules. A nonzero submodule N of M is weakly secondary submodule if $Nab \subseteq K$ where a, $b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t. $0 \neq M$ is a weakly secondary module if M is a weakly secondary submodule of itself. A non-zero submodule N is a secondary submodule of M if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or nilpotent (that is $Imf_a = Na = N$ or $Imf_a = Na^t = 0$) [1]. Equivalently, $0 \neq N$ is a secondary submodule of M if for every ideal I of R, $NI = N$ or $NI^t = 0$ for some positive integer t [1]. In this case, $ann_R(N)$ is a primary ideal of R (that is $\sqrt{ann_R(N)}$ is prime) [1]. A proper submodule K of M is classical primary if $Nab \subseteq K$ where $a, b \in R$ and N is a submodule of M then either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive

Corresponding author Ghaleb Ahmed

Email addresses: ghaleb.a.h@ihcoedu.uobaghdad.edu.iq

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integer t [4]. A proper submodule K of M is called completely irreducible when $K = \bigcap_{i\in\wedge} H_i$ where $\{H_i\}_{i\in\wedge}$ is a family of submodules of M implies that $K = H_i$ for some $i \in\wedge$ [2]. It is not hard to see that every submodule is an intersection of completely irreducible submodules of M consequently the intersection of all completely irreducible submodules of M is zero [2]. N is called simple (sometimes minimal) submodule of a module M if $N \neq 0$ and for each submodule L of M and N contains L properly implies $L = 0$ [5]. M is called a simple module if M is simple submodule of itself [5]. M is coquasi-dedekind if all nonzero endomorphism of M is epimorphism (in other word, $f(M) = M$ for every $0 \neq f \in S$) [6]. Let R be a commutative integral domain, M is called divisible module over R if $Ma = M$ for each $0 \neq a \in R$ [5]. A proper submodule N is maximal if it is not properly contained in any proper submodule of M [5]. A proper submodule N is called prime if $mr \in N$ implies $m \in N$ or $Mr \subseteq N$ [7]. A proper ideal I is prime if $ab \in I$ where $a, b \in R$ implies $a \in I$ or $b \in I$ [8]. Equivalently, a proper ideal I is prime if $AB \subseteq I$ where A and B are ideals of R implies $A \subseteq I$ or $B \subseteq I$ [8]. A ring in which every ideal is prime is called fully prime [9]. Equivalently, a ring R is fully prime if and only if it is fully idempotent and the set of ideals of R is totally ordered under inclusion [9]. A proper submodule N is called primary if $mr \in N$ implies $m \in N$ or $Mr^t \subseteq N$ for some positive integer t [4]. A proper ideal I is primary if $ab \in I$ where $a, b \in R$ implies $a \in I$ or $b^t \in I$ for some positive integer t [4]. A ring in which every ideal is primary is called generalized primary [10]. M is comultiplication provided for each submodule N of M, there exists an ideal I of R such that $[0:_M I] = ann_M(I) = \{m \in M \mid Im = 0\}$ is a submodule of M [11]. We able to take $I = [0:_R N] = ann_R(N) = \{r \in R \mid Nr = 0\}$ is an ideal of R [11]. M is called S-second if every $f \in S$ implies $f(M) = M$ or $f(M) = 0$ [12]. Also M is S-secondary if every $f \in S$ implies $f(M) = M$ or $f^t(M) = 0$ for some positive integer t [12]. M is indecomposable if $M \neq 0$ and it cannot be written as a direct sum of non-zero submodules (that is 0 and M are the only direct summands) [5]. M is called multiplication when each submodule N of M, we have $N = MI$ for an ideal I of R [13]. We able to take $I = [N :_R M] = \{r \in R \mid Mr \subseteq N\}$ [13]. M is a scalar module when for each $f \in End(M)$ there is $a \in R$ with $f(m) = ma$ for all $m \in M$ [14].

This paper consists of four sections. Within section two, we introduce and investigate the concept of weakly secondary submodules. Simultaneously some of characterizations of this concept are presented (Theorem 2.2 and Theorem 2.3). We provide many information (Remarks and Examples 2.5) and necessity features of this concept. The direct sum of weakly secondary submodules is investigated (Proposition 2.7). Among other results a new description of secondary submodules is given (Proposition 2.4). Section three includes (Theorem 3.1) the most important characterization of weakly secondary submodules used frequently in our work. Further (Theorem 3.7) another characterization of this concept we finish this section via **(**Corollary 3.10) which shows under what situation weakly secondary submodules to be secondary submodules. In section four we define S-weakly second modules. We present basic properties and characterizations of this modules (Remarks and Examples 4.4, Theorem 4.5 and Theorem 4.6). In what follows, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ and $Mat_n(R)$ we denote respectively, integers, rational numbers, the p-Prüfer group, the residue ring modulo n and an $n \times n$ matrix ring over R .

2. Weakly Secondary Submodules

Central features of this class of modules are presented in this section . We begin via our main definition.

Definition 2.1: A nonzero submodule N of M is weakly secondary submodule if $Nab \subseteq K$, where $a, b \in R$ and K is a submodule of M implies $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t.

Theorem 2.2: The following statements are equivalent

(1) N is a weakly secondary submodule of M.
(2) $N \neq 0$ and for each $a, b \in R$ and K is a finite intersection of completely irreducible submodules of M with $Nab \subseteq K$ implies either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $0 \neq N$ and K are submodules of M with $Nab \subseteq K$ where $a, b \in R$. Suppose $Na \nsubseteq K$ and $Nb^t \nsubseteq K$ for each positive integer t. As we mentioned before, $K = \cap_{i \in \wedge} H_i$ for some collection $\{H_i\}_{i \in \wedge}$ of completely irreducible submodules of M. We have $Na \nsubseteq \cap_{i \in \wedge} H_i$ and $Nb^t \nsubseteq \cap_{i \in \wedge} H_i$ for each positive integer t. So there exists $i, j \in \wedge$ such that $Na \nsubseteq H_i$ and $Nb^t \nsubseteq H_j$ for each positive integer t. But $Nab \subseteq K \subseteq H_i \cap H_j$ because $K \subseteq H_i$ for each $i \in \wedge$. By hypothesis $Na \subseteq H_i \cap H_j$ or $Nb^t \subseteq H_i \cap H_j$ for some positive integer t. Then $Na \subseteq H_i$ and $Nb^t \subseteq H_j$ which is a contradiction. Hence either $Na \subseteq K$ or $Nb^t \subseteq K$.

Theorem 2.3: the following statements are equivalent

(1) N is a weakly secondary of an R-module M.
(2) $N \neq 0$ and for each $a, b \in R$ implies either $Na = Nab$ or $Nb^t \subseteq Nab$ for some positive integer t.

Proof. (1) $\Rightarrow$ (2) First $N \neq 0$ because N is weakly secondary of M. Let $a, b \in R$ and K a submodule of M with $Nab \subseteq K$. Put $K = Nab$ then $Nab \subseteq Nab$ implies $Na \subseteq Nab$ and hence $Na = Nab$ or $Nb^t \subseteq Nab$ for some positive integer t.

(2) $\Rightarrow$ (1) Let $0 \neq N$ and K are submodules of M with $Nab \subseteq K$ where $a, b \in R$. By hypothesis $Na = Nab$ or $Nb^t \subseteq Nab$ thus $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t as desired.

Theorem 2.4: [15] the following statements are equivalent

(1) N is a secondary submodule of M.
(2) $N \neq 0$ and whenever $Na \subseteq K$ where $a \in R$ and K is a submodule of M implies either $N \subseteq K$ or $N\,a^t = 0$ for some positive integer t.

Remarks and Examples 2.5:

(1) Every secondary submodule is weakly secondary.

Proof. Let N be a secondary submodule of M and $a, b \in R$, K is a submodule of M with $Nab \subseteq K$ implies either $Nab = N$ or $N(ab)^n = 0$ for some positive integer n. If $Nab = N$ then $Na \subseteq N = Nab \subseteq K$ and $Nb \subseteq N = Nab \subseteq K$. In case $N(ab)^n = 0$ that is $Na^n b^n = 0$. Again either $Na^n = N$ or $N(a^n)^m = 0$ for some positive integer m, implies $Nb^n = 0 \subseteq K$ or $Na^t = 0 \subseteq K$, $t = nm$. Similarly if $Nb^n = N$ or $N(b^n)^m = 0$ for some positive integer m, we have $Na^n = 0 \subseteq K$ or $Nb^t = 0 \subseteq K$, $t = nm$ as desired.

(2) Weakly secondary submodules fail to be secondary. Consider $M = \mathbb{Z}_4 \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module where p is a prime number. By simple calculation we see M is weakly secondary but it is not secondary because $0_M \neq M.2^n = 0 \oplus \mathbb{Z}_{p^\infty} \neq M$ for each positive integer n.
(3) Clearly every weakly second submodule is weakly secondary while the converse is not true. $M = \mathbb{Z}_4 \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is weakly secondary but it is not weakly second since $M.2.2 = 0 \oplus \mathbb{Z}_{p^\infty} = K$ while $M.2 \nsubseteq K$. We would like to refer that weakly second submodules is studied in more detail by authors see [16].
(4) The secondary submodules and weakly second submodules concepts do not imply from each one to another [27].
(5) Clearly weakly second and weakly secondary concepts are coincide over Boolean rings.
(6) $\mathbb{Z}_{p^t}$ as $\mathbb{Z}$-module is secondary (and hence weakly secondary) for each a prime number p and t a positive integer.
(7) The following implication is clear simple submodule $\Rightarrow$ second submodule $\Rightarrow$ secondary (or weakly second) submodule $\Rightarrow$ weakly secondary submodule.
(8) The following implication is clear coquasi-dedekind module $\Rightarrow$ second module $\Rightarrow$ secondary (or weakly second) module $\Rightarrow$ weakly secondary module.
(9) If N is a maximal (and hence prime) submodule then N may not be weakly secondary. For example, $N = \mathbb{Z}_{12}.2$ is a maximal submodule in $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module but N is not weakly secondary since $N.2.3 = 0$ and neither $N.2 \neq 0$ nor $N.3 \neq 0$.
(10) Let N and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If N is weakly secondary of M then H need not be a weakly secondary submodule of M. For example, let $N = <\bar{2}>$ and $H = \mathbb{Z}_6 = M$ submodules of $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module where N is a simple

submodule so it is weakly secondary while H is not weakly secondary because $H2.3 = 0$ and $H.2^t =< \bar{2} >$ and $H.3^t =< \bar{3} >$ for each positive integer t.

(11) Let N and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If H is weakly secondary of M then N need not be a weakly secondary submodule of M. For example, let $N =< \frac{1}{p} + \mathbb{Z} > \oplus < \frac{1}{q} + \mathbb{Z} >$ and $H = M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ be submodules of $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ as $\mathbb{Z}$-module . Since H is a divisible module then H is a weakly secondary submodule of M but N is not a weakly secondary submodule of M because $p.q.N = 0_M$ while $N.p^t = 0 \oplus \mathbb{Z}_{q^\infty}$ and $N.q^t = \mathbb{Z}_{p^\infty} \oplus 0$ for each positive integer t. As another example, $\mathbb{Q}$ as $\mathbb{Z}$-module is divisible so it is weakly second but the submodule $\mathbb{Z}$ is not weakly second.

Proposition 2.6: Every nonzero homomorphic image of weakly secondary submodule is weakly secondary.

Proof. Let A and B be R-modules and $0 \neq f: A \to B$ an R-homomorphism. Let N be a weakly secondary submodule of A such that $f(N) \neq 0$. For each $a, b \in R$ then $f(N)ab = f(Nab) = f(Na) = f(N)a$ or $f(N)ab = f(Nab) \supseteq f(Nb^t) = f(N)b^t$ for some positive integer t.

Proposition 2.7: If $N = N_1 \oplus N_2$ is a weakly secondary submodule of $M = M_1 \oplus M_2$ such that $N_1 \neq 0_{M_1}$ and $N_2 \neq 0_{M_2}$. Then N_1 and N_2 are weakly secondary submodules of R-modules M_1 and M_2 respectively.

Proof. Let $a, b \in R$ then either $(N_1 \oplus N_2)ab = (N_1 \oplus N_2)a$ or $(N_1 \oplus N_2)ab \supseteq (N_1 \oplus N_2)b^t$ and hence $N_1.ab = N_1.a$ or $N_1.ab \supseteq N_1.b^t$ and either $N_2.ab = N_2.a$ or $N_2.ab \supseteq N_2.b^t$ for some positive integer t as required.

Corollary 2.8: Every non-zero direct summand of a weakly secondary module is weakly secondary.

Remarks and Examples 2.9:

(1) The direct sum of weakly secondary submodules need not be weakly secondary. For example, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ as $\mathbb{Z}$-modules are weakly secondary where p and q are prime numbers but $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is not weakly secondary $\mathbb{Z}$-module since $(\mathbb{Z}_p \oplus \mathbb{Z}_q)pq = 0 \oplus 0$ while $(\mathbb{Z}_p \oplus \mathbb{Z}_q)p^t = 0 \oplus \mathbb{Z}_q$ and $(\mathbb{Z}_p \oplus \mathbb{Z}_q)q^t = \mathbb{Z}_p \oplus 0$ for each positive integer t.

(2) It is clear if n is a square-free integer (a square-free integer is an integer which has a prime factorization has exactly one factor for each prime that appears in it. For example, 10 = 2 · 5 is square-free) then $\mathbb{Z}_n$ as $\mathbb{Z}$-module is not weakly secondary. $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module is not weakly secondary because $\mathbb{Z}_{12}.3.4 = 0$ but $\mathbb{Z}_{12}.3^t \neq 0 \neq \mathbb{Z}_{12}.4^t$ for each positive integer t and 12 is not square-free.

(3) Let $M = A \oplus B$ be a direct sum of two R-modules A and B. If N is a weakly secondary submodule of A then $N \oplus B$ may be not a weakly secondary submodule of M. For example $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module so it is weakly secondary while $\mathbb{Q} \oplus \mathbb{Z}$ is not a weakly secondary since $[\mathbb{Q} \oplus 6\mathbb{Z} :_{\mathbb{Z}} \mathbb{Q} \oplus \mathbb{Z}] = 6\mathbb{Z}$ is not a primary ideal of $\mathbb{Z}$ then by Theorem, $\mathbb{Q} \oplus \mathbb{Z}$ is not a weakly secondary $\mathbb{Z}$-module. In fact for any R-module M then $M \oplus \mathbb{Z}$ is not a weakly secondary $\mathbb{Z}$-module.

(4) $\mathbb{Q} \oplus \mathbb{Z}$, $\mathbb{Q} \oplus \mathbb{Z}_n$, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ and $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_n$ as $\mathbb{Z}$-modules are not weakly secondary by (2) and (4) where n is a square-free integer.

Proposition 2.10: If N is a weakly secondary submodule of M then $N \oplus N$ is weakly secondary of $M \oplus M$.

Proof. Firstly $N \oplus N \neq 0 \oplus 0$ because $N \neq 0$. Let $a, b \in R$ then $(N \oplus N)ab = Nab \oplus Nab$ but N is weakly secondary implies either $Nab = Na$ and hence $(N \oplus N)ab = (N \oplus N)a$ or $Nab \supseteq Nb^t$ implies $(N \oplus N)ab \supseteq (N \oplus N)b^t$ for some positive integer t as required.

Proposition 2.11: the following statements are equivalent

(1) N is a weakly secondary submodule M.

(2) $\frac{N}{H}$ is weakly secondary submodule of $\frac{M}{H}$ for each submodule H of M contained in N.

Proof. (1) $\Rightarrow$ (2) Let N be a weakly secondary submodule M and $\pi: M \to \frac{M}{H}$ be the natural epimorphism for each submodule H of M contained in N so by Proposition 2.6, $\pi(N) = \frac{N}{H}$ is a weakly secondary submodule $\frac{M}{H}$.

(2) $\Rightarrow$ (1) It is clear by taking $H = 0$

3. More Characterizations and Facts About Weakly Secondary Submodules

We start with important tool in our work to give another description for this class of modules.

Theorem 3.1: Let N be a submodule of M. the following statements are equivalent

(1) N is weakly secondary of M.

(2) $N \neq 0$ and $[K: N]$ is primary for each submodule $K \not\supseteq N$ in M.

Proof. (1) $\Rightarrow$ (2) Assume that N is a weakly secondary submodule of M and K a submodule of M with $N \nsubseteq K$. Thus $[K: N] \neq R$. Let $a, b \in R$ with $ab \in [K: N]$ and so $Nab \subseteq K$ then either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t so either $a \in [K: N]$ or $b^t \in [K: N]$ as required.

(2) $\Rightarrow$ (1) Let N and K be submodules of M such that $Nab \subseteq K$ where $a, b \in R$. In case $N \subseteq K$ then already $Na \subseteq K$ and $Nb^t \subseteq K$ for each positive integer t. If $N \nsubseteq K$ then $[K: N]$ is primary by hypothesis and $ab \in [K: N]$ implies $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t as desired.

Corollary 3.2: Every submodule of a module over a generalized prime ring is weakly secondary.

Proof. By Theorem 3.1.

Corollary 3.3: If N is a weakly secondary of M then $ann_R(N)$ is primary.

Proof. By Theorem 3.1.

Example 3.4: The opposite of Corollary 3.3 is not hold in general. $ann_R(N) = 0$ for every nonzero submodule N of the $\mathbb{Z}$-module $\mathbb{Z}$ but N is not weakly secondary.

Corollary 3.5: If N is a weakly secondary submodule of M, then for every submodule $K \not\supseteq N$ in M we have $\sqrt{[K: N]} = \sqrt{[K: Nb]}$ for each $b \in R$, $b \notin \sqrt{[K: N]}$.

Proof. Let $a \in \sqrt{[K: N]}$, thus $Na^m \subseteq K$ for some positive integer m. Thus for each $b \in R$, $Na^m b \subseteq K$ so $a \in \sqrt{[K: Nb]}$. Conversly, let $a \in \sqrt{[K: Nb]}$ so $Na^m b \subseteq K$ so $a^m b \in [K: N]$. By Theorem 3.1, $[K: N]$ is primary and $b^t \notin [K: N]$ for each positive integer t implies that $(a^m)^n \in [K: N]$ for some positive integer n it follows that $a \in \sqrt{[K: N]}$ as required.

Corollary 3.6: If N is a weakly secondary submodule of M, then $\sqrt{ann_R(N)} = \sqrt{ann_R(Nb)}$ for each $b \in R$ with $b^t \notin ann_R(N)$ for each positive integer t .

Proof. By Corollary 3.5.

Theorem 3.7: The following statements are equivalent

(1) μ is a weakly secondary submodule of M.

(2) The set $\{\sqrt{[\mathcal{Q}:_R \mu]}, \mathcal{Q}$ is a submodule of M with $\mathcal{Q} \not\supseteq \mu\}$ is a chain of prime ideals of R.

Proof. (1) $\Rightarrow$ (2) Initially $\sqrt{[\mathcal{Q}:_R \mu]}$ is prime for each submodule $\mathcal{Q} \not\supseteq \mu$ in M by Outcome 3.1. Let $\mathcal{Q}$ and ϱ be submodules of M, $\mathcal{Q} \not\supseteq \mu$ and $\varrho \not\supseteq \mu$ then $[\mathcal{Q}:_R \mu]$ and $[\varrho:_R \mu]$ are primary ideals of R. Suppose $\sqrt{[\mathcal{Q}:_R \mu]} \nsubseteq \sqrt{[\varrho:_R \mu]}$ and $\sqrt{[\varrho:_R \mu]} \nsubseteq \sqrt{[\mathcal{Q}:_R \mu]}$ this means there exist ideals I and J of R

with $I \subseteq \sqrt{[\mathcal{Q}:_R \mu]}$, $I \nsubseteq \sqrt{[\varrho:_R \mu]}$, $J \subseteq \sqrt{[\varrho:_R \mu]}$ and $J \nsubseteq \sqrt{[\mathcal{Q}:_R \mu]}$. So $\mu(IJ)^n \subseteq \mathcal{Q}$ and $\mu(IJ)^m \subseteq \varrho$ implies $(IJ)^t \subseteq [\mathcal{Q} \cap \varrho:_R \mu]$ and hence $IJ \subseteq \sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$ for some positive integers n, m and t respectively. Since $\mathcal{Q} \cap \varrho \nsupseteq \mu$ then $\sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$ is prime it follows $I \subseteq \sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$ or $J \subseteq \sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$. If $I \subseteq \sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$ we have $I^s \subseteq [\mathcal{Q}:_R \mu]$ and $I^r \subseteq [\varrho:_R \mu]$ for some positive integers s and r respectively. Similarly, if $J \subseteq \sqrt{[\mathcal{Q} \cap \varrho:_R \mu]}$ then $J^d \subseteq [\mathcal{Q}:_R \mu]$ and $J^k \subseteq [\varrho:_R \mu]$ for some positive integers d and k respectively.This means either $I \subseteq \sqrt{[\mathcal{Q}:_R \mu]}$ or $J \subseteq \sqrt{[\varrho:_R \mu]}$ which is a contradiction.

(2) $\Rightarrow$ (1) Let $\mu ab \subseteq \mathcal{Q}$, $a, b \in R$ and $\mathcal{Q}$ is a submodule of M. If $\mu \subseteq \mathcal{Q}$ then already we have the goal. Assume that $\mu \nsubseteq \mathcal{Q}$ then $ab \in [\mathcal{Q}:_R \mu] \subseteq \sqrt{[\mathcal{Q}:_R \mu]}$ which is prime it follows either $a \in \sqrt{[\mathcal{Q}:_R \mu]}$ or $b \in \sqrt{[\mathcal{Q}:_R \mu]}$ and hence $\mu a^n \subseteq \mathcal{Q}$ or $\mu b^m \subseteq \mathcal{Q}$ for some positive integers n and m respectively as wished.

Proposition 3.8: If $0 \neq N$ is comuliplication of M such that $ann_R(N)$ is a primary ideal of R then N is a secondary R-module.

Proof. Let $N \neq 0$. For every $a \in R$ we can define the endomorphism $f_a : N \to N$ by $f_a(n) = na$ for each $n \in N$. Thus $Imf_a = Na$. Because N is comultiplication implies $Na = ann_N(I)$ for an ideal I of R so $NaI = 0$ follows $aI \subseteq ann_R(N)$. But $ann_R(N)$ is primary so $Na^t = 0$ for some positive integer t or $NI = 0$. In case $Na^t \neq 0$ for each positive integer t then $NI = 0$ follows $Na = ann_N(I) = N$ as wanted.

Corollary 3.9: Let M be a comuliplication together with the annihilator of any nonzero submodule of M is primary then every nonzero submodule is secondary.

Proof. Because every submodule of a comultiplication module is comultiplication then by Proposition 3.8, the result is obtained.

Corollary 3.10: Let N be a nonzero comuliplication submodule of M. Then the following statements are equivalent

(1) N is a weakly secondary submodule of M.
(2) $ann_R(N)$ is a primary ideal of R.
(3) N is a secondary submodule of M.

Proof. (1) $\Rightarrow$ (2) From Corollary 3.3, (2) $\Rightarrow$ (3) it is known and (3) $\Rightarrow$ (1) is clear.

4. *S*-Weakly Secondary Modules

We define S-weakly secondary modules in this part. Firstly we provide a characterization and examples of S-secondary modules.

Theorem 4.1: The following statements are equivalent

(1) M is S-secondary.
(2) $M \neq 0$ and whenever $\zeta(M) \subseteq K$ where $\zeta \in S$ and K a submodule of M implies either $M = K$ or $\zeta^t(M) = 0$ for some positive integer t.

Proof. (1) $\Rightarrow$ (2) Assume M is an S-second R-module then $M \neq 0$. Let $\zeta(M) \subseteq K$ for some $\zeta \in S$ and K a submodule of M. By hypothesis either $\zeta(M) = M$ or $\zeta^t(M) = 0$ implies $M = K$ or $\zeta^t(M) = 0$.

(2) $\Rightarrow$ (1) By (2) we can choose $K = \zeta(M)$ where $\zeta \in S$ implies $\zeta(M) \subseteq \zeta(M)$ and hence $\zeta(M) = M$ or $\zeta^t(M) = 0$ for some positive integer t.

Remarks and Examples 4.2:

(1) Every S-secondary module is secondary.

Proof. Let M be S-secondary then for every $f \in S$, either $f(M) = M$ or $f^t(M) = 0$ for some positive integer t. For each $a \in R$, define $f_a: M \to M$ by $f_a(m) = ma$ for every $m \in M$ and it is well known $f_a \in S$ and $Imf_a = Ma$. By hypothesis $Ma = M$ or $Ma^t = 0$ as desired.

(2) The opposite of (2) is not valid in general. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is divisible and hence it is secondary but not S-second because there is an endomorphism $f = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Z}_2) & Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) & End_{\mathbb{Z}}(\mathbb{Z}_2) \end{pmatrix} \cong Mat_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ and $f(x, y) = (x, 0)$ for each $(x, y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ implies $\bar{0} \oplus \bar{0} \neq f^t(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \bar{0} \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for each positive integer t.

(3) Every S-Secondary module is indecomposable (that is when a module M has a decomposition then M is not S-secondary).

Proof. Let M be an S-secondary R-module then $M \neq 0$. Suppose that $M = A \oplus B$ for some non-zero R-modules A and B. So we can define the map $\zeta: M \to M$ by $\zeta(x, y) = (x, 0)$. Then $\zeta \in S$ and hence $0 \neq \zeta^t(M) = A \oplus 0 \neq M$ for each positive integer t and hence M is not S-secondary and this is a contradiction.

(4) The converse of (3) is not correct in general. For example, $\mathbb{Z}$ is indecomposable but not secondary and hence it is not S-secondary.

(5) It is clear that every coquasi-dedekind module is S-secondary.

(6) $\frac{\mathbb{Q}}{\mathbb{Z}} \cong \oplus \sum_p \mathbb{Z}_{p^\infty}$ is not S-secondary since if not then $\frac{\mathbb{Q}}{\mathbb{Z}}$ is indecomposable by (3), which is a contradiction and hence $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not coquasi-dedekind.

(7) Obviously every simple module is S-secondary.

Definition 4.3: A nonzero R-module M is called S-weakly secondary whenever $\zeta\vartheta(M) \subseteq K$, where $\zeta, \vartheta \in S$ and K a submodule of M implies either $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ for some positive integer t.

Remarks and Examples 4.4:

(1) Every S-weakly secondary module is weakly secondary.

Proof. Let M be an S-weakly secondary R-module then $M \neq 0$. Let $Mab \subseteq K$ for some $a, b \in R$ and K a submodule of M. Define the endomorphisms $f_a: M \to M$ by $f_a(m) = ma$ and $g_b: M \to M$ by $g_b(m) = mb$ for each $m \in M$. Then $fg(M) = f\big(g(M)\big) = f(Mb) = f(M)b = Mab \subseteq K$. By hypothesis either $f(M) \subseteq K$ or $g^t(M) \subseteq K$ that is $Ma \subseteq K$ or $Mb^t \subseteq K$ for some positive integer t as desired.

(2) Reversely of (1) fails in general, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is secondary (and hence weakly secondary) but it is not S-weakly secondary since if we take $f = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $g = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong Mat_2(\mathbb{Z}_2)$ implies $fg(M) = \{fg\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \bar{0} \end{pmatrix}$ for each $(\bar{x}, \bar{y}) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2\} = \bar{0} \oplus \bar{0}$ while $f^t(M) = \mathbb{Z}_2 \oplus \bar{0}$ and $g^t(M) = \bar{0} \oplus \mathbb{Z}_2$ for each positive integer t.

(3) Every S-weakly secondary module is indecomposable (that is when a module M has a decomposition then M is not S-weakly secondary).

Proof. Let M be an S-weakly secondary R-module then $M \neq 0$. Suppose that $M = A \oplus B$ for some non-zero R-modules A and B. So we can define the maps $\zeta: M \to M$ $\zeta(x, y) = (x, 0)$ and $\vartheta: M \to M$ by $\zeta(x, y) = (0, y)$ for each $(x, y) \in M$. It is clear that ζ, $\vartheta \in S$ implies $\zeta\vartheta(M) = \zeta(\vartheta(M)) = \zeta(0 \oplus B) = 0 \oplus 0$ but $\zeta^t(M) = A \oplus 0$ and $\vartheta^t(M) = 0 \oplus B$ for each positive integer t. Hence M is not S-weakly secondary which is a contradiction.

(4) The inverse of (3) is not hold in general, $\mathbb{Z}$ is indecomposable but not S-weakly secondary.

(5) Every S-secondary module is S-weakly secondary.

Proof. Let M be an S-secondary R-module and $\zeta, \vartheta \in S$, K is a submodule of M with $\zeta\vartheta(M) \subseteq K$ implies $\zeta\vartheta(M) = M$ or $(\zeta\vartheta)^t M = 0$ for some positive integer t. If $\zeta\vartheta(M) = M$ then $\zeta M \subseteq M = \zeta\vartheta(M) \subseteq K$ and $\vartheta M \subseteq M = \zeta\vartheta(M) \subseteq K$. In case $(\zeta\vartheta)^t M = 0$ that is $\zeta^t\vartheta^t(M) = 0$. Again $\vartheta^t M = M$ or $(g^t)^m M = 0$ for some positive integers t and m implies $\zeta^t M = 0 \subseteq K$ or $\vartheta^n(M) = 0 \subseteq K$ for some positive integer $n = tm$ as desired.

(6) Oppositely of (5) is not correct generally. Let F be a field and let R be the set of infinite matrices over F that have the form $\begin{pmatrix} A & & 0 \\ & a & \\ 0 & & a \\ & & & \ddots \end{pmatrix}$ where A is any finite matrix and a is any element of F. It is not hard to see that R is a ring with identity and the only nonzero proper ideal I of R is the subset of all matrices of R of the form $\begin{pmatrix} A & & 0 \\ & 0 & \\ 0 & & 0 \\ & & & \ddots \end{pmatrix}$ so is clear $I = I^2$ and hence I is prime [9], also it is obvious the zero ideal is prime and hence $R \cong End(R)$ is fully prime ring and hence it is generalized prime ring. Via Theorem 3.1, R is a weakly secondary which is not secondary.

(7) We have the implication Coquasi-dedekind modules $\Rightarrow S$-second modules $\Rightarrow$ S-secondary modules (or S-weakly second modules) $\Rightarrow S$-weakly secondary modules $\Rightarrow$ indecomposable modules.

Theorem 4.5: the following statements are equivalent

(1) M is an S-weakly secondary R-module.

(2) $M \neq 0$ and for each $\zeta, \vartheta \in S$ implies either $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) \supseteq \vartheta^t(M)$ for some positive integers t.

Proof. (1) $\Rightarrow$ (2) Assume that M is an S-weakly secondary R-module then $M \neq 0$. Let $\zeta, \vartheta \in S$ and $\zeta\vartheta(M) \subseteq K$ for submodule K of M. We can choose $K = \zeta\vartheta(M)$ so by (1) $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ and hence $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) \supseteq \vartheta^t(M)$ for some positive integers t.

(2) $\Rightarrow$ (1) Let $M \neq 0$ and $\zeta, \vartheta \in S$ with $\zeta\vartheta(M) \subseteq K$ for submodule K of M. By (2), $\zeta(M) = \zeta\vartheta(M) \subseteq K$ or $\vartheta^t(M) \subseteq \zeta\vartheta(M) \subseteq K$ for some positive integers t as desired.

Theorem 4.6: The following statements are equivalent

(1) M is an S-weakly secondary R-module.

(2) $M \neq 0$ and $[K :_S M]$ is primary for each proper submodule K of M.

Proof. (1) $\Rightarrow$ (2) Assume M is S-weakly secondary and K a proper submodule of M implies $[K :_S M] \neq R$. Let $\zeta, \vartheta \in S$ with $\zeta\vartheta \in [K :_S M]$ implies $\zeta\vartheta(M) \subseteq K$ then $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ for some positive integers t. So either $\zeta \in [K :_S M]$ or $\vartheta^t \in [K :_S M]$ as required.

(2) $\Rightarrow$ (1) Let K be submodule of M such that $\zeta\vartheta(M) \subseteq K$ where $\zeta, \vartheta \in S$. In case $M = K$ then already $\zeta(M) \subseteq K$ and $\vartheta(M) \subseteq K$. If $M \neq K$ then $[K :_S M]$ is primary by hypothesis and $\zeta\vartheta \in [K :_S M]$ implies $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ for some positive integers t as desired.

Corollary 4.7: If M is an S-weakly secondary R-module M then $ann_S(M) = \{f \in S : f(M) = 0\}$ is a primary ideal of S.

Proof. Directly from Theorem 4.6.

Examples 4.8:

(1) The opposite of Corollary 4.7 is not hold in general. For example, $ann_S(\mathbb{Z}) = 0$ is a prime ideal of $S = End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$ which is not weakly secondary and hence it is not S-weakly secondary .

(2) As another example of (1), let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be a ring, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ an idempotent in R and $M = eR = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ a module over R. We have $S = End_R(M) \cong eRe = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ is an integral domain implies $ann_S(M) = 0$ is a prime but M is not an S-weakly secondary R-module because if we take $f = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $g = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in S$ implies $fg(M) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} bc & bd \\ 0 & 0 \end{pmatrix}, a, b, c, d \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} abc & abd \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} ab\mathbb{Z} & ab\mathbb{Z} \\ 0 & 0 \end{pmatrix} \right\}$ but

$f(M) = \left\{\begin{pmatrix} a\mathbb{Z} & a\mathbb{Z} \\ 0 & 0 \end{pmatrix}\right\}$ and $g(M) = \left\{\begin{pmatrix} b\mathbb{Z} & b\mathbb{Z} \\ 0 & 0 \end{pmatrix}\right\}$ and hence $f^t(M) = \left\{\begin{pmatrix} a^t\mathbb{Z} & a^t\mathbb{Z} \\ 0 & 0 \end{pmatrix}\right\}$ and $g^t(M) = \left\{\begin{pmatrix} b^t\mathbb{Z} & b^t\mathbb{Z} \\ 0 & 0 \end{pmatrix}\right\}$ where t any positive integer. That is neither $fg(M) = f(M)$ nor $fg(M) \supseteq g^t(M)$.

Corollary 4.9: If M is an S-weakly secondary R-module M then for every proper submodule K of M we have $\sqrt{[K:_S M]} = \sqrt{[K:_S \vartheta(M)]}$ for each $\vartheta \in R$ with $\vartheta^t \notin [K:_S M]$ for each positive integer t.

Proof. Let $\zeta \in \sqrt{[K:_S M]}$ then $\zeta^m(M) \subseteq K$ for some positive integer m implies for each $\vartheta \in R$ $\zeta^m\vartheta(M) \subseteq K$ so $\zeta \in \sqrt{[K:_S \vartheta(M)]}$. Conversly, let $\zeta \in \sqrt{[K:\vartheta(M)]}$ then $\zeta^m\vartheta(M) \subseteq K$ for some positive integer m and so $\zeta^m\vartheta \in [K:_S M]$. By Theorem 4.6, $[K:_S M]$ is primary and $\vartheta^t \notin [K:_S M]$ for each positive integer t implies that $\zeta \in \sqrt{[K:_S M]}$ as required.

Corollary 4.10: If N is a weakly secondary submodule of M, then $\sqrt{ann_S(M)} = \sqrt{ann_S(\vartheta(M))}$ for each $\vartheta \in S$ with $\vartheta^t \notin [K:_S M]$ for each positive integer t .

Proof. By Corollary 4.9 .

Proposition 4.11: Every weakly secondary multiplication module is S-weakly secondary.

Proof. Let M be a weakly secondary multiplication R-module and $\zeta, \vartheta \in S$ with $\zeta\vartheta(M) \subseteq K$ for some K a submodule of M. Since M is multiplication then $\zeta\vartheta(M) = \zeta(JM) = J\zeta(M) = IJM$ for ideals I and J of R and hence $IJM \subseteq K$. By Theorem , either $IM \subseteq K$ or $J^tM \subseteq K$ then $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ for some positive integer t that is M is S-weakly secondary.

Proposition 4.12: Every weakly secondary scalar module is S-weakly secondary.

Proof. Let M be a weakly secondary scalar R-module and $\zeta, \vartheta \in S$ with $\zeta\vartheta(M) \subseteq K$ for some K a submodule of M. Since M is scalar then there exist $a, b \in R$ such that $\zeta(m) = am$ and $\vartheta(m) = mb$ for all $m \in M$. Then $Im\vartheta = MR_b$ and $Im\zeta = MR_a$ and hence $K \supseteq \zeta\vartheta(M) = \zeta(Mb) = MR_aR_b$ implies $MR_a \subseteq K$ or $M(R_b)^t \subseteq K$ it follows $\zeta(M) \subseteq K$ or $\vartheta^t(M) \subseteq K$ for some positive integer t as desired.

Theorem 4.13: the following statements are equivalent

(1) M is an S-weakly secondary R-module.

(2) The set $\{\sqrt{[Q:_S M]}, Q$ is a proper submodule of M $\}$ is a chain of prime ideals of R.

Proof. is a similar proof of Theorem 3.7

Proposition 4.14: Every summand of S-weakly secondary module is S-weakly secondary.

Proof. Let Q be a direct summand of S-weakly second $\mathfrak{M}$ then $\mathfrak{M} = Q \oplus \mu$ for some submodule μ of M. Let $\zeta, \vartheta \in End(N)$ with $\zeta\vartheta(Q) \subseteq \nabla$ for some ∇ a submodule of Q. We can define $\alpha(n + \hbar) = \zeta(n)$ and $\beta(n + \hbar) = \vartheta(n)$ where $n \in Q$and $\hbar \in \mu$. Visibly $\alpha, \beta \in S$, $\alpha(\mathfrak{M}) = \zeta(Q)$ and $\beta(\mathfrak{M}) = \vartheta(Q)$ implies $\alpha\beta(\mathfrak{M}) = \zeta\vartheta(Q) \subseteq \nabla$ it follows $\alpha(\mathfrak{M}) \subseteq \nabla$ or $\beta^t(\mathfrak{M}) \subseteq \nabla$ for some positive integers t and hence $\zeta(Q) \subseteq \nabla$ or $\vartheta^t(Q) \subseteq \nabla$ as desired.

References

[1] S. Yassemi, "The dual notion of prime submodules", Arch. Math. (Brno) 37 (2001) 273–278.

[2] H. Ansari-Toroghy and F. Farshadifar, "The dual notions of some generalizations of prime submodules", Comm. Algebra 39 (7), (2011), 2396-2416.

[3] M. Behboodi and H. Koohi, "Weakly prime submodules", Vietnam J. Math. 32 (2) (2004),185–195.

[4] M. Baziar and M. Behboodi, "Classical primary submodules and decomposition theory of modules", J. Algebra Appl. 8(3) (2009) 351-362.

[5] Wisbauer, R. (1991). "Foundations of Module and Ring Theory". Philadelphia: Gordon and Breach.

[6] Yaseen S.M., "Coquasi-Dedekind Modules", Ph.D Thesis, University of Baghdad, Baghdad, Iraq, 2003.

[7] Dauns, J. (1978). "Prime submodules". J. Reine Angew. Math. 298:156–181.

[8] M. D. Larsen, P. J. McCarthy, "Multiplicative Ideal Theory", Academic press, 1971.

[9] H. Tsutsui, "Fully prime rings", Comm. Algebra 24 (1996) 2981–2989.

[10] M. SATYANARAYANA. "Generalized Primary Rings", Math. Ann. t79, 109--1t4(1969).

[11] H. Ansari-Toroghy, F. Farshadifar, "The dual notion of multiplication modules", Taiwanese J. Math. 11 (2007) 1189–1201.

[12] Shireen Ouda, "S-Prime Submodules and Some Related Concepts" M.Sc. Thesis, University of Baghdad, Baghdad, Iraq, 2010.

[13] El-Bast Z. A., Smith P. F. "Multiplication modules", Commutative In Algebra, (16) (1988), pp: 755-779.

[14] Shihab B.N. "Scalar Reflexive Modules". Ph.D. Thesis, University of Baghdad, Baghdad, Iraq, 2004.

[15] H. Ansari-Toroghy, F. Farshadifar, S.S. Pourmortazavi and F. Khaliphe, On Secondary Modules", International Journal of Algebra, 6(16), (2012), 769 – 774.

[16] Zainab Saadi and Ghaleb Ahmed, "On Weakly Second Submodules", Iraqi Journal of Science, to appear.