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# Inverse Triangular and Hyperbolic Acceleration Methods of Second Kind for Improving the Numerical Integration Results

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ABSTRACT

The aims of this study are to introduce the acceleration methods which are called inverse triangular acceleration methods and inverse hyperbolic triangular acceleration methods, which we generally call Al-Tememe's acceleration methods of the second kind discovered by (Ali Hassan Mohammed). It is useful to improve the numerical results of continuous integrals in which the error is of the $4^{th}$ order, and regarding to accuracy, the number of used partial intervals and how fast is to get results especially to accelerate the results that can be got by using Simpson's method. Also, it is possible to utilize it in improving the numerical results of differential equations ,where the main error is of the forth order.

MSC.

## 1 . Introduction

There are several numerical methods used calculating single integrals that their integrands are bounded in their integration intervals ,such as

**1.**Trapezoidal Rule

**2.**Midpoint Rule

**3.**Simpson's Rule

Generally, these methods are called Newton–Cotes formulas.

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This study will introduce Simpson's method to find approximate values of single integrals of continuous integrands through using inverse triangular acceleration methods and inverse hyperbolic triangular acceleration methods, which come with in Al of Tememe's acceleration series- the second kind. We will compare these methods with respect to accuracy and the speed of approaching these values to the real values (analytical) of those integrals.

Let the integral J is defined as follows:

$$J = \int_{x_0}^{x_{2n}} f(x)dx \ldots \quad (1)$$

Where f(x) is a continuous integrand lies above X axis in the interval $[x_0, x_{2n}]$, and it is required to find the approximate value of J

Generally, Newton–Cotes formula for integration (1) can be written in the following form:

$$J = \int_{x_0}^{x_{2n}} f(x)dx = G(h) + E_G(h) + R_G$$

Whereas G (h) represents (Lagrangian – Approximation) the value of integration J, and G refers to the type of the rule, $E_G$ (h) is the correction terms that can be added to G(h) and $R_G$ is the remainder .The general formula of simpson rule is given by:-

$$S(h)=\frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \cdots + 2f(a+(2n-2)h) + 4f(a+(2n-1)h) + f(b)]$$

and the general formula for $E_G(h)$ is as follows:

$$E_S(h)= \frac{h^4}{180}( f_{x_{2n}}{}^{(3)} - f_{x_0}{}^{(3)})+ \frac{h^6}{1512} (f_{x_{2n}}{}^{(5)} - f_{x_0}{}^{(5)})+\ldots \qquad (E_S(h)=E_G(h))$$

see [1]

So, when the integrand of integration is a continuous function also their derivatives are its derivative is continuous at each point of integration intervals$[x_0, x_{2n}]$, it is possible to write error formula as:

$J-s(h)=A_1h^4+A_2h^6+A_3h^8+ \ldots$

where $A_1, A_2, A_3, \ldots$ are constants that their values do not depend on h but on the values of the function derivatives in the end of the integration interval.

## 2. Al-Tememe's inverse triangular acceleration of inverse and hyperbolic functions of the second kind

In this section, we will present the acceleration methods that come with Al-Tememe's series of acceleration and we call them by the inverse triangular acceleration methods and inverse hyperbolic acceleration methods.

It is mentioned that the error in Simpson's rule can be written as the following form:

$E= A_1h^4+ A_2h^6+\ldots \qquad (3)$

$=h^3(A_1h+A_2h^3+\ldots)$

$\cong h^3\sin^{-1}h \qquad$ since $\left(\sin^{-1}h=h+\frac{1}{6}h^3+\frac{3}{40}h^5+\frac{5}{112}h^7+\frac{35}{1152}h^9+\ldots\right)$ [2]

It is assumed that S(h) is the approximate value of integration in Simpson's rule, so: $J-S(h)\cong h^3 \sin h$.

Assuming that we calculate two numerical values for J, based on Simpson's rule as $S_1(h_1)$ when $h= h_1$, $S_2(h_2)$, when $h=h_2$, so:

J-s($h_1$) ≅ $h_1^3 \sin h_1$ (4)

J-s($h_2$) ≅ $h_2^3 \sin h_2$ (5)

From the equations (4) and (5), we get:

$$A^S_{\sin^{-1}} \cong \frac{(h_1^3 \sin^{-1} h_1)\, S_2(h_2) - (h_2^3 \sin^{-1} h_2)\, S_1(h_1)}{h_1^3 \sin^{-1} h_1 - h_2^3 \sin^{-1} h_2} \qquad (6)$$

The formula (6) is called Al-Tememe's inverse sine triangular acceleration of the second kind which is referred to by

($A^S_{\sin^{-1}}$)

Similarly, the second inverse triangular acceleration rule of cosine can be written ,while ,the error E can be written as :

$E=h^3(A_1h+A_2h^3+A_3h^5+\ldots)=h^3(\frac{\pi}{2} - \cos^{-1} h)$;

since ($\cos^{-1}h=\frac{\pi}{2}-\frac{1}{6}h^3-\frac{3}{40}h^5-\frac{5}{112}h^7-\ldots$) [2]

Similar to triangular acceleration inverse sine mentioned above, we get the following:

$$A^S_{\cos^{-1}} \cong \frac{h_1^4\left(\frac{\pi}{2} - \sin^{-1} h_1\right) S_2(h_2) - h_2^4\left(\frac{\pi}{2} - \sin^{-1} h_2\right) S_1(h_1)}{h_1^4\left(\frac{\pi}{2} - \sin^{-1} h_1\right) - h_2^4\left(\frac{\pi}{2} - \sin^{-1} h_2\right)} \qquad (7)$$

If we use the law in formula (7), we will get the true value of seven decimal when n=200 in several examples including, $I = \int_5^6 \frac{\ln(x)}{x} dx$ but it is expected that [3] of the inverse cosine will take the following form:

$$A^S_{\cos^{-1}} \cong \frac{h_1^4(\cos^{-1} h_1) S_2(h_2) - h_2^4(\cos^{-1} h_2)\, S_1(h_1)}{h_1^4(\cos^{-1} h_1) - h_2^4(\cos^{-1} h_2)} \qquad (8)$$

We call the formula (7) as Al-Timemi's inverse triangular acceleration cosine of the second class that referred to by ($A^S_{\cos^{-1}}$).

Similarly, we can find the third triangular acceleration law that we will call inverse triangular acceleration law of tangent, which is referred to by ($A^S_{\tan^{-1}}$):

$$A^S_{\tan^{-1}} \cong \frac{(h_1^3 \tan^{-1} h_1)\, S_2(h_2) - (h_2^3 \tan^{-1} h_2)\, S_1(h_1)}{h_1^3 \tan^{-1} h_1 - h_2^3 \tan^{-1} h_2} \qquad (9)$$

Since ($\tan^{-1}h=h-\frac{1}{3}h^3+\frac{1}{5}h^5-\frac{1}{7}h^7+\cdots$) [2]

$$A^S_{\cot^{-1}} \cong \frac{h_1^3\left(\frac{\pi}{2} - \cot^{-1} h_1\right) S_2(h_2) - h_2^3\left(\frac{\pi}{2} - \cot^{-1} h_2\right) S_1(h_1)}{h_1^3\left(\frac{\pi}{2} - \cot^{-1} h_1\right) - h_2^3\left(\frac{\pi}{2} - \cot^{-1} h_2\right)} \qquad (10)$$

Since ($\cot^{-1}h=\frac{\pi}{2}-h+\frac{1}{3}h^3-\frac{1}{5}h^5+\frac{1}{7}h^7-\frac{1}{9}h^9+\ldots$) [2]

If we use the law in (10), we will get true value of four decimal when n= 200 in several examples including $I = \int_5^6 \frac{\ln(x)}{x} dx$ ' but it is expected that the law of inverse cotangent will take the following formula:

$$A^S_{\cot^{-1}} \cong \frac{h_1^4(\cot^{-1} h_1) S_2(h_2) - h_2^4(\cot^{-1} h_2)\, S_1(h_1)}{h_1^4(\cot^{-1} h_1) - h_2^4(\cot^{-1} h_2)} \qquad (11)$$

Based on the same method that is followed in finding inverse triangular acceleration rules, we can also find the inverse hyperbolic acceleration rules:

$$A^S_{\sinh^{-1}} \cong \frac{(h_1^3 \sinh^{-1} h_1)\, S_2(h_2) - (h_2^3 \sinh^{-1} h_2)\, S_1(h_1)}{h_1^3 \sinh^{-1} h_1 - h_2^3 \sinh^{-1} h_2} \qquad (12)$$

$$A^{S}_{\tanh^{-1}} \cong \frac{(h_1^{3}\ \tanh^{-1} h_1)\ S_2(h_2) - (h_2^{3}\ \tanh^{-1} h_2)\ S_1(h_1)}{h_1^{3}\, tanh^{-1}\ h_1 - h_2^{3}\ \tanh^{-1} h_2} \qquad (13)$$

Since ($\sinh^{-1}h = h - \frac{1}{6}h^3 + \frac{3}{40}h^5 - \frac{5}{112}h^7 + \frac{35}{1152}h^9 - \ldots$) [2]

and ( $\tanh^{-1}h = h + \frac{1}{3}h^3 + \frac{1}{5}h^5 + \frac{1}{7}h^7 + \frac{1}{9}h^9 + \ldots$) [2]

## 3.Examples:

The following examples introduce some integrals that have continuous integrals in the of integration interval and we use inverse and hyperbolic triangular acceleration methods to improve the results of integrals:

**3.1**: $I = \int_5^6 \frac{\ln(x)}{x}dx$ and its exact value is 0.310055800794083 and it is rounded to 14 decimal.

**3.2**: $I = \int_3^4 \frac{1}{x}dx$ and its exact value is 0.28768207245178 and it is rounded to 14 decimal.

**3.3**: $I = \int_{1.5}^{2} \sqrt{x}\ dx$ . 3 and its exact value is 0.660873211772538 and it is rounded to 14 decimal.

## 4. Numerical Results

Clearly ,the integrals of integration dx $I=\int_5^6 \frac{\ln(x)}{x}dx$ is continuous in the integration interval [5,6], so, we use the formula of correction terms of Simpson's rule as mentioned above (equation3).

We have EPS=$10^{-12}$ (that is the absolute error of the subsequent value minus the previous value). The obtained results are shown in table (1). We get correct values through the accelerations of $A^{s}_{\tan^{-1}(h)}$ و $A^{S}_{\sin^{-1}(h)}$ $A^{s}_{\sinh^{-1}(h)}$ $A^{s}_{\tanh^{-1}(h)}$ and other accelerations are rounded to 9 decimal when n=14. Also, we get the same accuracy $A^{S}_{\cot^{-1}(h)}$ and $A^{S}_{\cos^{-1}(h)}$ when n=12,14.

Clearly, the integrals of integration $I=\int_3^4 \frac{1}{x}dx$ is continuous in the integration interval [3,4], so, we use the formula of correction terms of Simpson's rule as mentioned above (equation3).

The obtained results are shown in table (2). We obtain correct values through accelerating $A^{S}_{\cos^{-1}(h)}$ and other accelerations are rounded to 12 decimal when n=20 while the value by Simpson's without acceleration is correct to 8 decimal when n=20.

Clearly, the integrals of integration $\int_{1.5}^{2} \sqrt{x}dx$ is continuous in the integration interval [1,,2,5], so, we use the formula of correction terms of Simpson's rule as mentioned above (equation 3).

The obtained results are shown in table (3). We get correct values through accelerating $A^{S}_{\cos^{-1}(h)}$ and other accelerations are rounded to 12 decimal when n=14 while the value by Simpson's without acceleration is correct to 9 decimal when n=14.

## 5.Conclusion

We conclude from the mentioned tables 1,2 and 3 that these acceleration methods have the same efficiency and they can give high accurate in limited number of partial intervals.

| n | Simpson's rule | $A^S_{\cos^{-1}(h)}$ | $A^S_{\sin^{-1}(h)}$ | $A^S_{\tan^{-1}(h)}$ | $A^S_{\cot^{-1}(h)}$ | $A^S_{\sinh^{-1}(h)}$ | $A^S_{\tanh^{-1}(h)}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.31005515917445 | | | | | | |
| 4 | 0.31005575894700 | 0.31005579905099 | 0.31005579893143 | 0.31005579893264 | 0.31005579905099 | 0.31005579893224 | 0.31005579893102 |
| 6 | 0.31005579246037 | 0.31005580071936 | 0.31005580070980 | 0.31005580070985 | 0.31005580071936 | 0.31005580070983 | 0.31005580070978 |
| 8 | 0.31005579814966 | 0.31005580078478 | 0.31005580078299 | 0.31005580078299 | 0.31005580078478 | 0.31005580078299 | 0.31005580078298 |
| 10 | 0.31005579970948 | 0.31005580079215 | 0.31005580079164 | 0.31005580079164 | 0.31005580079215 | 0.31005580079164 | 0.31005580079164 |
| 12 | 0.31005580027065 | 0.31005580079354 | 0.31005580079335 | 0.31005580079335 | 0.31005580079354 | 0.31005580079335 | 0.31005580079335 |
| 14 | 0.31005580051142 | 0.31005580079390 | 0.31005580079382 | 0.31005580079382 | 0.31005580079390 | 0.31005580079382 | 0.31005580079382 |

$Table no.(1) calculating\ the\ integral \int_5^6 \frac{\ln(x)}{x}$ 0.310055800794083 dx by simpson's rule with the inverse triagular and hyperbolic methods of Al − Tememe acceleration of second kind

| n | Simpson's rule | $A^S_{\cos^{-1}(h)}$ | $A^S_{\sin^{-1}(h)}$ | $A^S_{\tan^{-1}(h)}$ | $A^S_{\cot^{-1}(h)}$ | $A^S_{\sinh^{-1}(h)}$ | $A^S_{\tanh^{-1}(h)}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.28769841269841 | | | | | | |
| 4 | 0.28768315018315 | 0.28768212964984 | 0.28768213269247 | 0.28768213266147 | 0.28768212965004 | 0.28768213267180 | 0.28768213270280 |
| 6 | 0.28768228762508 | 0.28768207505737 | 0.28768207530356 | 0.28768207530216 | 0.28768207505737 | 0.28768207530263 | 0.28768207530403 |
| 8 | 0.28768214079347 | 0.28768207278529 | 0.28768207283147 | 0.28768207283129 | 0.28768207278529 | 0.28768207283135 | 0.28768207283153 |
| 10 | 0.28768210049420 | 0.28768207252272 | 0.28768207253591 | 0.28768207253586 | 0.28768207252272 | 0.28768207253588 | 0.28768207253592 |
| 12 | 0.28768208598839 | 0.28768207247218 | 0.28768207247703 | 0.28768207247701 | 0.28768207247218 | 0.28768207247702 | 0.28768207247703 |
| 14 | 0.28768207976277 | 0.28768207245895 | 0.28768207246105 | 0.28768207246104 | 0.28768207245895 | 0.28768207246105 | 0.28768207246105 |
| 16 | 0.28768207673897 | 0.28768207245468 | 0.28768207245571 | 0.28768207245571 | 0.28768207245468 | 0.28768207245571 | 0.28768207245571 |
| 18 | 0.28768207512895 | 0.28768207245308 | 0.28768207245364 | 0.28768207245363 | 0.28768207245308 | 0.28768207245363 | 0.28768207245364 |
| 20 | 0.28768207420860 | 0.28768207245242 | 0.28768207245273 | 0.28768207245273 | 0.28768207245242 | 0.28768207245273 | 0.28768207245273 |

$Table(2) calcating\ the\ integral\ I =$
$\int_3^4 \frac{1}{x} dx$ 0.28768207245178 by simpson's rule with the inverse triagular and hyperbolic methods of Al − Tememe acceleration of second kind

| n | Simpson's rule | $A^S_{\cos^{-1}(h)}$ | $A^S_{\sin^{-1}(h)}$ | $A^S_{\tan^{-1}(h)}$ | $A^S_{\cot^{-1}(h)}$ | $A^S_{\sinh^{-1}(h)}$ | $A^S_{\tanh^{-1}(h)}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.66087175465782 | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 0.66087311806141 | 0.66087320909002 | 0.66087320895475 | 0.66087320895544 | 0.66087320909002 | 0.66087320895521 | 0.66087320895452 |
| 6 | 0.66087319315843 | 0.66087321165453 | 0.66087321164384 | 0.66087321164387 | 0.66087321165453 | 0.66087321164386 | 0.66087321164383 |
| 8 | 0.66087320587131 | 0.66087321175755 | 0.66087321175555 | 0.66087321175556 | 0.66087321175755 | 0.66087321175555 | 0.66087321175555 |
| 10 | 0.66087320935318 | 0.66087321176936 | 0.66087321176879 | 0.66087321176879 | 0.66087321176936 | 0.66087321176879 | 0.66087321176879 |
| 12 | 0.66087321060521 | 0.66087321177162 | 0.66087321177141 | 0.66087321177141 | 0.66087321177162 | 0.66087321177141 | 0.66087321177141 |
| 14 | 0.66087321114226 | 0.66087321177222 | 0.66087321177213 | 0.66087321177213 | 0.66087321177222 | 0.66087321177213 | 0.66087321177213 |

Table (3) calculating the integral
I=$\int_{1.5}^{2}\sqrt{x}dx = 0.660873211772538$ by simpson's rule withthe inverse triagular and hyperbolic methods of $\mathrm{Al}-\mathrm{Tememe}$ acceleration of second kind

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