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Strongly (E,F)-convexity with applications to optimization problems

Ammar A. Enad[a] , *Saba N. Majeed*[b]

[a] Department of Mathematics, College of Education for Pure Sciences (Ibn- AL Haitham), University of Baghdad, Baghdad, Iraq. *Email:* ammaralkaaby33@gmail.com

[b] Department of Mathematics, College of Education for Pure Sciences (Ibn- AL Haitham), University of Baghdad, Baghdad, Iraq. *Email:* saba.n.m@ihcoedu.uobaghdad.edu.iq

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ABSTRACT

In this paper, a new class of nonconvex sets and functions called strongly (E,F)-convex sets and strongly (E,F)-convex functions are introduced. This class is considered as a natural extension of strongly E-convex sets and functions introduced in the literature. Some basic and differentiability properties related to strongly (E,F)-convex functions are discussed. As an application to optimization problems, some optimality properties of constrained optimization problems are proved. In these optimization problems, either the objective function or the inequality constraints functions are strongly (E,F)-convex.

MSC : 46N10, 47N10, 90C48, 90C90, 49K27

1. Introduction and Preliminaries

The class of E-convex sets and E-convex functions, introduced first by Youness [18], is considered as one of the important class of generalized classical convex sets and convex functions in finite dimensional Eucledian space. A mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ make a main contribution in introducing this type of generalized convexity (see Definitions 1.2 and 1.7). Inspiring by Youness initial results, Youness and many other researchers are studied further, improved, generalized, and extended E-conexity. For instance, related to E-convex sets, Abou Tair and Sulaiman [2] and Suneja et. al [16] applied E-convex sets in proving some inequalities. Grace and Thangavelu [5] and Majeed and Abd Al-Majeed [9] defined E-convex hull, E-convex cone, E-affine sets, and studied some of their properties and characterizations. As an application of E-convexity into optimization problems, Youness and his collaborators studied optimality conditions for non-linear optimization problems, stability and duality in E-convex programming [19-20,12-13]. E-convexity results extended to a new class that includes semi E-convex functions [3-4]. For more results on E-convexity see e.g., [17,15,10,1]. Youness and Emam [21] extended the class of E-convex sets and E-convex functions into strongly E-convex sets and E-convex functions (see Definitions 1.3 and 1.8), respectively, and studied their properties (for more recent paper on strongly E-convex sets and strongly E-convex cone sets, see [11]). The class of semi E-convex functions is extended into the class of strongly semi E-conex functions by Youness and Emam [22]. E-convex sets and functions are also extended to another class called (E,F)-convex sets and (E,F)-

Corresponding author: *Saba Naser Majeed*

Email addresses: saba.n.m@ihcoedu.uobaghdad.edu.iq

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convex functions [6-7]. In this class, the effect of two mappings $E, F: \mathbb{R}^n \to \mathbb{R}^n$ are taking into account in defining the (E, F)-convex sets and functions (see Definitions 1.4 and 1.9). The results related to semi E-conexity mentioned earlier are also extended into semi (E, F)-convexity [8]. Some basic and optimality properties are discussed in [6-7]. By combining strongly E-convexity and (E, F)-convexity, we introduce in this paper the class of strongly (E, F)-convex sets and strongly (E, F)-convex functions and studying some of their properties. In section 2, the definitions of strongly (E, F)-convex sets and functions are introduced. Some examples related to the new sets and functions are illustrated. In section 3, some basic properties related to strongly (E, F)-convex functions are deduced. Finally, section 4 discusses some differentiability properties of strongly (E, F)-convex functions. In addition, some optimality properties of a constrained optimization problems are proved as an application of (E, F)-convexity to optimization problems in this section.

Assume that $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Throughout the paper, the following assumption is needed.

Assumption (A) Let $\emptyset \neq M \subseteq \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}$ be a real valued function. Assume that $E, F: \mathbb{R}^n \to \mathbb{R}^n$ are two mappings.

For the rest of the paper, M, f, E, and F are defined as Assumption in (A) unless otherwise stated. Next, we recall the necessary definitions and related concepts that is needed in the paper.

Definition 1.1 [14] The set M is called convex if, for each $m_1, m_2 \in M$ and for each $\lambda \in [0,1]$, we have $\lambda m_1 + (1-\lambda)m_2 \in M$.

Definition 1.2 [18] The set M is called E-convex if, for each $m_1, m_2 \in M$ and for each $\lambda \in [0,1]$, we have $\lambda E(m_1) + (1-\lambda)E(m_2) \in M$.

Definition 1.3 [21] The set M is called strongly E-convex if, for each $m_1, m_2 \in M$ and for each $\lambda \in [0,1]$, we have $\lambda(\alpha m_1 + E(m_1)) + (1-\lambda)(\alpha m_2 + E(m_2)) \in M$. for all $\alpha \in [0,1]$

Definition 1.4 [6] The set M is called (E, F)-convex if, for each $m_1, m_2 \in M$ and for each $\lambda \in [0,1]$, we have $\lambda E(m_1) + (1-\lambda)F(m_2) \in M$.

Remark 1.5

i. For the rest of the paper, $E(m_1)$ and$F(m_2)$ are written as Em_1 and Fm_2.

ii. In [19], the mappings E and F are considered as point to set maps. As a result, the set M is called (E, F)-convex if $\lambda E(m_1) + (1-\lambda)F(m_2) \subseteq M$. In this paper, however, we consider E and F as point to point mappings.

Definition 1.6 [14] The function f is said to be convex if M is convex and for each $m_1, m_2 \in M$, $\lambda \in [0,1]$, $f(\lambda m_1 + (1-\lambda)m_2) \leq \lambda f(m_1) + (1-\lambda)f(m_2)$.

Definition 1.7 [18] The function f is said to be E-convex if M is E-convex and for each $m_1, m_2 \in M$, $\lambda \in [0,1]$, $f(\lambda Em_1 + (1-\lambda)Em_2) \leq \lambda f(Em_1) + (1-\lambda)f(Em_2)$.

Definition 1.8 [21] The function f is called strongly E-convex if M is strongly E-convex and for each $m_1, m_2 \in M$, $\lambda, \alpha \in [0,1]$, we have
$f(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Em_2)) \leq \lambda f(Em_1) + (1-\lambda)f(Em_2)$.

Definition 1.9 [6] The function f is called (E, F)-convex if M is (E, F)-convex and for each $m_1, m_2 \in M, \lambda, \alpha \in [0,1]$, we have $f(\lambda Em_1 + (1-\lambda)Fm_2) \leq \lambda f(Em_1) + (1-\lambda)f(Fm_2)$.

Definition 1.10 [14] The epigraph of f is denoted by $epi\ f$ and defined as $epif = \{(m, \gamma) \in M \times \mathbb{R}: f(m) \leq \gamma\}$.

2. Strongly (E, F)-convex Sets and Functions

In this section, we define the class of strongly (E, F)-convex sets and the class of (E, F)-convex functions and we provide some examples related to the new definitions. Some properties of (E, F)-convex sets are also given.

Definition 2.1 M is said to be strongly (E, F)-convex set if for each $m_1, m_2 \in M$ and for each $\alpha, \lambda \in [0,1]$ we have $\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$.

An example of strongly (E, F)-convex set that is also (E, F)-convex set is given next.

Example 2.2 Let $M=\{(m_1,m_2)\in\mathbb{R}^2: m_1,m_2\geq 0\}$ and $E,F:\mathbb{R}^2\to\mathbb{R}^2$ such that $E(m_1,m_2)=(m_1,0)$ and $F(m_1,m_2)=(0,\frac{m_2}{2})$. First, we show that M is strongly (E,F)-convex set. Let $m=(m_1,m_2)\in M$ and $m^*=(m_1^*,m_2^*)\in M$ and $\alpha,\lambda\in[0,1]$ then

$$\lambda(\alpha m+Em)+(1-\lambda)(\alpha m^*+Fm^*)$$
$$=\lambda\big(\alpha(m_1,m_2)+(m_1,0)\big)+(1-\lambda)\left(\alpha(m_1^*,m_2^*)+\left(0,\frac{m_2^*}{2}\right)\right)$$
$$=\lambda(\alpha m_1+m_1,\alpha m_2)+(1-\lambda)\left(\alpha m_1^*,\alpha m_2^*+\frac{m_2^*}{2}\right)\in M.$$

Next, we prove that M is (E,F)-convex set, i.e., $\lambda Em+(1-\lambda)Fm^*\in M$.

$$\lambda E(m_1,m_2)+(1-\lambda)F(m_1^*,m_2^*)=\lambda(m_1,0)+(1-\lambda)\left(0,\frac{m_2^*}{2}\right)=\left(\lambda m_1,(1-\lambda)\frac{m_2^*}{2}\right)\in M \text{ as required. } \blacksquare$$

Remark 2.3 Every strongly (E,F)-convex set is (E,F)-convex set $(\alpha=0)$. The converse does not hold as we show in the next example.

Example 2.4 Let $M=\{(m_1,m_2)\in\mathbb{R}^2: -1\leq m_1\leq 1, -1\leq m_2\leq 1\}$ and $E,F:\mathbb{R}^2\to\mathbb{R}^2$ such that $E(m_1,m_2)=(\frac{m_1}{3},\frac{m_2}{2})$ and $F(m_1,m_2)=(\frac{2}{3}m_1,m_2)$. We show that M is (E,F)-convex set but not strongly (E,F)-convex. Let $m=(m_1,m_2)\in M$ and $m^*=(m_1^*,m_2^*)\in M$ and $\alpha,\lambda\in[0,1]$

$$\begin{aligned}\lambda Em+(1-\lambda)Fm^* &= \lambda E(m_1,m_2)+(1-\lambda)F(m_1^*,m_2^*)\\ &=\lambda\left(\frac{m_1}{3},\frac{m_2}{2}\right)+(1-\lambda)\left(\frac{2m_1^*}{3},m_2^*\right)\\ &=\left(\frac{\lambda m_1}{3}+(1-\lambda)\frac{2m_1^*}{3},\frac{\lambda m_1}{2}+(1-\lambda)m_2^*\right)\in M.\end{aligned}$$

This shows M is (E,F)-convex set, but M is not strongly (E,F)-convex set. Indeed, take If $\lambda=0$, $\alpha=1$, $(m_1^*,m_2^*)=(1,1)$. Then

$\lambda\big(\alpha(m_1,m_2)+E(m_1,m_2)\big)+(1-\lambda)\big(\alpha(m_1^*,m_2^*)+F(m_1^*,m_2^*)\big)=(1,1)+\left(\frac{2}{3},1\right)=\left(\frac{5}{3},2\right)\notin M$. This shows that M is not strongly (E,F)-convex set. $\blacksquare$

Proposition 2.5 If a set M is strongly (E,F)-convex set. Then $E(M)\subseteq M$ and $F(M)\subseteq M$.

Proof. Using the definition of strongly (E,F)-convex set, we have for any $m_1,m_2\in M,\lambda,\alpha\in[0,1]$,

$\lambda(\alpha m_1+Em_1)+(1-\lambda)(\alpha m_2+Fm_2)\in M$. When $\alpha=0$ and $\lambda=1$, we get $Em_1\in M$, i.e., $E(M)\subseteq M$. On the other hand, when $\alpha=\lambda=0$. Then $Fm_2\in M$, i.e., $F(M)\subseteq M$. $\blacksquare$

Definition 2.6 Let $M\times\mathbb{R}\subseteq\mathbb{R}^n\times\mathbb{R}$, $E,F:\mathbb{R}^n\to\mathbb{R}^n$ and $\bar{E},\bar{F}:\mathbb{R}\to\mathbb{R}$. The set $M\times\mathbb{R}$ is called strongly $(E,F)\times(\bar{E},\bar{F})$-convex if for $(m_1,\gamma),(m_2,\beta)\in M\times\mathbb{R}$ and $\alpha,\lambda\in[0,1]$, we get

$$(\lambda(\alpha m_1+Em_1)+(1-\lambda)(\alpha m_2+Fm_2),\lambda(\alpha\gamma+\bar{E}(\gamma))+(1-\lambda)(\alpha\beta+\bar{F}(\beta))\in M\times\mathbb{R}.$$

A characterization between the strongly (E,F)convexity of $M\subseteq\mathbb{R}^n$and $M\times\mathbb{R}$ is given next.

Proposition 2.7 M is a strongly (E,F)-convex if and only if $M\times\mathbb{R}$ is strongly $(E,F)\times(\bar{E},\bar{F})$-convex.

Proof. Assume that M is a strongly (E,F)-convex set, then for $m_1,m_2\in M$, $\alpha,\lambda\in[0,1]$, and $\gamma,\beta\in\mathbb{R}$ we have $\lambda(\alpha m_1+Em_1)+(1-\lambda)(\alpha m_2+Fm_2)\in M$ and $\lambda(\alpha\gamma+\bar{E}\gamma)+(1-\lambda)(\alpha\beta+\bar{F}\beta)\in\mathbb{R}$. Thus, $\big(\lambda(\alpha m_1+Em_1)+(1-\lambda)(\alpha m_2+Fm_2),\lambda(\alpha\gamma+\bar{E}\gamma)+(1-\lambda)(\alpha\beta+\bar{F}\beta)\big)\in M\times\mathbb{R}$. This shows $M\times\mathbb{R}$ is a strongly $(E,F)\times(\bar{E},\bar{F})$-convex. Similarly, the other direction follows directly. $\blacksquare$

Next, we introduce strongly (E,F)-convex function.

Definition 2.8 Let M is strongly (E,F)-convex set. A function f is said to be strongly (E,F)-convex function on M, if for each $m_1,m_2\in M,\lambda,\alpha\in[0,1]$, $f\big(\lambda(\alpha m_1+Em_1)+(1-\lambda)(\alpha m_2+Fm_2)\big)\leq\lambda f(Em_1)+(1-\lambda)f(Fm_2)$.

Example 2.9 Let $M=\mathbb{R}$ and $f,E,F:\mathbb{R}\to\mathbb{R}$ such that for each $m\in\mathbb{R}$

$$f(m)=\begin{cases}2 & if\ m\in[0,2]\\ 1 & otherwise\end{cases},$$

$E(m) = 0$ and $F(m) = \frac{1}{2}$. We show that f is a strongly (E,F)-convex function. For each $m_1, m_2 \in \mathbb{R}, \lambda, \alpha \in [0,1]$ we have $f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) = 1$ or 2. On the other hand, $\lambda f(Em_1) + (1-\lambda)f(Fm_2) = \lambda f(0) + (1-\lambda)f\left(\frac{1}{2}\right) = 2$. Thus, $f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda f(Em_1) + (1-\lambda)f(Fm_2)$ as required to show. ■

Remark 2.10 Every strongly (E,F)-convex function is (E,F)-convex $(\alpha = 0)$. The converse is not necessary true as it illustrated in the next example.

Example 2.11 Let $M = \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, and $E, F: \mathbb{R} \to \mathbb{R}$ such that

$$f(m) = \begin{cases} -1 & \text{if } m = 0 \\ -2 & \text{otherwise} \end{cases}, \quad E(m) = \begin{cases} 0 & \text{if } m = 0 \\ 2 & \text{otherwise} \end{cases}$$

and $F(m) = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{otherwise} \end{cases}$.

Let $m_1, m_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. First, we show that $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) \leq \lambda f(Em_1) + (1-\lambda)f(Fm_2)$. We consider four cases:

Case 1. If $m_1 = m_2 = 0$, then $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) = f\big(\lambda E(0) + (1-\lambda)F(0)\big) = f(0) = -1$ and

$\lambda f(Em_1) + (1-\lambda)f(Fm_2) = \lambda f\big(E(0)\big) + (1-\lambda)f\big(F(0)\big) = \lambda f(0) + (1-\lambda)f(0) = -\lambda - (1-\lambda) = -1.$

Case 2. If $m_1 \neq 0$ and $m_2 \neq 0$, then $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) = f(\lambda(2) + (1-\lambda)(1)) = -2$ and

$\lambda f(Em_1) + (1-\lambda)f(Fm_2) = \lambda f(2) + (1-\lambda)f(1) = \lambda(-2) + (1-\lambda)(-2) = -2.$

Case 3: If $m_1 = 0$ and $m_2 \neq 0$, then $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) = f(\lambda(0) + (1-\lambda)(1))$

$$= f(1-\lambda) = \begin{cases} -2 & if\ \lambda = 0 \\ -1 & if\ \lambda = 1 \end{cases},$$

On the other hand, $\lambda f(Em_1) + (1-\lambda)f(Fm_2) = \lambda f(0) + (1-\lambda)f(1) = \lambda(-1) + (1-\lambda)(-2) = \lambda - 2.$

Case 4: If $m_1 \neq 0$ and $m_2 = 0$, then $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) = f(\lambda(2) + (1-\lambda)(0))$

$$= f(2\lambda) = \begin{cases} -1 & if\ \lambda = 0 \\ -2 & if\ \lambda = 1 \end{cases},$$

and $\lambda f(Em_1) + (1-\lambda)f(Fm_2) = \lambda f(2) + (1-\lambda)f(0) = \lambda(-2) + (1-\lambda)(-1) = -1 - \lambda.$

From all cases, $f\big(\lambda(Em_1) + (1-\lambda)(Fm_2)\big) \leq \lambda f(Em_1) + (1-\lambda)f(Fm_2)$. Now, we show that f is not strongly (E,F)-convex function. Let $\alpha = 1, m_1 = -2, m_2 = 0, \lambda = 1$, then

$$f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) = f(-2 + E(-2)) = f(-2+2) = f(0) = -1,$$

On the other hand, $\lambda f(Em_1) + (1-\lambda)f(Fm_2) = f\big(E(-2)\big) = f(2) = -2.$

Thus, $f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) \geq \lambda f(Em_1) + (1-\lambda)f(Fm_2)$. This means f is not strongly (E,F)-convex function. ■

3. Some Properties of Strongly (E,F)-convex Functions

In this section, we discuss some basic properties of strongly (E,F)-convex functions such as closedness, supremum and composite properties. But first we start with the following proposition.

Proposition 3.1 If f strongly (E,F)-convex function on the strongly (E,F)-convex set M, then $f(\alpha m_1 + Em_1) \leq f(Em_1)$ and $f(\alpha m_2 + Fm_2) \leq f(Fm_2)$, for each $m_1, m_2 \in M$ and $\alpha \in [0,1]$.

Proof. From the assumptions on M and f, we have for each $m_1, m_2 \in M$ and for all $\lambda, \alpha \in [0,1]$, we have

$\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$ and $f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda f(Em_1) + (1-\lambda) f(Fm_2)$. Then for $\lambda = 1$, $f(\alpha m_1 + Em_1) \leq f(Em_1)$ and for $\lambda = 0$, we obtain $f(\alpha m_2 + Fm_2) \leq f(Fm_2)$. ■

Proposition 3.2 If $f_i: \mathbb{R}^n \to \mathbb{R}$, $i = 1,2,\dots,n$ are strongly (E,F)-convex functions on the strongly (E,F)-convex set M such that $f(m) = \sum_{i=1}^{n} a_i f_i(m)$, then f is strongly (E,F)-convex function on M for each $a_i \geq 0, i = 1,\dots,n$.

Proof. Since f_i, for all $i = 1,2,\dots,n$ are strongly (E,F)-convex function on a strongly (E,F)-convex set M, then for each $m_1, m_2 \in M$ and for all $\lambda, \alpha \in [0,1]$, we have $\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$ and

$$f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) = \sum_{i=1}^{n} a_i f_i(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2))$$

$$\leq \lambda \textstyle\sum_{i=1}^{n} a_i f_i(Em_1) + (1-\lambda) \sum_{i=1}^{n} a_i f_i(Fm_2)$$

$$= \lambda f(Em_1) + (1-\lambda) f(Fm_2)$$

This implies, f is strongly (E,F)-convex function on M. ■

Proposition 3.3 Let I be an index set and $f_i: \mathbb{R}^n \to \mathbb{R}$ for each $i \in I$ be a family of bounded above and strongly (E,F)-convex functions on the strongly (E,F)-convex set M. Then, $f(m) = \sup_{i \in \mathrm{I}} f_i\,(m)$ is a strongly (E,F)-convex on M.

Proof. From the assumptions, for each $m_1, m_2 \in M$ and for each $\lambda, \alpha \in [0,1]$, we have $\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$ and

$$f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) = \sup_{i \in \mathrm{I}} \{f_i\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big)\}$$

$$\leq \sup_{i \in \mathrm{I}} \{\lambda f_i(Em_1) + (1-\lambda) f_i(Fm_2)\}$$

where in the last inequality we used the fact that f_i is strongly (E,F)-convex for each $i \in I$ and f_i is bounded above for all $i \in I$. The inequality above yields

$$= \lambda \sup_{i \in \mathrm{I}} f_i(Em_1) + (1-\lambda) \sup_{i \in \mathrm{I}} f_i(Fm_2) = \lambda f(Em_1) + (1-\lambda) f(Fm_2).$$

This means f is a strongly (E,F)-convex on M. ■

Proposition 3.4 Let f be a strongly (E,F)-convex function on the strongly (E,F)-convex set M. Let $G: \mathbb{R} \to \mathbb{R}$ be a non-decreasing convex function, then $G \circ f$ is a strongly (E,F)-convex function on M.

Proof. Let $m_1, m_2 \in M$ and $\lambda, \alpha \in [0,1]$. From the assumptions on M and f, we get $\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$ and

$$f\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda f(Em_1) + (1-\lambda) f(Fm_2)$$

From the assumptions on G, the last inequality yields

$$(G \circ f)(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)) \leq G \circ (\lambda f(Em_1) + (1-\lambda) f(Fm_2))$$

$$\leq \lambda\,(G \circ f)(Em_1) + (1-\lambda)(G \circ f)(Fm_2)$$

This implies, $G \circ f$ is strongly (E, F)-convex function on M. ∎

Next, we show that, under mild condition, the epigraph of strongly(E, F)-convex functions is strongly(E, F)-convex set.

Proposition 3.5 Let M is strongly (E, F)-convex. Let $\bar{E}, \bar{F}: \mathbb{R} \to \mathbb{R}$ be two mappings such that $\bar{E}(f(m_1)) = f(Em_1)$ and $\bar{F}(f(m_2)) = f(Fm_2)$ for each $m_1, m_2 \in M$. If f is strongly (E, F)-convex on M, then $epif$ is strongly $(E, F) \times (\bar{E}, \bar{F})$-convex set on $M \times \mathbb{R}$.

Proof. Let $(m_1, \gamma), (m_2, \beta) \in epif$. From the assumption on M, we have for each $\alpha, \lambda \in [0,1]$

$\lambda(\alpha m_1 + Em_1) + (1 - \lambda)(\alpha m_2 + Fm_2) \in M$. Noting that $Em_1 \in M$ for $\alpha = 0$, $\lambda = 1$ and $Fm_2 \in M$ for $\alpha = \lambda = 0$. Since $f(m_1) \leq \gamma, f(m_2) \leq \beta$ then $f(Em_1) = \bar{E}(f(m_1)) \leq \bar{E}(\gamma)$ and $f(Fm_2) = \bar{F}(f(m_2)) \leq \bar{F}(\beta)$ where $\bar{E}(\gamma)$, $\bar{F}(\beta) \in \mathbb{R}$. Thus, $(Em_1, \bar{E}\gamma)$,$(Fm_2, \bar{F}\beta) \in epif$. Since f is strongly (E, F)-convex function on M. Then

$$f\big(\lambda(\alpha m_1 + Em_1) + (1 - \lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda f(Em_1) + (1 - \lambda)f(Fm_2)$$

$$\leq \lambda\bar{E}(\gamma) + (1 - \lambda)\bar{F}(\beta).$$

Thus, $\big(\lambda(\alpha m_1 + Em_1) + (1 - \lambda)(\alpha m_2 + Fm_2), \lambda\bar{E}(\gamma) + (1 - \lambda)\bar{F}(\beta)\big) \in epif$. This implies that $epif$ is strongly $(E, F) \times (\bar{E}, \bar{F})$-convex on $M \times \mathbb{R}$. ∎

4. Differentiability and Optimality Properties of Strongly (E, F)-convex Functions

In this section, we provide some necessary conditions for a differentiable function f to be strongly (E, F)-convex function. We also consider some optimality properties of non-linear optimization problems in which the objective function or the inequality constraints functions are strongly (E, F)-convex. Let us start with the following differentiability gradient property related to strongly (E, F)-convex functions.

Proposition 4.1 Let f be a differentiable strongly (E, F)-convex function on the strongly (E, F)-convex set M, then $f(Em_1) \geq f(Fm_2) + \langle\nabla f(Fm_2), Em_1 - Fm_2\rangle$ for each $m_1, m_2 \in M$,

where $\nabla f(.)$ denotes the gradient vector of f at a point belongs to M.

Proof. Since M is strongly (E, F)-convex and f is differentiable on M, then using Proposition 2.5, f is differentiable on $E(M) \subseteq M$ and $F(M) \subseteq M$. Consider $m_1, m_2 \in M$ and arbitrary $\lambda \in [0,1]$, and $\alpha \in (0,1]$. If $E(m_1) = F(m_2)$, then the gradient inequality directly satisfied. If $E(m_1) \neq F(m_2)$, then using the strongly (E, F)-convexity of f, we have $f\big(\lambda(\alpha m_1 + Em_1) + (1 - \lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda f(Em_1) + (1 - \lambda)f(Fm_2)$. That is,

$$f((\alpha m_2 + Fm_2) + \lambda((\alpha m_1 + Em_1) - (\alpha m_2 + Fm_2)) \leq f(Fm_2) + \lambda\big(f(Em_1) - f(Fm_2)\big)$$

By taking $\alpha \to 0^+$, we get $f\big(Fm_2 + \lambda(Em_1 - Fm_2)\big) \leq f(Fm_2) + \lambda\big(f(Em_1) - f(Fm_2)\big)$. Re-arranging the last inequality yields, $\frac{f(Fm_2 + \lambda(Em_1 - Fm_2)) - f(Fm_2)}{\lambda} \leq f(Em_1) - f(Fm_2)$. Taking the limit to both sides of the above inequality ($as\ \lambda \longrightarrow 0^+$) yields, $\lim_{\lambda \to 0^+} \frac{f(Fm_2 + \lambda(Em_1 - Fm_2)) - f(Fm_2)}{\lambda} \leq f(Em_1) - f(Fm_2)$. The left-hand side of the last inequality is the directional derivative of f at Fm_2 in the direction of $(Em_1 - Fm_2)$. Thus it becomes $\langle\nabla f(Fm_2), Em_1 - Fm_2\rangle \leq f(Em_1) - f(Fm_2)$. Re-arranging last expression, we get

$$f(Em_1) \geq f(Fm_2) + \langle\nabla f(Fm_2), Em_1 - Fm_2\rangle. \quad ∎$$

Proposition 4.2 Let f be a differentiable strongly (E, F)-convex function on the strongly (E, F)-convex set M, then $\langle\nabla f(Fm_2) - \nabla f(Em_1), Fm_2 - Em_1\rangle \geq 0$ for each $m_1, m_2 \in M$

Proof. From Proposition 4.1, we have $f(Em_1) \geq f(Fm_2) + \langle \nabla f(Fm_2), Em_1 - Fm_2 \rangle$ and $f(Fm_2) \geq f(Em_1) + \langle \nabla f(Em_1), Fm_2 - Em_1 \rangle$ for each $m_1, m_2 \in M$. Adding and re-arranging the above two inequalities implies $\langle -\nabla f(Fm_2) + \nabla f(Em_1), Fm_2 - Em_1 \rangle \leq 0$. i.e., $\langle \nabla f(Fm_2) - \nabla f(Em_1), Fm_2 - Em_1 \rangle \geq 0$ as required. ■

Using the second derivative of f, another necessary condition for f to be strongly (E, F)-convex is shown below.

Proposition 4.3 If f be a differentiable strongly (E, F)-convex function on the strongly (E, F)-convex set M. Then the Hessian matrices $H(Em_1) = \nabla^2 f(Em_1)$ and $H(Fm_2) = \nabla^2 f(Fm_2)$ are positive semi definite for all $m_1, m_2 \in M$.

Proof. Suppose $H(Em_1)$ is not positive semi definite for some $m_1 \in M$. Hence there exists $m_2 \in M$ such that

$$(Fm_2 - Em_1)\langle H(Em_1), Fm_2 - Em_1 \rangle < 0 \qquad (1)$$

Consider some point lies on the line segment joining Em_1 and Fm_2, namely, $m^* = \lambda Em_1 + (1-\lambda)Fm_2$, $\lambda \in (0,1)$. Since M is strongly (E, F)-convex set, then $m^* \in M$. Using second order truncated Taylor's series, we have

$$f(Fm_2) = f(Em_1) + \langle \nabla f(Em_1), Fm_2 - Em_1 \rangle + \frac{1}{2}(Fm_2 - Em_1)\langle H(m^*), Fm_2 - Em_1 \rangle \qquad (2)$$

Choose m^* sufficiently close to Em_1, we can use $f \in C^2$(continuity of second order patrials) such that $\frac{1}{2}(Fm_2 - Em_1)^T H(m^*)(Fm_2 - Em_1) < 0$ where the last inequality follows from (1). Therefore, (2) becomes $f(Fm_2) < f(Em_1) + \langle \nabla f(Em_1), (Fm_2 - Em_1) \rangle$. By Proposition 4.1, this contradicts the strongly (E, F)-convexity of f over M. Therefore, $H(Em_1)$ must be positive semi definite. In a similar manner, one can obtain the same conclusion if $H(Fm_2) = \nabla^2 f(Fm_2)$ is positive semi definite. ■

For the rest of this section we apply strongly (E, F)-convexity into non-linear optimization problems. Thus, let us consider the following nonlinear optimization problem which we denoted by (P).

$$\min\ f(m)$$

$$\text{subject to} \quad m \in M,$$

where M and f are assumed as in the Assumption (A). Let $g_i: \mathbb{R}^n \to \mathbb{R}$ be a real valued function for each $i = 1, \dots, r$ such that $M = \{m \in \mathbb{R}^n: g_i(m) \leq 0 \text{ for each } i = 1, \dots, r\}$.

Definition 4.4 In the Problem (P)

1. The set of all global minimum (or optimal solutions) is denoted by $argmin_M f$ and is defined as $argmin_M f = \{m^* \in M: f(m^*) \leq f(m), for\ each\ m \in M\}$.
2. A point $m^* \in \mathbb{R}^n$ is said to be local minimium if there exists $\varepsilon > 0$ such that $f(m^*) \leq f(m)$ for each $m \in B(m^*, \varepsilon) \cap M$, where $B(m^*, \varepsilon) = \{m \in \mathbb{R}^n: ||m - m^*|| < \varepsilon\}$ is the neighborhood of m^* with radius ε.

Next, we prove that, under simple conditions, the constraint set M of Problem (P) is strongly (E, F)-convex set.

Proposition 4.5 Let g_i are strongly (E, F)-convex functions for each $i = 1,2, \dots, r$ such that M and g_i are defined as in Problem (P). If $E(M) \subseteq M$ and $F(M) \subseteq M$ then M is strongly (E, F)-convex set.

Proof. Since $g_i(m)$, $i = 1,2, \dots, r$ are strongly (E, F)-convex function then, for each $m_1, m_2 \in M$, $\alpha, \lambda \in [0,1]$, we have

$$g_i\big(\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2)\big) \leq \lambda g_i(Em_1) + (1-\lambda) g_i(Fm_2) \leq 0\ ,\ i = 1,2, \dots, r,$$

where in the right most inequality we employed the assumptions $E(M) \subseteq M$ and $F(M) \subseteq M$. Hence, $\lambda(\alpha m_1 + Em_1) + (1-\lambda)(\alpha m_2 + Fm_2) \in M$. Thus, M is strongly (E, F)-convex set. ■

Theorem 4.6 Consider Problem (P) such that M be a strongly (E, F)-convex set and $f(\alpha m + Fm) \leq f(m)$ for each $m \in M$, $\alpha \in [0,1]$. If $m^* \in M$ is solution of the following problem denoted by (P_1)

$$\min f(\alpha m_1 + Em_1)$$

subject to $m_1 \in M$

Then $\alpha m^* + Em^*$ is an optimal solution of Problem (P).

Proof. On Contrary, assume that $\alpha m^* + Em^*$ is not a solution of problem (P), then there is $m_2 \in M$ such that $f(m_2) < f(\alpha m^* + Em^*)$. From the assumption, $f(\alpha m_2 + Fm_2) \leq f(m_2) < f(\alpha m^* + Em^*)$ for each $m_2 \in M$, which contradicts the optimality of m^* for problem (P_1). Hence $\alpha m^* + Em^*$ is an optimal solution of problem (P). ■

Theorem 4.7 Let M be a strongly (E,F)-convex set and f is a strongly (E,F)-convex function on M and $f(Fm_2) \leq f(m_2)$ and $f(Em_1) \leq f(m_1)$ for each $m_1, m_2 \in M$. If $m^* = E(z^*) \in E(M)$ is a local minimum of problem (P), then m^* is global minimum of problem (P) on M.

Proof. Let $m^* = E(z^*) \in E(M)$ be a non-global minimum of problem (P) on M, then, there is $m_2 \in M$ such that $f(m_2) \leq f(m^*) = f(Ez^*)$. Since f is a strongly (E,F)-convex function and $f(Fm_2) \leq f(m_2)$ for each $m_2 \in M$, it implies that

$$f\big(\lambda(\alpha m_2 + Fm_2) + (1-\lambda)(\alpha z^* + Ez^*)\big) \leq \lambda f(Fm_2) + (1-\lambda) f(Ez^*)$$

$$\leq \lambda f(m_2) + (1-\lambda) f(m^*) \leq f(m^*)$$

By putting $\alpha = 0$, we get $f(\lambda(Fm_2) + (1-\lambda)m^*) \leq f(m^*)$. For any small $\lambda \in (0,1)$, which contradicts the local optimality of m^* for problem (P). Hence, m^* is a global minimum of problem (P) on M. ■

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التحدب بقوة من النوع (E, F) وتطبيقاته على مشاكل الامثلية

عمار عبد الكاظم عناد   صبا ناصر مجيد

قسم الرياضيات، كلية التربية للعلوم الصرفة إبن الهيثم، جامعة بغداد، بغداد، العراق.

المستخلص

في هذا البحث قمنا بتعريف نوع جديد من المجموعات والدوال الغير المحدبة والمسماة بالمجموعات والدوال المحدبة بقوة من نوع (E, F) والتي تعتبر توسيع طبيعي للمجاميع والدوال المحدبة بقوة من النوع E. قمنا بدراسة بعض الخواص الاساسية وخواص بعض الدوال المحدبة بقوة من النوع (E, F). درسنا ايضاً خواص مشاكل الامثلية الغير الخطية والتي تكون دالة الهدف فيها او دوال القيود محدبة بقوة من النوع (E, F)

الكلمات المفتاحية: المجاميع المحدبة بقوة من النوع E، الدوال المحدبة بقوة من النوع E ، المجاميع المحدبة بقوة من النوع (E, F)، الدوال المحدبة بقوة من النوع (E, F)