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Neutral Stochastic Functional Differential Equations with Infinite Delay under state space C_r: existence and uniqueness

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ABSTRACT

In this paper, we studied the neutral stochastic functional differential equations with infinite delay (NSFDEwID for short).The existence and uniqueness of solutions to NSFDEwID at the state space C_r have been addressed under the local Lipschitz condition and Linear growth condition . we proved lemma 3.4 because it is presented in many articles without prove.

MSC.

1 . Introduction

stochastic differential equation play an important role in many branches of the natural sciences and engineering. Neutral stochastic functional differential equations (NSFDEs) involve derivatives with delays , also depends on past and present values. The (NSFDEs) have been invistigated by many authors, see [2, 5, 7, 8, 9, 13, 14, 17]. Many articles studied The existence and uniqueness of solutions to NSFDEs by imposing Lipschitz condition, for example see [3, 14, 19,20]. Anguraj et al. [1] have established the impulsive NSFDEs under non-Lipschitz condition and Lipschitz condition. A. Lin et al. [11] studied the neutral impulsive stochastic integro-differential equations with infinite delay via fractional operators. In recent years, there is an increasing interest in the theory of existence and uniqueness of solutions of (NSFDEwID). H. Bin Chen[6] and W.Lin et al. [12]have proved the existence and uniqueness for the solution of neutral stochastic functional differential equations with infinite delay(NSFDEwID). Bao and Hou [4], have investigated the existence and uniqueness of mild solutions to stochastic neutral partial functional differential equations under a non-Lipschitz condition and a weakened linear growth condition. Mohammed [15] proved that the solution maps of SFDEs with finite delay on appropriate phase spaces have Markov

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property. Wu et al. [18] investigates existence and uniqueness of solutions, Markov properties, and ergodicity of SFDEs with infinite delay by using phase state C_r . Ren and Xia [16] have studied existence and uniqueness of solutions with infinite delay at phase space $BC((-\infty;0];\mathbb{R}^d)$ which denotes the family of bounded continuous $\mathbb{R}^d$-value functions with norm $\|\varphi\| sup_{-\infty<\theta\leq 0}|\varphi(\theta)|$ under non-Lipschitz condition.The aim of this paper is to investigate the existence and uniqueness of NSFDEwID under local Lipschitz condition and Linear growth condition which can be obtained in the state space C_r .This paper is organized as follows, some useful preliminaries are introduced In Section2 .In Section3, we addressed existence and uniqueness of maximal local strong solutions of (2) by local Lipschitz condition and Linear growth condition.

2. Preliminaries

Throughout this paper, we adopt the symbols as follows. R^d denotes the usual d-dimensional Euclidean space, $|\cdot|$ norm in R^d. If A is a vector or a matrix, its transpose is denoted by A^T ; and, $|A|=\sqrt{trace(A^TA)}$ its trace norm. Denote by X^TY the inner product of X, Y $\in \mathbb{R}^d$. We choose the state space with the fading memory to be C_r defined as follows: for given positive number r,

$$C_r=\{\varphi\in C((-\infty,0);R^d): \|\varphi\|_r=\sup_{-\infty<\theta\leq 0} e^{r\theta}|\varphi(\theta)|<\infty), \tag{1}$$

where $C((-\infty,0);R^d)$ denotes the family of all bounded continuous R^d–value functions φ defined on $(-\infty,0)$ to R^d with the norm $\|\varphi\|_r$. $M^2([a,b];R^d)$ is a family of process $\{\varphi(t)\}_{a\leq t\leq b}$ in $\mathcal{L}^2([a,b];\mathbb{R}^d)$ such that $\mathbb{E}\int_a^b|\varphi(t)|^2dt<\infty$. For $\varphi\in M^2$, let $\|\varphi\|_r^2:=(\int_{-\infty}^0 e^{2r\theta}|\varphi(\theta)|^2)^{\frac{1}{2}}$. Then M^2 is a Hibert space equppied with the norm $\|\cdot\|_r$ and $(C_r,\|\cdot\|_r)$ is a Banach space which is introduced in [10], contains the Banach space of bounded and continuous functions and for any $0<r_1\leq r_2<\infty, C_{r_1}\subset C_{r_2}$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,+\infty]}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets).

Let I_B denote the indicator function of a set B. Consider a d-dimensional neutral stochastic functional differential equations with infinite delay

$$d\{x(t)-D(x_t)\}=b(x_t)dt+\sigma(x_t)dw(t),\quad on\quad t\geq 0, \tag{2}$$

with the initial data:

$$x_0=\xi=\{\xi(\theta):-\infty<\theta\leq 0\}\in C_r, \tag{3}$$

where

$$x_t=x(t+\theta):-\infty<\theta\leq 0$$

and $b,D:C_r\to\mathbb{R}^d$; $\sigma:C_r\to\mathbb{R}^{d\times m}$ are Borel measurable, $w(t)$ is an m-dimensional Brownian motion. It should be pointed out that $x(t)\in R^d$ is a point, while $x_t\in C_r$ is a continuous function on the interval $(-\infty,0]$ taking values in R^d.

Definition 2.1[14]: $\mathcal{R}^d$-value stochastic process $x(t)$ defined on $-\infty<t\leq T$ is called the solution of (2) with initial data (3), if $x(t)$ has the following properties:

(i) $x(t)$is continuous and $\{x(t)\}_{t_0\leq t\leq T}$ is $\mathcal{F}_t$-adapted;

(ii) $\{b(x_t)\}\in\mathcal{L}^1([0.T];\mathcal{R}^d)$ and $\{\sigma(x_t)\}\in\mathcal{L}^2([0.T];\mathcal{R}^{d\times m})$

(iii) $x_{t_0}=\xi$, for each $t_0\leq t\leq T$,

$$x(t) = D(x_t) + x(0) - D(x_0) + \int_0^t b(x_s)\, ds + \int_0^t \sigma(x_s) ds$$

Asolution $x(t)$ is called as a unique if any other solution $\bar{x}(t)$is indistinguishable with $x(t)$, that is

$$P(x(t) = \bar{x}(t)\text{ , for all } -\infty < t \leq T) = 1$$

3. Wellposedness under Local Lipschitz Condition

To address existence and uniqueness of maximal local strong solutions of (2), let us first impose the following conditions for b, σ and the neutral term D:

(H1) (The local Lipschitz condition) For any $n > 0$, there exists a $k_n$such that

$$|b(\phi) - b(\varphi)| \vee |\sigma(\phi) - \sigma(\varphi)| \leq k_n \parallel \phi - \varphi \parallel_r, \tag{4}$$

where $\phi, \varphi \in C_r$ with $\parallel \varphi \parallel_r \vee \parallel \phi \parallel_r \leq n$.

Remark 3.1: [19] If we define two stopping times*:* $\tau_n = inf\{t \geq 0, |x(t)| > n\}$ *, and* $\rho_n = inf\{t \geq 0, \parallel x_t \parallel_r > n\}$*, then* $\tau_n = \rho_n$*.*

(H2) There is a $k \in (0,1)$ such that for all $\phi, \varphi \in C_r$,

$$|D(\phi) - D(\varphi)| \leq k \parallel \phi - \varphi \parallel_r \text{ And and } D(0) = 0 \tag{5}$$

Remark 3.2: Note that **(H2)** implies that for any $\varphi \in C_r$, $|D(\varphi)| \leq k \parallel \varphi \parallel_r$, if we suppose $D(0) = 0$.

(H3) (Linear growth condition) For any $\phi \in C_r$, there exists a positive number c such that:

$$|b(\phi)| \vee |\sigma(\phi)| \leq c(1 + \parallel \phi \parallel_r). \tag{6}$$

Keeping in mind the end goal to demonstrate the theorem of existence and uniqueness of the solution to the equation (2) under the local Lipschitz condition, we initially set up these two lemmas.

Lemma 3.3: Let **(H2)** and **(H3)** hold. Let x(t) be a solution to equation (2) with initial data (3). Then

$$\begin{aligned}\mathbb{E}\Big(\sup_{-\infty<t\leq T}(e^{rt}|x(t)|)^2\Big) \leq\ & \mathbb{E}\parallel \xi \parallel_r^2 + \Big[\Big(\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2} + \frac{3e^{2rT}(1+cT(T+1))}{(1-k)(1-\sqrt{k})}\Big)\mathbb{E}\parallel \xi \parallel_r^2 \\ & + \frac{3cTe^{2rT}(T+1)}{(1-k)(1-\sqrt{k})}\Big] \times \exp\Big(\frac{3ce^{2rT}T(T+1)}{(1-k)(1-\sqrt{k})}\Big).\end{aligned} \tag{7}$$

Where, particularly, $x(t) \in M^2((-\infty, T]; R^d)$.

Proof: For every integer $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [0, T]: \parallel x_t \parallel_r \geq n\},$$

it is clear that $\tau_n \uparrow T$ a.s. . Set $x_t^0 = \xi$ and $x^n(t) = x(t \wedge \tau_n)$ for $t \in [0, T]$. Then for $0 \leq t \leq T$

$$x^n(t) = D(x_t^n) - D(\xi) + \xi(0) + \int_0^{t\wedge\tau_n} b(x_s^n) ds + \int_0^{t\wedge\tau_n} \sigma(x_s^n) dw(s)$$

Let

$$J^n(t) = \xi(0) + \int_0^{t\wedge\tau_n} b(x_s^n)ds + \int_0^{t\wedge\tau_n} \sigma(x_s^n)dw(s).$$

Then, applying (for any a, b and $0 < k < 1$ we have $(a+b)^2 \leq \frac{a^2}{k} + \frac{b^2}{1-k}$) twice one derives that:

$$\begin{aligned} e^{2rt}|x^n(t)|^2 &\leq \frac{e^{2rt}}{k}|D(x_t^n) - D(\xi)|^2 + \frac{e^{2rt}}{1-k}|J^n(t)|^2 \\ &\leq ke^{2rt} \parallel x_t^n - \xi \parallel_r^2 + \frac{e^{2rt}}{1-k}|J^n(t)|^2 \\ &\leq \sqrt{k}e^{2rt} \parallel x_t^n \parallel_r^2 + \frac{ke^{2rt}}{1-\sqrt{k}} \parallel \xi \parallel_r^2 + \frac{e^{2rt}}{1-k}|J^n(t)|^2. \end{aligned} \tag{8}$$

Since

$$\begin{aligned} e^{2rt} \parallel x_t^n \parallel_r^2 &= e^{2rt} \sup_{-\infty<\theta\leq 0} (e^{r\theta}|x_t^n(\theta)|)^2 \\ &= \sup_{-\infty<\theta\leq 0} (e^{r(t+\theta)}|x^n(t+\theta)|)^2 \\ &= \sup_{-\infty<s\leq t} (e^{rs}|x^n(s)|)^2 \\ &\leq \sup_{-\infty<s\leq 0} (e^{rs}|x^n(s)|)^2 + \sup_{0\leq s\leq t} (e^{rs}|x^n(s)|)^2 \\ &= \parallel \xi \parallel_r^2 + \sup_{0\leq s\leq t} (e^{rs}|x^n(s)|)^2. \end{aligned} \tag{9}$$

So, by substituting (9) into (8) one can get that:

$$\begin{aligned} e^{2rt}|x^n(t)|^2 &\leq \sqrt{k} \parallel \xi \parallel_r^2 + \sqrt{k} \sup_{0\leq s\leq t} (e^{rs}|x^n(s)|)^2 \\ &\quad + \frac{ke^{2rt}}{1-\sqrt{k}} \parallel \xi \parallel_r^2 + \frac{e^{2rt}}{1-k}|J^n(t)|^2 \\ &\leq \frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})}(\parallel \xi \parallel_r^2) + \sqrt{k}(\sup_{0\leq s\leq t} (e^{rs}|x^n(s)|)^2 \\ &\quad + \frac{e^{2rt}}{1-k}|J^n(t)|^2. \end{aligned} \tag{10}$$

Hence, by taking the expectation on both sides of (10) with the Holder Inequality, one can get that:

$$\mathbb{E}(\sup_{0\leq t\leq T} e^{2rt}|x^n(t)|^2) \leq \frac{\sqrt{k}+k(e^{2rT}-1)}{(1-\sqrt{k})^2} \mathbb{E} \parallel \xi \parallel_r^2 + \frac{1}{(1-k)(1-\sqrt{k})} \mathbb{E}(\sup_{0\leq t\leq T} e^{2rt}|J^n(t)|^2). \tag{11}$$

On the other hand, by Holder Inequality, the Burkholder -Davis-Gundy inequality and **(H3)** (Linear growth condition) we can get that:

$$\begin{aligned} \mathbb{E}(\sup_{0\leq t\leq T} e^{2rt}|J^n(t)|^2) &\leq 3e^{2rT}\mathbb{E}(\sup_{0\leq t\leq T} [|\xi(0)|^2 + |\int_0^{t\wedge\tau_n} b(x_s^n)ds|^2 + |\int_0^{t\wedge\tau_n} \sigma(X_s^n)dw(s)|^2]) \\ &\leq 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3e^{2rT}\mathbb{E}(\sup_{0\leq t\leq T} (\int_0^{t\wedge\tau_n} |b(x_s^n)|)^2 ds) + 12\mathbb{E} \int_0^{T\wedge\tau_n} |\sigma(x_s^n)|^2 ds \\ &\leq 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3Te^{2rT} \int_0^T \mathbb{E}|b(x_s^n)|^2 ds + 12e^{2rT} \int_0^T \mathbb{E}|\sigma(x_s^n)|^2 ds \\ &\leq 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3ce^{2rT}(T+4) \int_0^T \mathbb{E}(1 + \parallel x_s^n \parallel_r^2) ds \\ &= 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3cTe^{2rT}(T+4) + 3ce^{2rT}(T+4) \int_0^T \mathbb{E} \parallel x_s^n \parallel_r^2 ds \\ &\leq 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3cTe^{2rT}(T+4) + 3ce^{2rT}(T+4) \int_0^T \mathbb{E}(\sup_{-\infty<\theta\leq 0} e^{2r\theta}|x^n(s+\theta)|^2) ds \\ &\leq 3e^{2rT}\mathbb{E} \parallel \xi \parallel_r^2 + 3cTe^{2rT}(T+4) \\ &\quad + 3ce^{2rT}e^{-2rs}(T+4) \int_0^T \mathbb{E}(\parallel \xi \parallel_r^2 + \sup_{0\leq u\leq T} e^{2ru}|x^n(u)|_r^2) du \\ &\leq 3e^{2rT}(1 + cT(T+4))\mathbb{E} \parallel \xi \parallel_r^2 + 3cTe^{2rT}(T+4) \\ &\quad + 3ce^{2rT}(T+4) \int_0^T \mathbb{E}(\sup_{0\leq u\leq T} e^{2ru}|x^n(u)|^2) du. \end{aligned} \tag{12}$$

Substituting (12) into (11) yields that

$$\mathbb{E}(\sup_{0\le t\le T}(e^{rt}|x^n(t)|)^2) \le (\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2}+\frac{3e^{2rT}(1+cT(T+4))}{(1-k)(1-\sqrt{k})})\mathbb{E}\parallel \xi \parallel_r^2 + \frac{3cTe^{2rT}(T+4)}{(1-k)(1-\sqrt{k})}+\frac{3ce^{2rT}((T+4)}{(1-k)(1-\sqrt{k})}\int_0^T \mathbb{E}(\sup_{0\le u\le T}(e^{ru}|x^n(u)|)^2)du. \tag{13}$$

Hence, by the Gronwall inequality:

$$\mathbb{E}(\sup_{0\le t\le T} e^{2rt}|x^n(t)|^2) \le [(\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2}+\frac{3e^{2rT}(1+cT(T+4))}{(1-k)(1-\sqrt{k})})\mathbb{E}\parallel \xi \parallel_r^2 + \frac{3cTe^{2rT}(T+4)}{(1-k)(1-\sqrt{k})}]\times \exp(\frac{3ce^{2rT}T(T+4)}{(1-k)(1-\sqrt{k})}). \tag{14}$$

Note that:

$$\begin{aligned}\mathbb{E}(\sup_{-\infty<t\le T}(e^{rt}|x^n(t)|)^2) &\le \mathbb{E}(\sup_{-\infty<t\le 0}(e^{rt}|x^n(t)|)^2)+\mathbb{E}(\sup_{0\le t\le T}(e^{rt}|x^n(t)|)^2)\\ &\le \mathbb{E}\parallel \xi \parallel_r^2 + [(\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2}+\frac{3e^{2rT}(1+cT(T+1))}{(1-k)(1-\sqrt{k})})\mathbb{E}\parallel \xi \parallel_r^2\\ &\quad + \frac{3cTe^{2rT}(T+1)}{(1-k)(1-\sqrt{k})}]\times \exp(\frac{3ce^{2rT}T(T+1)}{(1-k)(1-\sqrt{k})}).\end{aligned} \tag{15}$$

That’s means:

$$\mathbb{E}(\sup_{-\infty<t\le T}(e^{rt}|x(t\wedge\tau_n)|)^2) \le \mathbb{E}\parallel \xi \parallel_r^2 + [(\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2}+\frac{3e^{2rT}(1+cT(T+1))}{(1-k)(1-\sqrt{k})})\mathbb{E}\parallel \xi \parallel_r^2 + \frac{3cTe^{2rT}(T+1)}{(1-k)(1-\sqrt{k})}]\times \exp(\frac{3ce^{2rT}T(T+1)}{(1-k)(1-\sqrt{k})}). \tag{16}$$

If $n \to \infty$ then (16) implies the following inequality and the proof is complete:

$$\mathbb{E}(\sup_{-\infty<t\le T}(e^{rt}|x(t)|)^2) \le \mathbb{E}\parallel \xi \parallel_r^2 + [(\frac{\sqrt{k}+k(e^{2rt}-1)}{(1-\sqrt{k})^2}+\frac{3e^{2rT}(1+cT(T+1))}{(1-k)(1-\sqrt{k})})\mathbb{E}\parallel \xi \parallel_r^2 + \frac{3cTe^{2rT}(T+1)}{(1-k)(1-\sqrt{k})}]. \tag{17}$$

To prove the Theorem 3.5 we utilize the standard argument, for instance, see [14, 3], which is the truncation procedure, so we preclude it. For accomplishment, we present the accompanying Lemma.

Lemma 3.4: For any n_0 sufficiently large and $n \ge n_0$ define the truncation functions b_n and σ_n as follows:

$$b_n(\phi) = \begin{cases} b(\phi) & for \quad \parallel \phi \parallel_r \le n,\\ b(\frac{n\phi}{\parallel\phi\parallel_r}) & for \quad \parallel \phi \parallel_r > n;\end{cases} \tag{18}$$

$$\sigma_n(\phi) = \begin{cases} \sigma(\phi) & for \quad \parallel \phi \parallel_r \le n,\\ \sigma(\frac{n\phi}{\parallel\phi\parallel_r}) & for \quad \parallel \phi \parallel_r > n.\end{cases} \tag{19}$$

Then b_n and σ_n satisfy the global Lipschitz and the linear growth conditions.

Proof: By (18) , $b_n(\phi) = b(\frac{\parallel\phi\parallel_r\wedge n}{\parallel\phi\parallel_r}\phi)$ then by the assumption $(H1)$ for any $\parallel \phi \parallel_r$ and $\parallel \varphi \parallel_r$ belongs to C_r we have four cases:

Case1: If $\parallel \phi \parallel_r, \parallel \varphi \parallel_r \le n$:$|b_n(\phi)-b_n(\varphi)| = |b(\phi)-b(\varphi)| = \parallel \phi-\varphi \parallel_r \le k_n \parallel \varphi-\phi \parallel_r$.

Case 2: Now let $\| \phi \|_r, \| \varphi \|_r > n$:

$$\begin{aligned}|b_n(\phi)-b_n(\varphi)| &= |b(\tfrac{n\phi}{\|\phi\|_r})-b(\tfrac{n\varphi}{\|\varphi\|_r})| \le k_n \| \tfrac{n\phi}{\|\phi\|_r}-\tfrac{n\varphi}{\|\varphi\|_r}\|_r = nk_n \| \tfrac{\phi}{\|\phi\|_r}-\tfrac{\varphi}{\|\varphi\|_r}\|_r\\ &= nk_n \| \tfrac{\phi\|\varphi\|_r-\phi\|\phi\|_r+\phi\|\phi\|_r-\varphi\|\phi\|_r}{\|\phi\|_r\|\varphi\|_r}\|_r \le k_n \| \tfrac{\phi(\|\varphi\|_r-\|\phi\|_r)+\|\phi\|_r(\phi-\varphi)}{\|\phi\|_r}\|_r\\ &\le 2k_n \tfrac{\|\phi\|_r\|\|\varphi\|_r-\|\phi\|_r\|_r+\|\phi\|_r\|\phi-\varphi\|_r}{\|\phi\|_r} \le 2k_n \|\phi-\varphi\|_r.\end{aligned}$$

Case 3: Suppose that $\| \phi \|_r > n, \| \varphi \|_r \le n$:

$$\begin{aligned}|b_n(\phi)-b_n(\varphi)| &= |b(\tfrac{n\phi}{\|\phi\|})-b(\varphi)| \le k_n \| \tfrac{n\phi}{\|\phi\|_r}-\varphi\|_r = k_n \tfrac{\|n\phi-\phi\|\phi\|_r+\phi\|\phi\|_r-\varphi\|\phi\|_r\|_r}{\|\phi\|_r}\\ &\le 2k_n(\| n-\|\phi\|_r\|_r + \|\phi-\varphi\|_r) \le 2k_n(\|\|\varphi\|_r - \|\phi\|_r\|_r + \|\phi-\varphi\|_r)\\ &\le 2k_n \|\phi-\varphi\|_r.\end{aligned}$$

Case 4: Suppose that $\| \phi \|_r \le n, \| \varphi \|_r > n$:

$$\begin{aligned}|b_n(\phi)-b_n(\varphi)| &= |b(\phi)-b(\tfrac{\varphi}{\|\varphi\|_r})| \le k_n \| \phi-\tfrac{n\varphi}{\|\varphi\|_r}\|_r = k_n \tfrac{\|\phi\|\varphi\|_r-\varphi\|\varphi\|_r+\varphi\|\varphi\|_r-n\varphi\|_r}{\|\varphi\|_r}\\ &\le 2k_n(\|\phi-\varphi\|_r + \|\|\varphi\| - n\|_r) \le 2k_n(\|\phi-\varphi\|_r + \|\|\varphi\| - \|\phi\|_r\|_r)\\ &\le 2k_n \|\phi-\varphi\|_r.\end{aligned}$$

Also we note that by the assumption $(H3)$:

$$|b_n(x)|^2 = |b(\tfrac{|x|\wedge n}{|x|}x)|^2 \le c(1+|\tfrac{|x|\wedge n}{|x|}x|^2) = c(1+(\tfrac{|x|\wedge n}{|x|})^2|x|^2).$$

Sense $\frac{|x|\wedge n}{|x|} \le 1$ thus $|b_n(x)|^2 \le c(1+|x|^2)$. The proof is therefore complete.

We note that, for any initial data $\xi \in C_r$, by $(H1)$, coefficients b_n and σ_n satisfy the uniform Lipschitz condition, which reveals the linear growth condition.

Theorem 3.5: Assume that (H1), (H2) and (H3) hold then there exists a unique Maximal Local Strong solution $x(t)$ to equation (2) with the initial data (3) in $M^2((-\infty, T]; R^d)$.

Proof: For each $n \ge 1$ define truncation functions b_n and σ_n as in the equations (18) and (19), respectively, then by the Lemma 3.4 they satisfy the uniform Lipschitz and the linear growth conditions. So that, for any initial data $\xi \in C_r$, $t \ge 0$ the NSFDEwID equation,

$$x^n(t) = D(x_t^n) - D(\xi) + \xi(0) + \int_0^t b_n(x_s^n)ds + \int_0^t \sigma_n(x_s^n)dw(s), \tag{20}$$

has a unique solution $x^n(t) \in M^2((-\infty, T]; R^d)$.

Define the stopping time $\tau_a = T \wedge inf\{t \ge 0: \| x_t^n \|_r \ge n\}$. We can show that

$x^n(t) = x^{n+1}(t)$ if $0 \le t \le \tau_n$. Taking the expectation, and by Holder inequality, it deduces that

$$\begin{aligned}\mathbb{E}|x^{n+1}(t) \quad & -x^n(t)|^2 = \mathbb{E}|D(x_t^{n+1}) - D(x_t^n) + \int_0^t [b_{n+1}(x_s^{n+1}) - b_n(x_s^n)]ds\\ &+ \int_0^t [\sigma_{n+1}(x_s^{n+1}) - \sigma_n(x_s^n)]dw(s)|^2\\ &\le k\mathbb{E}\| x_t^{n+1} - x_t^n \|_r^2 + \tfrac{2}{1-k}\mathbb{E}|\int_0^t [b_{n+1}(x_s^{n+1}) - b_n(x_s^n)]ds|^2\\ &+ \tfrac{2}{1-k}\mathbb{E}|\int_0^t [\sigma_{n+1}(x_s^{n+1}) - \sigma_n(x_s^n)]dw(s)|^2\\ &\le k\mathbb{E}\| x_t^{n+1} - x_t^n \|_r^2 + \tfrac{4t}{1-k}\mathbb{E}\int_0^t [|b_{n+1}(x_s^{n+1}) - b_{n+1}(x_s^n)|^2 + |b_{n+1}(x_s^n) - b_n(x_s^n)|^2]ds\\ &+ \tfrac{4}{1-k}\mathbb{E}\int_0^t [|\sigma_{n+1}(x_s^{n+1}) - \sigma_{n+1}(x_s^n)|^2 + |\sigma_{n+1}(x_s^n) - \sigma_n(x_s^n)|^2]ds.\end{aligned} \tag{21}$$

For $0 \le t \le \tau_n$, we have known that: $b_{n+1}(x_s^n) = b_n(x_s^n) = b(x_s^n)$ and

$\sigma_{n+1}(x_s^n) = \sigma_n(x_s^n) = \sigma(x_s^n)$ again by $x^{n+1}(s) = x^n(s) = \xi(s), s \in (-\infty, 0)$. We get that:

$$\begin{aligned}\mathbb{E}(\sup_{0<s\le t}|x^{n+1}(t) \quad -x^n(t)|^2) &\le ke^{-2rt}\mathbb{E}(\sup_{0<s\le t}|x^{n+1}(s) - x^n(s)|^2)\\ &+\frac{4k_n(T+1)e^{-2rt}}{1-k}\int_0^T \mathbb{E}(\sup_{0<s\le t}|x^{n+1}(s) - x^n(s)|^2)ds,\end{aligned} \tag{22}$$

thus,

$$\mathbb{E}(\sup_{0<s\le t}|x^{n+1}(s) - x^n(s)|^2) \le \frac{4k_n(T+1)e^{-2rt}}{(1-ke^{-2rt})(1-k)}\int_0^T \mathbb{E}(\sup_{0<s\le t}|x^{n+1}(s) - x^n(s)|^2)ds. \tag{23}$$

From Grownwall inequality, one can see that

$$\mathbb{E}(\sup_{0<s\le t}|x^{n+1}(s) - x^n(s)|^2) = 0, \tag{24}$$

this means, for all $0 \le t \le \tau_n$, we always have

$$x^{n+1}(t) = x^n(t). \tag{25}$$

It then deduces τ_n is increasing, that is as $n \to \infty, \tau_n \to T$ a.s. By $(H1)$, for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$ such that $\tau_n = T$ as $n \ge n_0$. Now define $x(t)$ by

$$x(t) = x_{n_0}(t), t \in [0, T]. \tag{26}$$

By (25), $x(t \wedge \tau_n) = x^n(t \wedge \tau_n)$, and it therefore follows from (20) that

$$\begin{aligned}x(t \wedge \tau_n) \quad &= D(x_{t\wedge\tau_n}) - D(\xi) + \xi(0) + \int_0^{t\wedge\tau_n} b_n(x_s)ds + \int_0^{t\wedge\tau_n} \sigma_n(x_s)dw(s)\\ &= D(x_{t\wedge\tau_n}) - D(\xi) + \xi(0) + \int_0^{t\wedge\tau_n} b(x_s)ds + \int_0^{t\wedge\tau_n} \sigma(x_s)dw(s).\end{aligned} \tag{27}$$

Letting $n \to \infty$ we see that $x(t)$ is a solution of equation (2), that is

$$x(t \wedge T) = D(x_{t\wedge T}) - D(\xi) + \xi(0) + \int_0^{t\wedge T} b(x_s)ds + \int_0^{t\wedge T} \sigma(x_s)dw(s), \tag{28}$$

consequently,

$$x(t) = D(x_t) - D(\xi) + \xi(0) + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dw(s), \tag{29}$$

which, by Lemma 3.3, belongs to $M^2((-\infty, T]; R^d)$.

The proof of existence is complete. The uniqueness can be proved via a stopping procedure. This completes the proof.

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