



# ***A New Hermite Homotopy Perturbation Technique for Solving Fuzzy Integral Equations***

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## ARTICLE INFO

### Article history:

Received: 16 /06/ 2019

Rrevised form: 16 /07/ 2019

Accepted : 18 /08/ 2019

Available online: 24 /09/2019

### Keywords:

Hermite polynomial;Modified homotopy perturbation method;Fuzzy Fredholm integral equation; Fuzzy Volterra integral equation.

## ABSTRACT

The proposed new technique in this article is based on modification Homotopy Perturbation method (HPM) using Hermite polynomials for approximate solution to fuzzy integral equations of the second kind for both Volterra and Fredholm types. Our method produces accurate results which are illustrated throughout some examples.

MSC.

DOI : 10.29304/jqcm.2019.11.4.620

## 1 . Introduction

The concept of fuzzy sets and fuzzy functions were firstly studied by [6,25]. Then the definition of fuzzy numbers and its applications in approximate problems and control theory were presented in [26,5]. One of the most important applications of fuzzy seta is fuzzy integral equations which play roles in variant areas [3,4,17]. In recent years, numerical and approximate methods have been suggested for solving fuzzy integral equations. For examples, In [11], an analytical and numerical methods are used to solve fuzzy Volterra integral equations while the Taylor expansion and the variational iteration methods were applied to second kind fuzzy linear Volterra integral equation[12]. In [1], the fuzzy Gauss quadrature formula was utilized to solve fuzzy integral equation approximately. Furthermore, a new method was proposed by [10] which is based on artificial neural networks for approximate results to Fredholm fuzzy integral equations of the second kind. In [7] , hybrid of block-pulse functions

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Communicated by Qusuay Hatim Egaar

and Taylor series were employed to solve linear Fredholm fuzzy integral equations of the second kind. Other works for approximate solutions to fuzzy integral equations can be found in [16,19,22] .This research is an attempt to propose the developments of the homotopy perturbation method [13,18,21,23,27] with Hermite polynomials by the use of an accelerating parameter for the approximate solution to both second kind fuzzy Fredholm and Volterra integral equation .

The structure of this paper is organized as follows: in section 2, basic definitions and notions of fuzzy numbers and fuzzy continuous function are listed. Section 3 gives the definition of Hermite polynomials and some of their important properties while fuzzy integral equation is introduced in section 4. Section 5 provides an approximate solution with Homotopy perturbation technique for fuzzy integral equation. The description of the proposed new method for solving both Fredholm and Volterra fuzzy integral equation are described in section 6. Illustrated results with some examples and the conclusion are introduced in section 7 and 8 respectively.

**2. Preliminaries and notations**

Some necessary definitions and mathematical preliminaries are given in the present section which are farther used in the next sections.

**Definition2.1[15]** :A fuzzy number  $\tilde{u}$  is a fuzzy interval with the following properties:

- 1- $\tilde{u}$  is normal, i.e., $\exists x_0 \in R$  such that  $U(x_0) = 1$ .
- 2-  $\tilde{u}$  is convex, i.e,  $\forall a, b \in R$  and  $\delta \in [0,1]$ ,  $\tilde{u}(\delta a + (1 - \delta)b) \geq \min\{\tilde{u}(a), \tilde{u}(b)\}$ .
- 3-  $\tilde{u}$  is upper semi-continuous on  $R$ .
- 4-  $\text{supp } \tilde{u} = \{U(y) > 0 \mid y \in R\}$  is the support of the  $\tilde{u}$ .

The set of all fuzzy numbers is denoted by  $E^1$  . The  $r$ -cut of a fuzzy interval  $\tilde{u} \in E^1$  with  $r \in (0,1]$  is :

$$[\tilde{u}]_r = \begin{cases} \{\tilde{u}(y) \geq 0 \mid y \in R\} & \text{if } r \in (0,1] \\ \text{cl}(\text{supp}\tilde{u}) & \text{if } r = 1 \end{cases}$$

**Definition 2.2 [2]:** A is a pair  $(\underline{u}, \bar{u})$  of functions  $\underline{u}(r), \bar{u}(r)$  represent fuzzy number  $\tilde{u}$  where  $r \in [0,1]$  and the following points are satisfied

- a)  $\underline{u}(r)$ , is increasing left continuous and bounded monotonic function,
- b)  $\bar{u}(r)$  is decreasing left continuous bounded monotonic function,
- c)  $r \in [0,1]$  ,  $\bar{u}(r) \geq \underline{u}(r)$  .

we define addition, subtraction, scalar product by  $k$  respectively for fuzzy number  $\tilde{u} = (\underline{u}(r), \bar{u}(r))$  and  $\tilde{v} = (\underline{v}(r), \bar{v}(r))$  ,  $r \in [0,1]$  , and scalar  $k$  as follow:

- addition:  $(\underline{u} + \underline{v})(r) = \underline{u}(r) + \underline{v}(r)$  ,  $(\overline{u + v})(r) = \bar{u}(r) + \bar{v}(r)$  .
- subtraction:  $(\underline{u} - \underline{v})(r) = \underline{u}(r) - \underline{v}(r)$  ,  $(\overline{u - v})(r) = \bar{u}(r) - \bar{v}(r)$

• scalar product:  $ku(r) = \begin{cases} (k\underline{u}(r), k\bar{u}(r)) & k \geq 0 \\ (k\bar{u}(r), k\underline{u}(r)) & k < 0 \end{cases}$

**Definition 2.3 [20]:** let  $\tilde{u} = (\underline{u}(r), \bar{u}(r))$  and  $\tilde{v} = (\underline{v}(r), \bar{v}(r))$  be a fuzzy number .The distance between  $\tilde{u}$  and  $\tilde{v}$  is

$$D(\tilde{u}, \tilde{v}) = \max\{\sup_{0 \leq r \leq 1} |\underline{u}(r) - \underline{v}(r)|, \sup_{0 \leq r \leq 1} |\bar{u}(r) - \bar{v}(r)| \}$$

and satisfy following properties

- (i)  $D(\tilde{u} + \tilde{z}, \tilde{v} + \tilde{z}) = D(\tilde{u}, \tilde{v})$ ,  $\forall \tilde{u}, \tilde{v}, \tilde{z} \in E^1$ ,
- (ii)  $D(k\tilde{u}, k\tilde{v}) = |k|D(\tilde{u}, \tilde{v})$ ,  $\forall k \in R, \tilde{u}, \tilde{v} \in E^1$ ,
- (iii)  $D(\tilde{u} + \tilde{v}, \tilde{z} + \tilde{e}) \leq D(\tilde{u}, \tilde{z}) + D(\tilde{v}, \tilde{e})$ ,  $\forall \tilde{u}, \tilde{v}, \tilde{z}, \tilde{e} \in E^1$

(iv)  $(D, E^1)$  is a complete metric space.

**Definition2.4 [24]:** Let  $\check{f} : [a, b] \subseteq R \rightarrow E^1$  be a fuzzy valued function.  $\check{f}$  is called fuzzy continuous in  $t_0 \in [a, b]$  if there is  $\sigma > 0$  for each  $\mu > 0$  such that  $D(\check{f}(t), \check{f}(t_0)) < \mu$  whenever  $t \in [a, b]$  and  $|t - t_0| < \sigma$ .

### 3.Hermite Polynomials [14]

Let  $t \in (-\infty, +\infty)$ , then Hermite polynomials  $H_n(x)$  have the explicit form given by :

$$H_n(x) = n! \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i (2x)^{n-2i}}{i! (n-2i)!}$$

and they are satisfied the three terms recurrence formula

$$H_{n+1}(x) = 2xH_n(x) - 2n H_{n-1}(x), \text{ for } n=2,3,\dots$$

where  $H_0(x) = 1, H_1(x) = 2x$

Some of Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

### 4.Fuzzy Integral Equations

There are three main types of fuzzy integral equations. In this section, the integral equations which are discussed are Fredholm and Volterra integral equations. Second kind Fredholm integral equation is given by [8]

$$\gamma(x) = g(x) + \lambda \int_a^b k(x, t) \gamma(t) dt \quad (1)$$

Where  $\lambda$  is a positive parameter and  $k(x, t)$  is represent kernel function for  $a \leq x, t \leq b$  and  $g(x)$  is a function of  $x : a \leq x \leq b$ .

Now with respect to definition (2.2), we introduce parametric form of a Fuzzy Fredholm Integral Equation of the second kind (FFIE-2). Let  $(\underline{g}(x, r), \bar{g}(x, r))$  and  $(\underline{\gamma}(x, r), \bar{\gamma}(x, r))$ ,  $r \in [0, 1]$  and  $x \in [a, b]$  are parametric form of  $g(x)$  and  $\gamma(x)$  then, parametric form of second kind fuzzy Fredholm integral equation is

$$\underline{\gamma}(x, r) = \underline{g}(x, r) + \lambda \int_a^b \underline{U}(t, r) dt \quad (2)$$

$$\bar{\gamma}(x, r) = \bar{g}(x, r) + \lambda \int_a^b \bar{U}(t, r) dt$$

$$\underline{U}(t, r) = \begin{cases} k(x, t) \underline{\gamma}(t, r), & k(x, t) \geq 0 \\ k(x, t) \bar{\gamma}(t, r), & k(x, t) < 0 \end{cases}$$

and

$$\bar{U}(t,r) = \begin{cases} k(x,t)\bar{\gamma}(t,r), & k(x,t) \geq 0 \\ k(x,t)\underline{\gamma}(t,r), & k(x,t) < 0 \end{cases}$$

for each  $t \geq 0$ ,  $b \geq x$  and  $r \in [0,1]$ .

## 5.Homotopy Perturbation Method for Solving Fuzzy Integral Equations

Consider the nonlinear Volterra fuzzy integral equation as follows

$$\underline{\gamma}(x,r) = \underline{g}(x,r) + \int_a^x k(x,t)[\underline{\gamma}(t,r)]^q dt \quad (3)$$

$$\bar{\gamma}(x,r) = \bar{g}(x,r) + \int_a^x k(x,t)[\bar{\gamma}(t,r)]^q dt$$

Rewrite eq.(3) as :

$$L(\underline{u}) = \underline{u}(x,r) - \underline{g}(x,r) - \int_a^x k(x,t)[\underline{u}(t,r)]^q dt = 0. \quad (4)$$

$$L(\bar{u}) = \bar{u}(x,r) - \bar{g}(x,r) - \int_a^x k(x,t)[\bar{u}(t,r)]^q dt = 0$$

with solution  $\underline{u}(x,r) = \underline{\gamma}(x,r)$ ,  $\bar{u}(x,r) = \bar{\gamma}(x,r)$

Homotopy  $H(\underline{u}, p)$ ,  $H(\bar{u}, p)$  define as follows:

$$\begin{cases} H(\underline{u}, 0) = F(\underline{u}), & H(\underline{u}, 1) = L(\underline{u}) \\ H(\bar{u}, 0) = F(\bar{u}), & H(\bar{u}, 1) = L(\bar{u}) \end{cases} \quad (5)$$

Where  $F(\underline{u})$ ,  $F(\bar{u})$  are functional operators with solutions, say  $\underline{u}_0$ ,  $\bar{u}_0$ . We choose a convex homotopy

$$\begin{cases} H(\underline{u}, p) = (1-p)F(\underline{u}) + pL(\underline{u}) = 0 \\ H(\bar{u}, p) = (1-p)F(\bar{u}) + pL(\bar{u}) = 0 \end{cases} \quad (6)$$

where  $p \in (0,1]$

$$\underline{u} = \sum_{n=0}^{\infty} p^n \underline{u}_n \quad (7)$$

$$\bar{u} = \sum_{n=0}^{\infty} p^n \bar{u}_n$$

One can get the approximate solution when  $p \rightarrow 1$ ,

$$\underline{\gamma}(x,r) = \lim_{p \rightarrow 1} \underline{u} = \sum_{n=0}^{\infty} \underline{u}_n \quad (8)$$

$$\bar{\gamma}(x,r) = \lim_{p \rightarrow 1} \bar{u} = \sum_{n=0}^{\infty} \bar{u}_n$$

Taking  $F(\underline{u}) = \underline{u}(x, r) - \underline{g}(x, r)$  and  $F(\bar{u}) = \bar{u}(x, r) - \bar{g}(x, r)$ , with the aid of eqns.(6-7) and after equating powers of  $p$ , yields

$$\begin{cases} \underline{u}_0(x, r) = \underline{g}(x, r) \\ \bar{u}_0(x, r) = \bar{g}(x, r) \end{cases}$$

$$\underline{u}_{n+1}(x, r) = \int_a^x k(x, t)H_{n1}(t, r)dt$$

$$\bar{u}_{n+1}(x, r) = \int_a^x k(x, t)H_{n2}(t, r)dt$$

where the  $H_n$ 's are the so-called He's polynomials [9] which can be calculated by using the formula

$$H_{n1}(t, r) = H_n(\underline{u}_0, \underline{u}_1, \underline{u}_2, \dots) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( \left( \sum_{k=0}^n p^k \underline{u}_k \right)^n \right)_{p=0}$$

$$H_{n2}(t, r) = H_n(\bar{u}_0, \bar{u}_1, \bar{u}_2, \dots) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( \left( \sum_{k=0}^n p^k \bar{u}_k \right)^n \right)_{p=0}$$

In the same way the homotopy method can be explained for the fuzzy Fredholm integral equation.

### 6. Description of the New Method

The main goal of this section is to present a modification of MHPM based on Hermit polynomial and a parameter depended on the HPM. To realize our goal, we first reconfigure Eq. (3) as follows:

$$\underline{\gamma}(x, r) = \sum_{m=0}^N \beta_m \underline{\omega}_m(x, r) - \sum_{m=0}^N \beta_m \underline{\omega}_m(x, r) + \underline{g}(x, r) + \int_a^b k(x, t)[\underline{\gamma}(t, r)]^q dt \tag{9}$$

$$\bar{\gamma}(x, r) = \sum_{m=0}^N \beta_m \bar{\omega}_m(x, r) - \sum_{m=0}^N \beta_m \bar{\omega}_m(x, r) + \bar{g}(x, r) + \int_a^b k(x, t)[\bar{\gamma}(t, r)]^q dt$$

By using the HPM, we let

$$\begin{cases} F(\underline{u}) = \underline{u}(x, r) - \sum_{m=0}^N \beta_m \underline{\omega}_m(x, r) \\ F(\bar{u}) = \bar{u}(x, r) - \sum_{m=0}^N \beta_m \bar{\omega}_m(x, r) \end{cases}$$

So a new convex homotopy perturbation can be define as follow:

$$H_\beta(\underline{u}, p) = \underline{u}(x, r) - p \underline{g}(x, r) + (p - 1) \left[ \sum_{m=0}^N \beta_m \underline{\omega}_m(x, r) \right] - p \int_a^b k(x, t)[\underline{u}(t, r)]^q dt = 0 \tag{10}$$

$$H_\beta(\bar{u}, p) = \bar{u}(x, r) - p \bar{g}(x, r) + (p - 1) \left[ \sum_{m=0}^N \beta_m \bar{\omega}_m(x, r) \right] - p \int_a^b k(x, t)[\bar{u}(t, r)]^q dt = 0 \tag{11}$$

where  $\beta = [\beta_m]$  and  $\beta_m, m = 0, 1, 2, \dots, N$  are said to be accelerating components of the parameter, and

$$\begin{cases} \underline{\omega}(x, r) = [\underline{\omega}_m(x, r)] \\ \bar{\omega}(x, r) = [\bar{\omega}_m(x, r)] \end{cases} \quad m = 0, 1, 2, \dots, N$$

are selective functions taken from Hermite polynomials .

### 7. Numerical Examples

In this section, three examples were considered to explain the new method of solving fuzzy integral equations.

**Example 1:** Consider Fredholm fuzzy integral equation with

$$\underline{g}(x, r) = 3(r^2 - 2)(9x^2 - 10) + x^3(r^5 + 2r) - r(r^4 + 2)(3x^2 + 2)$$

$$\overline{g}(x, r) = 3(r^2 - 2)(3x^2 + 2) - x^3(3r^2 - 6) - r(r^4 + 2)(9x^2 - 10)$$

and kernel  $k(x, t) = 3(2 - t^2 + x^2)$   $0 \leq t, x \leq 2$  and  $a = 0, b = 2$ .

The exact solution in this case is given by

$$\underline{\gamma}(x, r) = x^3(r^5 + 2r)$$

$$\overline{\gamma}(x, r) = x^3(6 - 3r^2)$$

We apply  $H_\beta(u, p)$  method for upper and lower case respectively to approximate the solutions. We will choose in this example,  $\underline{\omega}_1(x, r) = rH_1(x) = 2rx$  and  $\underline{\omega}_2(x, r) = rH_3(x) = r(8x^3 - 12x)$  for lower case.

The homotopy equation, Eq. (10), becomes

$$H_\beta(\underline{u}, p) = \underline{u}(x, r) - p r \underline{g}(x, r) + (p - 1) (2\beta_0 r x + \beta_1 r (8x^3 - 12x)) - p \int_0^2 3(2 - t^2 + x^2) [\underline{u}(t, r)] dt = 0 \quad (12)$$

with the aid of eqns.(7-12) and after equating powers of  $p$ , yields

$$p^0: \underline{u}_0(x, r) - (2\beta_0 r x + \beta_1 r (8x^3 - 12x)) = 0 \Rightarrow \underline{u}_0(x, r) = 2\beta_0 r x + 8\beta_1 r x^3 - 12\beta_1 r x$$

$$p^1: \underline{u}_1(x, r) - \underline{g}(x, r) + (2\beta_0 r x + \beta_1 r (8x^3 - 12x)) - \int_0^2 3(2 - t^2 + x^2) \underline{u}_0(t, r) dt = 0$$

$$\Rightarrow \underline{u}_1(x, r) = (60 - 30r^2 - 2r^5 - 4r - 64r\beta_1) + (12\beta_1 r - 2\beta_0 r)x + (27r^2 - 54 - 3r^5 - 6r + 12\beta_0 r + 24 r\beta_1)x^2 + (r^5 + 2r - 8\beta_1 r)x^3$$

$$p^2: \underline{u}_2(x, r) - \int_0^2 3(2 - t^2 + x^2) \underline{u}_1(t, r) dt = 0$$

$$\Rightarrow \underline{u}_2(x, r) = \int_0^2 3(2 - t^2 + x^2) \underline{u}_1(t, r) dt$$

and in general

$$\underline{u}_{n+1}(x, r) = \int_a^b H_{n1}(t, r) dt$$

So, to find  $\beta_0$  and  $\beta_1$  such that  $\underline{u}_1 = 0$ , we should have

$$\begin{cases} 60 - 30r^2 - 2r^5 - 4r - 64r\beta_1 = 0 \\ 12\beta_1 r - 2\beta_0 r = 0 \\ 27r^2 - 54 - 3r^5 - 6r + 12\beta_0 r + 24 r\beta_1 = 0 \\ r^5 + 2r - 8\beta_1 r = 0 \end{cases}$$

thus  $\beta_0 = 6\beta_1$  and  $\beta_1 = \frac{r^5 + 2r}{8r}$ .

Therefore, we obtain

$$\underline{\gamma}(x, r) = \underline{u}_0(x, r) = 2\beta_0 r x + 8\beta_1 r x^3 - 12\beta_1 r x = (r^5 + 2r)x^3$$

which is the same as the exact solutions for lower case.

Now, we choose  $\overline{\omega}_1(x, r) = r^2 H_1(x) = 2r^2 x$  and  $\overline{\omega}_2(x, r) = r^3 H_2(x) = r^3(8x^3 - 12x)$  for upper case.

Therefore, Eq. (11) can be written:

$$H_\beta(\overline{u}, p) = \overline{u}(x, r) - p \overline{g}(x, r) + (p - 1)(2\beta_0 r^2 x + \beta_1 r^3(8x^3 - 12x)) - p \int_0^2 3(2 - t^2 + x^2) [\overline{u}(t, r)] dt = 0$$

rearrange the above equation to obtain

$$p^0: \overline{u}_0(x, r) - (2\beta_0 r^2 x + \beta_1 r^3(8x^3 - 12x)) = 0$$

$$\Rightarrow \overline{u}_0(x, r) = (2\beta_0 r^2 x + \beta_1 r^3(8x^3 - 12x))$$

$$p^1: \overline{u}_1(x, r) - \overline{g}(x, r) + (2\beta_0 r^2 x + \beta_1 r^3(8x^3 - 12x)) - \int_0^2 3(2 - t^2 + x^2) \overline{u}_0(t, r) dt = 0$$

$$\bar{u}_1(x, r) = (6r^2 - 12 + 10r^5 + 20r + 64\beta_1 r^3) + (12\beta_1 r^3 - 2\beta_0 r^2)x + (9r^2 - 18 - 9r^5 - 18r + 12\beta_0 r^2 + 24\beta_1 r^3)x^2 + (6 - 3r^2 - 8\beta_1 r^3)x^3$$

$$p^2: \bar{u}_2(x, r) - \int_0^2 3(2 - t^2 + x^2) \bar{u}_1(t, r)dt = 0$$

$$\bar{u}_2 = \int_0^2 3(2 - t^2 + x^2) \bar{u}_1(t, r)dt$$

and in general

$$\bar{u}_{n+1}(x, r) = \int_a^b H_{n2}(t, r)dt$$

Now we find  $\beta_0$  and  $\beta_1$  in such a way that  $\bar{u}_1 = 0$ , if  $\bar{u}_1 = 0$  then  $\bar{u}_2 = \bar{u}_3 = \dots = 0$ , and the exact solution obtained by  $\bar{y}(x, r) = \bar{u}_0(x, r)$ ; therefore for each values of x we have

$$\begin{cases} 6r^2 - 12 + 10r^5 + 20r + 64\beta_1 r^3 = 0 \\ 12\beta_1 r^3 - 2\beta_0 r^2 = 0 \\ 9r^2 - 18 - 9r^5 - 18r + 12\beta_0 r^2 + 24\beta_1 r^3 = 0 \\ 6 - 3r^2 - 8\beta_1 r^3 = 0 \end{cases}$$

then  $\beta_0 = 6r\beta_1$  and  $\beta_1 = \frac{6-3r^2}{8r^3}$ .

Thus, the solution would be as follows:

$$\bar{y}(x, r) = \bar{u}_0(x, r) = (2\beta_0 r^2 x + \beta_1 r^3(8x^3 - 12x)) = (6 - 3r^2)x^3.$$

**Example 2:** Consider Fredholm fuzzy integral equation with

$$g(x, r) = r(\frac{1}{2} x - \frac{1}{3})$$

$$\bar{g}(x, r) = (2 - r)(\frac{1}{2} x - \frac{1}{3})$$

and kernel  $k(x, t) = x + t$   $0 \leq x, t \leq 1$  and  $a = 0, b = 1$ . The exact solution in this case is given by

$$\begin{aligned} \underline{y}(x, r) &= rx \\ \bar{y}(x, r) &= (2 - r)x \end{aligned}$$

The approximate solution can be obtained by the method  $H_\beta(\underline{u}, p)$  method with  $\underline{\omega}(x, r) = rH_1(x) = 2rx$  for lower case.

Therefore, Eq. (10) can be written in the following form:

$$H_\beta(\underline{u}, p) = \underline{u}(x, r) - p r g(x, r) + (p - 1)(2\beta r x) - p \int_0^1 (x + t) [\underline{u}(t, r)] dt = 0 \tag{13}$$

By the same steps as listed in example 1, one can get

$$p^0: \underline{u}_0(x, r) - 2\beta r x = 0 \Rightarrow \underline{u}_0(x, r) = 2\beta r x$$

$$p^1: \underline{u}_1(x, r) - r(\frac{1}{2} x - \frac{1}{3}) + 2\beta r x - \int_0^1 (x + t) \underline{u}_0(t, r) dt = 0$$

$$\Rightarrow \underline{u}_1(x, r) = (\frac{2}{3}\beta r - \frac{1}{3}r) + (\frac{1}{2}r - \beta r)x$$

$$p^2: \underline{u}_2(x, r) - \int_0^1 (x + t) \underline{u}_1(t, r) dt = 0 \Rightarrow \underline{u}_2(x, r) = \int_0^1 (x + t) \underline{u}_1(t, r) dt$$

and in general

$$\underline{u}_{n+1}(x, r) = \int_a^b H_{n1}(t, r)dt$$

Now we find  $\beta$  in such a way that  $\underline{u}_1 = 0$ , if  $\underline{u}_1 = 0$  then  $\underline{u}_2 = \underline{u}_3 = \dots = 0$ . Thus for each values of x we should have

$$\begin{cases} \frac{2}{3}\beta r - \frac{1}{3}r = 0 \\ \frac{1}{2}r - \beta r = 0 \end{cases}$$

thus  $\beta = \frac{1}{2}$

So we have

$\underline{\gamma}(x, r) = \underline{u}_0(x, r) = 2\beta r x = r x$  , which is the same as the exact solutions for lower case.

Now , we choose  $\bar{\omega}(x, r) = r^2 H_1(x) = 2r^2 x$  for upper case.

Therefore, Eq. (11) can be written as follows:

$$H_\beta(\bar{u}, p) = \bar{u}(x, r) - p\bar{g}(x, r) + (p - 1)2\beta r^2 x - p \int_0^1 (x + t)[\bar{u}(t, r)]dt = 0 \quad (14)$$

with the aid of eqns.(7-14) and after equating powers of p , yields

$$p^0: \bar{u}_0(x, r) - 2\beta r^2 x = 0 \Rightarrow \bar{u}_0(x, r) = 2\beta r^2$$

$$p^1: \bar{u}_1(x, r) - (2 - r) \left( \frac{1}{2} x - \frac{1}{3} \right) + 2\beta r^2 x - \int_0^1 (x + t) \bar{u}_0(t, r) dt = 0$$

$$\Rightarrow \bar{u}_1(x, r) = \left( \frac{1}{3} r - \frac{2}{3} + \frac{2}{3} \beta r^2 \right) + \left( 1 - \frac{1}{2} r - \beta r^2 \right) x$$

$$p^2: \bar{u}_2(x, r) - \int_0^1 (x + t) \bar{u}_1(t, r) dt = 0$$

$$\Rightarrow \bar{u}_2(x, r) = \int_0^1 (x + t) \bar{u}_1(t, r) dt$$

and in general

$$\bar{u}_{n+1}(x, r) = \int_a^b H_{n2}(t, r) dt$$

Now we find  $\beta$  in such a way that  $\bar{u}_1 = 0$ , if  $\bar{u}_1 = 0$  then  $\bar{u}_2 = \bar{u}_3 = \dots = 0$ . Thus for each values of x we should have

$$\begin{cases} \frac{1}{3}r - \frac{2}{3} + \frac{2}{3}\beta r^2 = 0 \\ 1 - \frac{1}{2}r - \beta r^2 = 0 \end{cases}$$

thus  $\beta = \frac{1}{r^2} - \frac{1}{2r}$  and the solution is :

$\bar{\gamma}(x, r) = \bar{u}_0(x, r) = 2\beta r^2 x = 2 \left( \frac{1}{r^2} - \frac{1}{2r} \right) r^2 x = (2 - r)x$  , which is the same as the exact solutions for upper case.

**Example 3:** Consider Volterra fuzzy integral equation with

$$\underline{g}(x, r) = rx - x^2 \left( \frac{2}{3} rx^3 - \frac{4}{3} x^3 - \frac{1}{2} rx^2 + x^2 + \frac{1}{12} r - \frac{1}{12} \right)$$

$$\bar{g}(x, r) = (2 - r)x - x^2 \left( \frac{2}{3} rx^3 - \frac{1}{2} rx^2 + \frac{1}{12} r - \frac{1}{12} \right)$$

and kernel  $k(x, t) = x^2(1 - 2t)$   $0 \leq x, t \leq x$  and  $a = 0, b = 1$ . The exact solution in this case is given by

$$\begin{aligned} \underline{\gamma}(x, r) &= rx \\ \bar{\gamma}(x, r) &= (2 - r)x \end{aligned}$$

we choose,  $\underline{\omega}_1(x, r) = rH_0(x) = r$  and  $\underline{\omega}_2(x, r) = rH_1(x) = 2rx$  for lower case.

The homotopy equation, Eq. (10), becomes



$$H_{\beta}(\underline{u}, p) = \underline{u}(x, r) - p r \underline{g}(x, r) + (p - 1) (\beta_0 r + 2\beta_1 r x) - p \int_0^x x^2(1 - 2t)[\underline{u}(t, r)]dt = 0 \quad (15)$$

now we substitute (7) into (15) , we get

$$p^0: \underline{u}_0(x, r) - (\beta_0 r + 2\beta_1 r x) = 0 \Rightarrow \underline{u}_0(x, r) = \beta_0 r + 2\beta_1 r x$$

$$p^1: \underline{u}_1(x, r) - \underline{g}(x, r) + (\beta_0 r + 2\beta_1 r x) - \int_0^x x^2(1 - 2t)\underline{u}_0(t, r)dt = 0$$

$$\Rightarrow \underline{u}_1(x, r) = (-\beta_0 r) + (r - 2\beta_1 r)x + \left(\frac{1}{12} - \frac{1}{12}r\right)x^2 + (\beta_0 r)x^3 + \left(\frac{1}{2}r - 1 + \beta_1 r - \beta_0 r\right)x^4 + \left(\frac{4}{3} - \frac{2}{3}r - \frac{4}{3}\beta_1 r\right)x^5$$

$$p^2: \underline{u}_2(x, r) - \int_0^x x^2(1 - 2t) \underline{u}_1(t, r)dt = 0$$

$$\Rightarrow \underline{u}_2(x, r) = \int_0^x x^2(1 - 2t) \underline{u}_1(t, r)dt$$

and in general

$$\underline{u}_{n+1}(x, r) = \int_a^b H_{n1}(t, r)dt$$

So, to find  $\beta_0$  and  $\beta_1$  such that  $\underline{u}_1 = 0$  , we should have

$$\begin{cases} -\beta_0 r = 0 \\ r - 2\beta_1 r = 0 \\ \frac{1}{12} - \frac{1}{12}r = 0 \\ \beta_0 r = 0 \\ \frac{1}{2}r - 1 + \beta_1 r - \beta_0 r = 0 \\ \frac{4}{3} - \frac{2}{3}r - \frac{4}{3}\beta_1 r = 0 \end{cases}$$

thus  $\beta_0 = 0$  and  $\beta_1 = \frac{1}{2}$ .

Therefore, we obtain

$$\underline{\gamma}(x, r) = \underline{u}_0(x, r) = \beta_0 r + 2\beta_1 r x = r x$$

which is the same as the exact solutions for lower case.

Now , we choose  $\bar{\omega}_1(x, r) = r^2 H_0(x) = r^2$  and  $\bar{\omega}_2(x, r) = r^2 H_1(x) = 2r^2 x$  for upper case.

Therefore, Eq. (11) can be written in the following form:

$$H_{\beta}(\bar{u}, p) = \bar{u}(x, r) - p \bar{g}(x, r) + (p - 1)(\beta_0 r^2 + 2\beta_1 r^2 x) - p \int_0^x x^2(1 - 2t)[\bar{u}(t, r)]dt = 0$$

we have

$$p^0: \bar{u}_0(x, r) - (\beta_0 r^2 + 2\beta_1 r^2 x) = 0$$

$$\Rightarrow \bar{u}_0(x, r) = \beta_0 r^2 + 2\beta_1 r^2 x .$$

$$p^1: \bar{u}_1(x, r) - \bar{g}(x, r) + (\beta_0 r^2 + 2\beta_1 r^2 x) - \int_0^x x^2(1 - 2t) \bar{u}_0(t, r)dt = 0$$

$$\bar{u}_1(x, r) = (-\beta_0 r^2) + (2 - r - 2\beta_1 r^2)x + \left(\frac{1}{12} - \frac{1}{12}r\right)x^2 + (\beta_0 r^2)x^3 + \left(\frac{1}{2}r + \beta_1 r^2 - \beta_0 r^2\right)x^4 + \left(-\frac{2}{3}r - \frac{4}{3}\beta_1 r^2\right)x^5$$

$$p^2: \bar{u}_2(x, r) - \int_0^x x^2(1 - 2t) \bar{u}_1(t, r)dt = 0$$

$$\bar{u}_2 = \int_0^x x^2(1-2t) \bar{u}_1(t,r) dt$$

and in general

$$\bar{u}_{n+1}(x,r) = \int_a^b H_{n2}(t,r) dt$$

Now we find  $\beta_0$  and  $\beta_1$  in such a way that  $\bar{u}_1 = 0$ , if  $\bar{u}_1 = 0$  then  $\bar{u}_2 = \bar{u}_3 = \dots = 0$ , we should have

$$\begin{cases} -\beta_0 r = 0 \\ 2 - r - 2\beta_1 r^2 = 0 \\ \frac{1}{12} - \frac{1}{12} r = 0 \\ \beta_0 r = 0 \\ \frac{1}{2} r + \beta_1 r^2 - \beta_0 r^2 = 0 \\ -\frac{2}{3} r - \frac{4}{3} \beta_1 r^2 = 0 \end{cases}$$

then  $\beta_0 = 0$  and  $\beta_1 = \frac{2-r}{2r^2}$ .

Hence  $\bar{y}(x,r) = \bar{u}_0(x,r) = (\beta_0 r^2 + 2\beta_1 r^2 x) = 0 + 2r^2 \left(\frac{2-r}{2r^2}\right) x = (2-r)x$ , which is the same as the exact solutions for upper case.

## 8. Conclusions

New perturbation algorithm was proposed coupled with the homotopy approach with the aid of Hermite polynomials for solving both Volterra and Fredholm fuzzy integral equations. In this method, a new homotopy  $H_\beta(u,p)$  was erect where  $\beta = [\beta_m]$  represent a parameter depended on HMP. This parameter accorded fast convergent rate where only one iteration leads to exact solutions. This method realizes accurate results which are illustrated throughout some examples.

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