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Exact sequences of FW_6-second and third pair of hooks representation modules over a field of characteristic 3

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ABSTRACT

In this paper we will prove a short sequences of FW_6-second pair of hooks representation modules M^1_F , M^2_F corresponding to the partitions ((4,1),(1)),((1),(4,1)) and FW_6-third pair of hooks representation modules N^1_F , N^2_F corresponding to the partitions ((3,1,1),(1)),((1),(3,1,1)), are exact and split when F is a field of characteristic equal to 3.

MSC : 20C20

1- Introduction

The hook representation modules of the symmetric groups have been given first in 1971 by M. H. Peel in [1], and later in 1975 in [2]. In 1977 M.A. Eyad presents in [3] the analogues of some results in [1] and [2] for the Weyl groups of type B_n. In 1997 F.N. Jinan in[4] present the analogues of more result in the symmetric groups for the Weyl groups of type B_n.
The purpose of this paper is to prove the sequences of FW_6 – pair of hooks representation modules are exact and split. When F is a field of characteristic 3.

$$0 \longrightarrow U^1_F \xrightarrow{\ \theta\ } N^1_F \xrightarrow{\ \phi\ } M^1_F \longrightarrow 0$$

$$0 \longrightarrow U^2_F \xrightarrow{\ h\ } N^2_F \xrightarrow{\ \psi\ } M^2_F \longrightarrow 0$$

2- The Weyl group W_n

Let $x_1, \ldots, x_n$ be independent indeterminate over a field F of characteristic p where (p=0 or prime number not equal to 2). let W_n be the set of all bijection mapping w from the set $\{\pm x_1, \ldots, \pm x_n\}$ to itself such that $w(-x_i) = -w(x_i)$, i = 1,…,n . The pair (W_n , o) forms a group known as the Weyl groups [4]. The elements w of W_n is called a permutation and can be represented as

$$w = \begin{pmatrix} x_1 \ \ldots \ x_n & -x_1 \ \ldots \ -x_n \\ w(x_1) \ldots w(x_n) & -w(x_1) \ldots -w(x_n) \end{pmatrix}$$

The order W_n is denoted by $|W_n|$ which is equal to 2n! .

3-The group algebra FW_n

Let $F[x_1, \dots, x_n]$ be the set of all polynomials in $x_1, \dots, x_n$ with coefficients in the field F , any permutation $w \in W_n$ can be write as a one mapping from $F[x_1, \dots, x_n]$ onto $F[x_1, \dots, x_n]$ by defining $wf(x_1, \dots, x_n) = f(wx_1, \dots, wx_n)$ for each Polynomial $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$. That is w changes each variable x_i by the variable $w(x_i)$ in each polynomial $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$. Now since $cf(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ for each polynomial $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ and for each $c \in F$, then the multiplication of a permutation $w \in W_n$ by a scalar $c \in F$ is a function $cwf(x_1, \dots, x_n) = cf(wx_1, \dots, wx_n)$ for each $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$.
Let FW_n be the set of all c-linear combination of the permutations $w_i \in Wn$, i.e. $FW_n=\{\Sigma\, c_i w_i \,/\, c_i \in F\}$.

4- Example

Let Z_7 be the field of integers modulo 7 and let
$f(x_1, x_2, x_3) = x_1^3 x_2^2 + 4x_1 x_3^8 + x_1^2 x_2^2 x_3^2 \in Z_7[x_1, x_2, x_3]$, and let $w_1 = (x_1 \;\; -x_2)$ and $w_2 = (x_1 \;\; x_2 \;\; -x_3)$ which belong to W_3 and let $c_1 = 3$ and $c_2 = 6$ which belong to Z_7 . then

$$(c_1 w_1 + c_2 w_2)(f(x_1, x_2, x_3))$$

$$= 3(x_1 \;\; -x_2)(f(x_1, x_2, x_3)) + 6(x_1 \;\; x_2 \;\; -x_3)(f(x_1, x_2, x_3))$$

$$= 3(6x_1^2 x_2^3 + 3x_2 x_3^8 + x_1^2 x_2^2 x_3^2) + 6(x_2^3 x_3^2 + 4x_2 x_1^8 + x_1^2 x_2^2 x_3^2)$$

$$= 4x_1^2 x_2^3 + 2x_2 x_3^8 + 3x_1^2 x_2^2 x_3^2 + 6x_2^3 x_3^2 + 3x_2 x_1^8 + 6x_1^2 x_2^2 x_3^2$$

$$= 4x_1^2 x_2^3 + 2x_2 x_3^8 + 2x_1^2 x_2^2 x_3^2 + 6x_2^3 x_3^2 + 3x_2 x_1^8 \in Z_7[x_1, x_2, x_3]$$

Finaly it be useful to introduce the following denotations:-

- μ_r denotes to the monomial $x_i\, x_j\, x_k\, x_p\, x_q$, where $1 \le i < j < k < p < q \le 6$ and $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.
  In another word μ_r denotes to the monomial $x_1, x_2, x_3, x_4, x_5, x_6$ in which x_r is omit, where $r \in \{1, 2, 3, 4, 5, 6\}$.
- $\mu_{r,s}$ denote to the monomial μ_r in which x_s is of degree 3, where $s \in \{I, j, k, p, q\}$. note that $\mu_r x_s^2 = \mu_{r,s}$.
- $\mu_{r,s,t}$ denote to the monomial μ_r in which x_s , x_t are of degree 3 and 5 respectively, where $s, t \in \{i, j, k, p, q\}$. note that $\mu_r x_s^2 x_t^4 = \mu_{r,s,t}$.
- $\beta_{r,s}$, $\beta_{r,s,t}$ denoted to the monomials $X_r X_s^2$ and $X_r X_s^2 X_t^4$ respectively, where $s, t \in \{i, j, k, p, q\}$ and $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

5- Partitions and tableaux

The pairs of partitions of the number 6 are

((6) , ( ))	(( ) , (6))
((5,1) , ( ))	(( ) , (5,1))
((4,1,1) , ( ))	(( ) , (4,1,1))
((3,1,1,1) , ( ))	(( ) , (3,1,1,1))
((2,1,1,1,1) , ( ))	(( ) , (2,1,1,1,1))
((1,1,1,1,1,1) , ( ))	(( ) , (1,1,1,1,1,1))
((5) , (1))	((1) , (5))
((4,1) , (1))	((1) , (4,1))
((3,1,1) , (1))	((1) , (3,1,1))
((2,1,1,1) , (1))	((1) , (2,1,1,1))
((1,1,1,1,1) , (1))	((1) , (1,1,1,1,1))
((4) , (2))	((2) , (4))
((3,1) , (2))	((2) , (3,1))
((2,1,1) , (2))	((2) , (2,1,1))
((1,1,1,1) , (2))	((2) , (1,1,1,1))
((4) , (1,1))	((1,1) , (4))
((3,1) , (1,1))	((1,1) , (3,1))
((2,1,1) , (1,1))	((1,1) , (2,1,1))
((1,1,1,1) , (1,1))	((1,1) , (1,1,1,1))
((3) , (3))	((2,1) , (2,1))

((3) , (2))
((3) , (1,1,1))
((2,1) , (1,1,1))
((1,1,1) , (1,1,1))
((2,1) , (3))
((1,1,1) , (3))
((1,1,1) , (2,1))

• A row standard tableau is a tableau such that the variables
Occur in increasing order along the rows from left to right so there are exactly 30 row standard tableaux corresponds to the pair of hook partition ((4,1) , (1)).

$x_1 x_2 x_3 x_4$ x_6 x_5	$x_1 x_2 x_3 x_5$ x_6 x_4	$x_1 x_2 x_3 x_6$ x_4 x_5	$x_2 x_3 x_5 x_6$ x_4 x_1
$x_1 x_2 x_3 x_4$ x_5 x_6	$x_1 x_2 x_3 x_5$ x_4 x_6	$x_1 x_2 x_3 x_6$ x_5 x_4	$x_3 x_4 x_5 x_6$ x_2 x_1
$x_1 x_2 x_4 x_5$ x_3 x_6	$x_1 x_2 x_4 x_6$ x_5 x_3	$x_1 x_2 x_5 x_6$ x_4 x_3	$x_2 x_4 x_5 x_6$ x_3 x_1
$x_1 x_2 x_4 x_5$ x_6 x_3	$x_1 x_2 x_4 x_6$ x_3 x_5	$x_1 x_2 x_5 x_6$ x_3 x_4	$x_2 x_4 x_5 x_6$ x_1 x_3
$x_1 x_3 x_4 x_5$ x_6 x_2	$x_1 x_3 x_4 x_6$ x_5 x_2	$x_1 x_3 x_5 x_6$ x_4 x_2	$x_3 x_4 x_5 x_6$ x_1 x_2
$x_1 x_3 x_4 x_5$ x_2 x_6	$x_1 x_3 x_4 x_6$ x_2 x_5	$x_1 x_3 x_5 x_6$ x_2 x_4	$x_2 x_3 x_5 x_6$ x_1 x_4
$x_1 x_4 x_5 x_6$ x_3 x_2	$x_2 x_3 x_4 x_5$ x_6 x_1	$x_2 x_3 x_4 x_6$ x_5 x_1	x_1
$x_1 x_4 x_5 x_6$ x_2 x_3	$x_2 x_3 x_4 x_5$ x_1 x_6	$x_2 x_3 x_4 x_6$ x_1 x_5	

x_1 x_1 x_1

6- FW_6 – modules

We are interesting in the following FW_6 – modules which we are dealing with in this paper. (see [3] , [5])

i- M^1_F and M^2_F are the second pair of hooks representation modules corresponding to the pairs of partition ((4,1) , (1)) and ((1) , (4,1)) respectively. M^1_F is generated over FW_6 by $\mu_{6,5}$ and consists of all polynomials in $x_1 , x_2 , x_3 , x_4 , x_5 , x_6$ of the form:

$$\sum_{1 \leq i < j < k < p < q \leq 6} \mu_r \sum_{s=i}^{q} c_{i,j,k,p,q,s}\, x_s^2 \text{ , where } c_{i,j,k,p,q,s} \in F$$

M^2_F is generated over FW_6 by $\beta_{6,5}$ and consists of all polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$\sum_{1\le i<j<k<p<q\le 6} x_r \sum_{s=i}^{q} c_{i,j,k,p,q,s}\, x_s^2 \text{ , where } c_{i,j,k,p,q,s} \in F$$

and $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

ii- N^1_F and N^2_F are the third pair of hooks representation modules corresponding to the pairs of partition ((3,1,1) , (1)) and ((1) , (3,1,1)) respectively.

N^1_F is generated over FW_6 by $\mu_{6,4,5}$ and consists of all polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$\sum_{1\le i<j<k<p<q\le 6} \mu_r \sum_{s=i}^{q} \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s}\, x_s^2 x_t^4 \text{ , where } c_{i,j,k,p,q,s,t} \in F$$

N^2_F is generated over FW_6 by $\beta_{6,4,5}$ and consists of all polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$\sum_{1\le i<j<k<p<q\le 6} x_r \sum_{s=i}^{q} \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s}\, x_s^2\, x_t^4$$

where $c_{i,j,k,p,q,s,t} \in F$ and $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

iii- U^1_F is the FW_6 – sub module of the module N^1_F , generated by $\mu_6\,(x^2_1x^4_3 - x^2_1x^4_2)$. and consists of all polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$\sum_{1\le i<j<k<p<q\le 6} \mu_r \sum_{s=i}^{q} \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s}\, x_s^2\, x_t^4$$

$$\text{where } c_{i,j,k,p,q,s,t} \in F \text{ , and } \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s,t} = 0$$

iv- U^2_F is the FW_6 – sub module of the module N^2_F , generated by $x_6\,(x^2_1x^4_3 - x^2_1x^4_2)$. and consists of all polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$ of the form:

$$\sum_{1\le i<j<k<p<q\le 6} x_r \sum_{s=i}^{q} \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s}\, x_s^2\, x_t^4$$

where $c_{i,j,k,p,q,s,t} \in F$, $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

$$\text{and } \sum_{\substack{t=i\\ t\ne s}}^{q} c_{i,j,k,p,q,s,t} = 0$$

7- Basis and dimensions of FW_6 – modules

•The set:-

$B^1 = \{\mu_{1,6}, \mu_{1,5}, \mu_{1,4}, \mu_{1,3}, \mu_{1,2}, \mu_{2,1}, \mu_{2,3}, \mu_{2,4}, \mu_{2,5}, \mu_{2,6}, \mu_{3,1}, \mu_{3,2}, \mu_{3,4}, \mu_{3,5}$
$, \mu_{3,6}, \mu_{4,1}, \mu_{4,2}, \mu_{4,3}, \mu_{4,5}, \mu_{4,6}, \mu_{5,1}, \mu_{5,2}, \mu_{5,3}, \mu_{5,4}, \mu_{5,6}, \mu_{6,1}, \mu_{6,2}, \mu_{6,3}$
$, \mu_{6,4}, \mu_{6,5}\}$.

is F – basis of the module M^1_F , and $\dim_F M^1_F = 30$.

•The set:-

$B^2 = \{\beta_{1,2}, \beta_{1,3}, \beta_{1,4}, \beta_{1,5}, \beta_{1,6}, \beta_{2,1}, \beta_{2,3}, \beta_{2,4}, \beta_{2,5}, \beta_{2,6}, \beta_{3,1}, \beta_{3,2}, \beta_{3,4},$
$\beta_{3,5}, \beta_{3,6}, \beta_{4,1}, \beta_{4,2}, \beta_{4,3}, \beta_{4,5}, \beta_{4,6}, \beta_{5,1}, \beta_{5,2}, \beta_{5,3}, \beta_{5,4}, \beta_{5,6}, \beta_{6,1},$
$\beta_{6,2}, \beta_{6,3}, \beta_{6,4}, \beta_{6,5}\}$.

is F – basis of the module M^2_F, and $\dim_F M^2_F = 30$.

•The set:-

$C^1 = \{\mu_{1,2,3}, \mu_{1,3,2}, \mu_{1,2,4}, \mu_{1,4,2}, \mu_{1,2,5}, \mu_{1,5,2}, \mu_{1,2,6}, \mu_{1,6,2}, \mu_{1,3,4}, \mu_{1,4,3}, \mu_{1,3,5}, \mu_{1,5,3}, \mu_{1,3,6}, \mu_{1,6,3}, \mu_{1,4,5}, \mu_{1,5,4}, \mu_{1,4,6}, \mu_{1,6,4},$
$\mu_{1,5,6}, \mu_{1,6,5}, \mu_{2,1,3}, \mu_{2,3,1}, \mu_{2,1,4}, \mu_{2,4,1}, \mu_{2,1,5}, \mu_{2,5,1}, \mu_{2,1,6}, \mu_{2,6,1}, \mu_{2,3,4}, \mu_{2,4,3}, \mu_{2,3,5}, \mu_{2,5,3}, \mu_{2,3,6}, \mu_{2,6,3}, \mu_{2,1,3},$
$\mu_{2,4,5}, \mu_{2,5,4}, \mu_{2,4,6}, \mu_{2,6,4}, \mu_{2,5,6}, \mu_{2,6,5}, \mu_{3,1,2}, \mu_{3,2,1}, \mu_{3,1,4}, \mu_{3,4,1}, \mu_{3,1,5}, \mu_{3,5,1}, \mu_{3,1,6}, \mu_{3,6,1}, \mu_{3,2,4}, \mu_{3,4,2}, \mu_{3,2,5},$
$\mu_{3,5,2}, \mu_{3,2,6}, \mu_{3,6,2}, \mu_{3,4,5}, \mu_{3,5,4}, \mu_{3,4,6}, \mu_{3,6,4}, \mu_{3,5,6}, \mu_{3,6,5}, \mu_{4,1,2}, \mu_{4,2,1}, \mu_{4,1,3}, \mu_{4,3,1}, \mu_{4,1,5}, \mu_{4,5,1}, \mu_{4,1,6}, \mu_{4,6,1},$
$\mu_{4,2,3}, \mu_{4,3,2}, \mu_{4,2,5}, \mu_{4,5,2}, \mu_{4,2,6}, \mu_{4,6,2}, \mu_{4,3,5}, \mu_{4,5,3}, \mu_{4,3,6}, \mu_{4,6,3}, \mu_{4,5,6}, \mu_{4,6,5}, \mu_{5,1,2}, \mu_{5,2,1}, \mu_{5,1,3}, \mu_{5,3,1}, \mu_{5,1,4},$
$\mu_{5,4,1}, \mu_{5,1,6}, \mu_{5,6,1}, \mu_{5,2,3}, \mu_{5,3,2}, \mu_{5,2,4}, \mu_{5,4,2}, \mu_{5,2,6}, \mu_{5,6,2}, \mu_{5,3,4}, \mu_{5,4,3}, \mu_{5,3,6}, \mu_{5,6,3}, \mu_{5,4,6}, \mu_{5,6,4}, \mu_{6,1,2}, \mu_{6,2,1},$
$\mu_{6,1,3}, \mu_{6,3,1}, \mu_{6,1,4}, \mu_{6,4,1}, \mu_{6,1,5}, \mu_{6,5,1}, \mu_{6,2,3}, \mu_{6,3,2}, \mu_{6,2,4}, \mu_{6,4,2}, \mu_{6,2,5}, \mu_{6,5,2}, \mu_{6,3,4}, \mu_{6,4,3}, \mu_{6,3,5}, \mu_{6,5,3}, \mu_{6,4,5},$
$\mu_{6,5,4}\}$.

is F – basis of the module N^1_F, and $\dim_F N^1_F = 120$.

•The set:-

$C^2 = \{\beta_{3,1,2}, \beta_{3,2,1}, \beta_{2,1,3}, \beta_{2,3,1}, \beta_{1,2,3}, \beta_{1,3,2}, \beta_{4,1,2}, \beta_{4,2,1}, \beta_{2,1,4}, \beta_{2,4,1}, \beta_{1,2,4}, \beta_{1,4,2}, \beta_{5,1,2}, \beta_{5,2,1}, \beta_{2,1,5}, \beta_{2,5,1}, \beta_{1,2,5}, \beta_{1,5,2},$
$\beta_{6,1,2}, \beta_{6,2,1}, \beta_{2,1,6}, \beta_{2,6,1}, \beta_{1,2,6}, \beta_{1,6,2}, \beta_{4,1,3}, \beta_{4,3,1}, \beta_{3,1,4}, \beta_{3,4,1}, \beta_{1,3,4}, \beta_{1,4,3}, \beta_{5,1,3}, \beta_{5,3,1}, \beta_{3,1,5}, \beta_{3,5,1}, \beta_{1,3,5}, \beta_{1,5,3},$
$\beta_{6,1,3}, \beta_{6,3,1}, \beta_{3,1,6}, \beta_{3,6,1}, \beta_{1,3,6}, \beta_{1,6,3}, \beta_{5,1,4}, \beta_{5,4,1}, \beta_{4,1,5}, \beta_{4,5,1}, \beta_{1,4,5}, \beta_{1,5,4}, \beta_{6,1,4}, \beta_{6,4,1}, \beta_{4,1,6}, \beta_{4,6,1}, \beta_{1,4,6}, \beta_{1,6,4},$
$\beta_{6,1,5}, \beta_{6,5,1}, \beta_{1,4,5}, \beta_{1,5,4}, \beta_{6,1,4}, \beta_{6,4,1}, \beta_{4,1,6}, \beta_{4,6,1}, \beta_{1,4,6}, \beta_{1,6,4}, \beta_{6,1,5}, \beta_{6,5,1}, \beta_{5,1,6}, \beta_{5,6,1}, \beta_{1,5,6}, \beta_{1,6,5}, \beta_{4,2,3}, \beta_{4,3,2},$
$\beta_{3,2,4}, \beta_{3,4,2}, \beta_{2,4,3}, \beta_{2,3,4}, \beta_{5,2,3}, \beta_{5,3,2}, \beta_{3,2,5}, \beta_{3,5,2}, \beta_{2,5,3}, \beta_{2,3,5}, \beta_{5,2,4}, \beta_{5,4,2}, \beta_{4,2,5}, \beta_{4,5,2}, \beta_{2,5,4}, \beta_{2,4,5}, \beta_{6,2,3}, \beta_{6,3,2},$
$\beta_{3,2,6}, \beta_{3,6,2}, \beta_{2,6,3}, \beta_{2,3,6}, \beta_{6,2,4}, \beta_{6,4,2}, \beta_{4,2,6}, \beta_{4,6,2}, \beta_{2,6,4}, \beta_{2,4,6}, \beta_{6,2,5}, \beta_{6,5,2}, \beta_{5,2,6}, \beta_{5,6,2}, \beta_{2,6,5}, \beta_{2,5,6}, \beta_{5,3,4}, \beta_{5,4,3},$
$\beta_{4,3,5}, \beta_{4,5,3}, \beta_{3,5,4}, \beta_{3,4,5}, \beta_{6,3,4}, \beta_{6,4,3}, \beta_{4,3,6}, \beta_{4,6,3}, \beta_{3,6,4}, \beta_{3,4,6}, \beta_{6,3,5}, \beta_{6,5,3}, \beta_{5,3,6}, \beta_{5,6,3}, \beta_{3,6,5}, \beta_{3,5,6}, \beta_{6,4,5}, \beta_{6,5,4},$
$\beta_{5,4,6}, \beta_{5,6,4}, \beta_{4,6,5}, \beta_{4,5,6}\}$.

is F – basis of the module N^2_F, and $\dim_F N^2_F = 120$.

•The set:-

$D_1 = \{\mu_4(x_1^2x_3^4 - x_1^2x_2^4), \quad \mu_5(x_1^2x_3^4 - x_1^2x_2^4), \quad \mu_6(x_1^2x_3^4 - x_1^2x_2^4),$
$\mu_3(x_1^2x_4^4 - x_1^2x_2^4), \quad \mu_5(x_1^2x_4^4 - x_1^2x_2^4), \quad \mu_6(x_1^2x_4^4 - x_1^2x_2^4),$
$\mu_3(x_1^2x_5^4 - x_1^2x_2^4), \quad \mu_4(x_1^2x_5^4 - x_1^2x_2^4), \quad \mu_6(x_1^2x_5^4 - x_1^2x_2^4),$
$\mu_3(x_1^2x_6^4 - x_1^2x_2^4), \quad \mu_4(x_1^2x_6^4 - x_1^2x_2^4), \quad \mu_5(x_1^2x_6^4 - x_1^2x_2^4),$
$\mu_4(x_2^2x_3^4 - x_2^2x_1^4), \quad \mu_5(x_2^2x_3^4 - x_2^2x_1^4), \quad \mu_6(x_2^2x_3^4 - x_2^2x_1^4),$
$\mu_3(x_2^2x_4^4 - x_2^2x_1^4), \quad \mu_5(x_2^2x_4^4 - x_2^2x_1^4), \quad \mu_6(x_2^2x_4^4 - x_2^2x_1^4),$
$\mu_3(x_2^2x_5^4 - x_2^2x_1^4), \quad \mu_4(x_2^2x_5^4 - x_2^2x_1^4), \quad \mu_6(x_2^2x_5^4 - x_2^2x_1^4),$
$\mu_3(x_2^2x_6^4 - x_2^2x_1^4), \quad \mu_4(x_2^2x_6^4 - x_2^2x_1^4), \quad \mu_5(x_2^2x_6^4 - x_2^2x_1^4),$
$\mu_4(x_3^2x_2^4 - x_3^2x_1^4), \quad \mu_5(x_3^2x_2^4 - x_3^2x_1^4), \quad \mu_6(x_3^2x_2^4 - x_3^2x_1^4),$

$\mu_2(x_3^2x_4^4 - x_3^2x_1^4)$, $\mu_5(x_3^2x_4^4 - x_3^2x_1^4)$, $\mu_6(x_3^2x_4^4 - x_3^2x_1^4)$,

$\mu_2(x_3^2x_5^4 - x_3^2x_1^4)$, $\mu_4(x_3^2x_5^4 - x_3^2x_1^4)$, $\mu_6(x_3^2x_5^4 - x_3^2x_1^4)$,

$\mu_2(x_3^2x_6^4 - x_3^2x_1^4)$, $\mu_4(x_3^2x_6^4 - x_3^2x_1^4)$, $\mu_5(x_3^2x_6^4 - x_3^2x_1^4)$,

$\mu_3(x_4^2x_2^4 - x_4^2x_1^4)$, $\mu_5(x_4^2x_2^4 - x_4^2x_1^4)$, $\mu_6(x_4^2x_2^4 - x_4^2x_1^4)$,

$\mu_2(x_4^2x_5^4 - x_4^2x_1^4)$, $\mu_3(x_4^2x_5^4 - x_4^2x_1^4)$, $\mu_6(x_4^2x_5^4 - x_4^2x_1^4)$,

$\mu_2(x_4^2x_6^4 - x_4^2x_1^4)$, $\mu_3(x_4^2x_6^4 - x_4^2x_1^4)$, $\mu_5(x_4^2x_6^4 - x_4^2x_1^4)$,

$\mu_3(x_5^2x_2^4 - x_5^2x_1^4)$, $\mu_4(x_5^2x_2^4 - x_5^2x_1^4)$, $\mu_6(x_5^2x_2^4 - x_5^2x_1^4)$,

$\mu_2(x_5^2x_3^4 - x_5^2x_1^4)$, $\mu_4(x_5^2x_3^4 - x_5^2x_1^4)$, $\mu_6(x_5^2x_3^4 - x_5^2x_1^4)$,

$\mu_2(x_5^2x_4^4 - x_5^2x_1^4)$, $\mu_3(x_5^2x_4^4 - x_5^2x_1^4)$, $\mu_6(x_5^2x_4^4 - x_5^2x_1^4)$,

$\mu_2(x_5^2x_6^4 - x_5^2x_1^4)$, $\mu_3(x_5^2x_6^4 - x_5^2x_1^4)$, $\mu_4(x_5^2x_6^4 - x_5^2x_1^4)$,

$\mu_3(x_6^2x_2^4 - x_6^2x_1^4)$, $\mu_4(x_6^2x_2^4 - x_6^2x_1^4)$, $\mu_5(x_6^2x_2^4 - x_6^2x_1^4)$,

$\mu_2(x_6^2x_3^4 - x_6^2x_1^4)$, $\mu_4(x_6^2x_3^4 - x_6^2x_1^4)$, $\mu_5(x_6^2x_3^4 - x_6^2x_1^4)$,

$\mu_2(x_6^2x_4^4 - x_6^2x_1^4)$, $\mu_3(x_6^2x_4^4 - x_6^2x_1^4)$, $\mu_5(x_6^2x_4^4 - x_6^2x_1^4)$,

$\mu_2(x_6^2x_5^4 - x_6^2x_1^4)$, $\mu_3(x_6^2x_5^4 - x_6^2x_1^4)$, $\mu_4(x_6^2x_5^4 - x_6^2x_1^4)$,

$\mu_2(x_1^2x_4^4 - x_2^2x_3^4)$, $\mu_2(x_1^2x_5^4 - x_2^2x_3^4)$, $\mu_2(x_1^2x_6^4 - x_2^2x_3^4)$,

$\mu_1(x_2^2x_4^4 - x_2^2x_3^4)$, $\mu_1(x_2^2x_5^4 - x_2^2x_3^4)$, $\mu_1(x_2^2x_6^4 - x_2^2x_3^4)$,

$\mu_1(x_3^2x_4^4 - x_3^2x_2^4)$, $\mu_1(x_3^2x_5^4 - x_3^2x_2^4)$, $\mu_1(x_3^2x_6^4 - x_3^2x_2^4)$,

$\mu_1(x_4^2x_3^4 - x_4^2x_2^4)$, $\mu_1(x_4^2x_5^4 - x_4^2x_2^4)$, $\mu_1(x_4^2x_6^4 - x_4^2x_2^4)$,

$\mu_1(x_5^2x_3^4 - x_5^2x_2^4)$, $\mu_1(x_5^2x_4^4 - x_5^2x_2^4)$, $\mu_1(x_5^2x_6^4 - x_5^2x_2^4)$,

$\mu_1(x_6^2x_3^4 - x_6^2x_2^4)$, $\mu_1(x_6^2x_4^4 - x_6^2x_2^4)$, $\mu_1(x_6^2x_5^4 - x_6^2x_2^4)\}$

is F – basis of the module $U^1{}_F$, and $\dim_F U^1{}_F = 90$.

• The set:-

$D_1 = \{x_4(x_1^2x_3^4 - x_1^2x_2^4)$, $x_5(x_1^2x_3^4 - x_1^2x_2^4)$, $x_6(x_1^2x_3^4 - x_1^2x_2^4)$,

$x_3(x_1^2x_4^4 - x_1^2x_2^4)$, $x_5(x_1^2x_4^4 - x_1^2x_2^4)$, $x_6(x_1^2x_4^4 - x_1^2x_2^4)$,

$x_3(x_1^2x_5^4 - x_1^2x_2^4)$, $x_4(x_1^2x_5^4 - x_1^2x_2^4)$, $x_6(x_1^2x_5^4 - x_1^2x_2^4)$,

$x_3(x_1^2x_6^4 - x_1^2x_2^4)$, $x_4(x_1^2x_6^4 - x_1^2x_2^4)$, $x_5(x_1^2x_6^4 - x_1^2x_2^4)$,

$x_4(x_2^2x_3^4 - x_2^2x_1^4)$, $x_5(x_2^2x_3^4 - x_2^2x_1^4)$, $x_6(x_2^2x_3^4 - x_2^2x_1^4)$,

$x_3(x_2^2x_4^4 - x_2^2x_1^4)$, $x_5(x_2^2x_4^4 - x_2^2x_1^4)$, $x_6(x_2^2x_4^4 - x_2^2x_1^4)$,

$x_3(x_2^2x_5^4 - x_2^2x_1^4)$, $x_4(x_2^2x_5^4 - x_2^2x_1^4)$, $x_6(x_2^2x_5^4 - x_2^2x_1^4)$,

$x_3(x_2^2x_6^4 - x_2^2x_1^4)$, $x_4(x_2^2x_6^4 - x_2^2x_1^4)$, $x_5(x_2^2x_6^4 - x_2^2x_1^4)$,

$x_4(x_3^2x_2^4 - x_3^2x_1^4)$, $x_5(x_3^2x_2^4 - x_3^2x_1^4)$, $x_6(x_3^2x_2^4 - x_3^2x_1^4)$,

$$x_2(x_3^2x_4^4 - x_3^2x_1^4)\ ,\quad x_5(x_3^2x_4^4 - x_3^2x_1^4)\ ,\quad x_6(x_3^2x_4^4 - x_3^2x_1^4),$$
$$x_2(x_3^2x_5^4 - x_3^2x_1^4)\ ,\quad x_4(x_3^2x_5^4 - x_3^2x_1^4)\ ,\quad x_6(x_3^2x_5^4 - x_3^2x_1^4),$$
$$x_2(x_3^2x_6^4 - x_3^2x_1^4)\ ,\quad x_4(x_3^2x_6^4 - x_3^2x_1^4)\ ,\quad x_5(x_3^2x_6^4 - x_3^2x_1^4),$$
$$x_3(x_4^2x_2^4 - x_4^2x_1^4)\ ,\quad x_5(x_4^2x_2^4 - x_4^2x_1^4)\ ,\quad x_6(x_4^2x_2^4 - x_4^2x_1^4),$$
$$x_2(x_4^2x_3^4 - x_4^2x_1^4)\ ,\quad x_5(x_4^2x_3^4 - x_4^2x_1^4)\ ,\quad x_6(x_4^2x_3^4 - x_4^2x_1^4),$$
$$x_2(x_4^2x_5^4 - x_4^2x_1^4)\ ,\quad x_3(x_4^2x_5^4 - x_4^2x_1^4)\ ,\quad x_6(x_4^2x_5^4 - x_4^2x_1^4),$$
$$x_2(x_4^2x_6^4 - x_4^2x_1^4)\ ,\quad x_3(x_4^2x_6^4 - x_4^2x_1^4)\ ,\quad x_5(x_4^2x_6^4 - x_4^2x_1^4),$$
$$x_3(x_5^2x_2^4 - x_5^2x_1^4)\ ,\quad x_4(x_5^2x_2^4 - x_5^2x_1^4)\ ,\quad x_6(x_5^2x_2^4 - x_5^2x_1^4),$$
$$x_2(x_5^2x_3^4 - x_5^2x_1^4)\ ,\quad x_4(x_5^2x_3^4 - x_5^2x_1^4)\ ,\quad x_6(x_5^2x_3^4 - x_5^2x_1^4),$$
$$x_2(x_5^2x_4^4 - x_5^2x_1^4)\ ,\quad x_3(x_5^2x_4^4 - x_5^2x_1^4)\ ,\quad x_6(x_5^2x_4^4 - x_5^2x_1^4),$$
$$x_2(x_5^2x_6^4 - x_5^2x_1^4)\ ,\quad x_3(x_5^2x_6^4 - x_5^2x_1^4)\ ,\quad x_4(x_5^2x_6^4 - x_5^2x_1^4),$$
$$x_3(x_6^2x_2^4 - x_6^2x_1^4)\ ,\quad x_4(x_6^2x_2^4 - x_6^2x_1^4)\ ,\quad x_5(x_6^2x_2^4 - x_6^2x_1^4),$$
$$x_2(x_6^2x_3^4 - x_6^2x_1^4)\ ,\quad x_4(x_6^2x_3^4 - x_6^2x_1^4)\ ,\quad x_5(x_6^2x_3^4 - x_6^2x_1^4),$$
$$x_2(x_6^2x_4^4 - x_6^2x_1^4)\ ,\quad x_3(x_6^2x_4^4 - x_6^2x_1^4)\ ,\quad x_5(x_6^2x_4^4 - x_6^2x_1^4),$$
$$x_2(x_6^2x_5^4 - x_6^2x_1^4)\ ,\quad x_3(x_6^2x_5^4 - x_6^2x_1^4)\ ,\quad x_4(x_6^2x_5^4 - x_6^2x_1^4),$$
$$x_2(x_1^2x_4^4 - x_1^2x_3^4)\ ,\quad x_2(x_1^2x_5^4 - x_1^2x_3^4)\ ,\quad x_2(x_1^2x_6^4 - x_1^2x_3^4),$$
$$x_1(x_2^2x_4^4 - x_2^2x_3^4)\ ,\quad x_1(x_2^2x_5^4 - x_2^2x_3^4)\ ,\quad x_1(x_2^2x_6^4 - x_2^2x_3^4),$$
$$x_1(x_3^2x_4^4 - x_3^2x_2^4)\ ,\quad x_1(x_3^2x_5^4 - x_3^2x_2^4)\ ,\quad x_1(x_3^2x_6^4 - x_3^2x_2^4),$$
$$x_1(x_4^2x_3^4 - x_4^2x_2^4)\ ,\quad x_1(x_4^2x_5^4 - x_4^2x_2^4)\ ,\quad x_1(x_4^2x_6^4 - x_4^2x_2^4),$$
$$x_1(x_5^2x_3^4 - x_5^2x_2^4)\ ,\quad x_1(x_5^2x_4^4 - x_5^2x_2^4)\ ,\quad x_1(x_5^2x_6^4 - x_5^2x_2^4),$$
$$x_1(x_6^2x_3^4 - x_6^2x_2^4)\ ,\quad x_1(x_6^2x_4^4 - x_6^2x_2^4)\ ,\quad x_1(x_6^2x_5^4 - x_6^2x_2^4)\}$$

is F – basis of the module U^2_F , and $\dim_F U^2_F = 120$.

8- FW_6 – homomorphisms

We are interesting in the following FW_6 – homomorphisms throughout this paper. (see [3] , [5])

i- $\Phi: N^1_F \longrightarrow M^1_F$, defined by:-

$$\Phi\left(\mu_{r,s,t}\right) = \frac{1}{5!}\sum_{k=1}^{6}\frac{\partial^4}{\partial x_k^4}\left(\mu_{r,s,t}\right). \\ = \mu_{r,s}$$

for all s, t, ∈ {i, j, k, p, q}, and $\{x_r\} \cup \{x_i , x_j , x_k , x_p , x_q\} = \{x_1 , x_2 , x_3 , x_4 , x_5 , x_6\}$.

ii- $\Psi: N^2_F \longrightarrow M^2_F$, defined by:-

$$\Psi(\beta_{r,s,t}) = \frac{1}{4!}\sum_{k=1}^{6}\frac{\partial^4}{\partial x_k^4}(\beta_{r,s,t}).$$
$$= \beta_{r,s}$$

for all s, t, ∈ {i, j, k, p, q}, and $\{x_r\} \cup \{x_i, x_j, x_k, x_p, x_q\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

9- Exact sequences:

Theorem (1):

The following sequence of FW_6 – modules is exact and split.

$$0 \longrightarrow U^1{}_F \xrightarrow{\theta} N^1{}_F \xrightarrow{\phi} M^1{}_F \longrightarrow 0$$

Proof: θ is one – to – one map. (inclusion map)

Since $\text{Im}\,\phi = \phi(N^1{}_F)$

$= \phi(FW_6\, \mu_{6,4,5})$

$= FW_6\, \phi(\mu_{6,4,5})$

$= FW_6\, \mu_{6,4}$

$= M^1{}_F$

Thus ϕ_1 is onto .

Now, $\phi(\mu_6(x_1^2x_3^4 - x_1^2x_2^4)) = \phi(\mu_6(x_1^2x_3^4 - \mu_6\, x_1^2x_2^4))$

$= \phi(\mu_{6,1,3} - \mu_{6,1,2})$

$= \phi(\mu_{6,1,3} - \phi(\mu_{6,1,2})$

$= \mu_{6,1} - \mu_{6,1} = 0$

Furthermore $\mu_6(x_1^2x_3^4 - x_1^2x_2^4)$ generates $U^1{}_F$.

Hence $U^1{}_F \subset \ker\phi$

Also $\dim_F \ker\phi = \dim_F N^1{}_F - \dim_F M^1{}_F$

$= 120 - 30 = 90$

$= \dim_F U^1{}_F$

Hence $\ker\phi = U^1{}_F$

But $\text{Im}\,\theta = U^1{}_F$ since θ is inclusion map.

Therefore Im θ = ker ϕ, and the sequence is exact.

To prove that the sequence split, first define

FW_6 – homomorphism $f: M^1{}_F \longrightarrow N^1{}_F$ by :-

$$f(\mu_r\, x_s^2) = \mu_r\, x_s^2 \sum_{\substack{t=i\\ t\neq s}}^{q} x_t^4 \quad , \qquad \text{for all } r, s\,.$$

where s ∈ {i, j, k, p, q}, and {r} ∪ { i, j, k, p, q} = {1, 2, 3, 4, 5, 6}

Then $\phi\, f(\mu_r\, x_s^2) = \phi(\mu_r\, x_s^2 \sum_{\substack{t=i\\ t\neq s}}^{q} x_t^4)$

$= \phi(\mu_r\, x_s^2\, x_a^4 + \mu_r\, x_s^2\, x_b^4 + \mu_r\, x_s^2\, x_c^4\ \mu_r\, x_s^2\, x_d^4)$

$= \mu_{r,s} + \mu_{r,s} + \mu_{r,s} + \mu_{r,s}$ where $\{a, b, c, d, s\} = \{i, j, k, p, q\}$
$= \mu_{r,s}$
$= \mu_r x_s^2$

Which proves that ϕf is the identity on M^1_F .
Therefore the sequence split. ▯

Theorem (2):

The following sequence of FW_6 – modules is exact and split.

$$0 \longrightarrow U^2_F \xrightarrow{h} N^2_F \xrightarrow{\Psi} M^2_F \longrightarrow 0$$

Proof: h is one – to – one map. (inclusion map)

Also $\operatorname{Im} \Psi = \Psi (N^2_F)$
$= \Psi (FW_6\ \beta_{6,4,5})$
$= FW_6\ \beta_{6,4}$
$= FW_6\ \mu_{6,4}$
$= M^2_F$

Thus we need only to prove the exactness at N^2_F .

$$\Psi (x_6 (x_1^2 x_3^4 - x_1^2 x_2^4)) = \Psi (x_6 (x_1^2 x_3^4 - x_1^2 x_2^4))$$
$= \Psi (\beta_{6,1,3} - \beta_{6,1,2})$
$= \Psi (\beta_{6,1,3}) - \Psi (\beta_{6,1,2})$
$= \beta_{6,1} - \beta_{6,1}$
$= o$

Furthermore $x_6 (x_1^2 x_3^4 - x_1^2 x_2^4)$ generates U^2_F
that is, $U^2_F \subset \ker \Psi$
We prove the converse by counting dimensions
$\dim_F \ker \Psi = \dim_F N^2_F - \dim_F M^2_F$
$= 120 – 30 = 90$
$= \dim_F U^2_F$
Then $\ker \Psi = U^2_F$, that is:
$\operatorname{Im} h = \ker \Psi$
Hence the sequence is exact.
Now as a first step to prove that the sequence split, define

FW_6 – homomorphism $g: M^2_F \longrightarrow N^2_F$ by:-

$$g(\beta_{r,s}) = \beta_{r,s} \sum_{\substack{t=i \\ t \neq s}}^{q} x_t^4 \quad , \qquad \text{for all } r, s .$$

where $s \in \{i, j, k, p, q\}$, and $\{r\} \cup \{i, j, k, p, q\} = \{1, 2, 3, 4, 5, 6\}$

$$\text{Then } \Psi g(\beta_{r,s}) = \Psi (\beta_{r,s} \sum_{\substack{t=i \\ t \neq s}}^{q} x_t^4)$$

$= \Psi (\beta_{r,s} x_a^4 + \beta_{r,s} x_b^4 + \beta_{r,s} x_c^4 + \beta_{r,s} x_d^4)$
$= \beta_{r,s} + \beta_{r,s} + \beta_{r,s} + \beta_{r,s}$ where $\{a, b, c, d, s\} = \{i, j, k, p, q\}$
$= \beta_{r,s}$ (since F of characteristic 3)

Which proves that Ψg is the identity on M^2_F .

Therefore the sequence split. □

References

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