Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

A Differential Sandwich Theorems for Analytic Functions Associated with Convolution Structure

Najah Ali Jiben Al-Ziadi [a] , ***Waggas Galib Atshan*** [b]

[a] *Department of Mathematics, College of Education , University of Al-Qadisiyah, Diwaniya-Iraq , Email: najah.ali@qu.edu.iq.*

[b] *Department of Mathematics, College of Computer Science and Information Technology , University of Al-Qadisiyah, Diwaniya-Iraq , Email:waggas.galib@qu.edu.iq.*

ARTICLE INFO

Article history:
Received: 18 / 8 / 2019
Rrevised form: / /
Accepted : 2 / 9 / 2019
Available online: 20 / 12 / 2019

Keywords:

Analytic function

Hadamard product

Differential subordination

Differential superordination

Generalized operator.

ABSTRACT

In this paper, by making use of the generalized operator, we introduce and study subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Our results extended corresponding previously known results.

MSC. 30C45, 30C50, 30C80.

DOI:10.29304/jqcm.2019.11.4.641

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}). \tag{1.1}$$

Also, let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

*Corresponding author Najah Ali Jiben Al-Ziadi

Email addresses: najah.ali@qu.edu.iq.

Communicated by Qusuay Hatim Egaar

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Let $f, g \in \mathcal{H}$, the function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in \mathcal{H}$ and $\varphi(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q \prec p$ for all the function p satisfying (1.3). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.3) is called the best subordinant. Recently Miller and Mocanu [9] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For the functions $f \in \mathcal{A}$ given by (1.2) and $g \in \mathcal{A}$ defined by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the convolution (or Hadamard product) of f and g by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)$.

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta \geq 0, \alpha \in \mathbb{R}$ with $\alpha + \beta > 0$ and $f \in \mathcal{A}$. The generalized operator $I_{\alpha,\beta}^{m}$ (see [14]) is defined by

$$I_{\alpha,\beta}^{m} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta}{\alpha + \beta}\right)^{m} a_n z^n. \tag{1.4}$$

It follows from (1.4) that

$$\beta z \left(I_{\alpha,\beta}^{m} f(z)\right)' = (\alpha + \beta) I_{\alpha,\beta}^{m+1} f(z) - \alpha I_{\alpha,\beta}^{m} f(z), \quad \beta > 0. \tag{1.5}$$

Note that the generalized operator $I_{\alpha,\beta}^{m}$ unifies many operators of $\mathcal{A}$.

In particular:

(1) $I_{\alpha,1}^{m} f(z) = I_{\alpha}^{m} f(z), \ \alpha > -1$ (see Cho and Srivastava [8] and Cho and Kim [7]).

(2) $I_{1-\beta,\beta}^{m} f(z) = D_{\beta}^{m} f(z), \ \beta \geq 0$ (see Al-Oboudi [1]).

(3) $I_{c+1-\beta,\beta}^{m} f(z) = I_{c,\beta}^{m} f(z), \ c > -1, \beta \geq 0$ (see Catas [6]).

Using the results of Miller and Mocanu [9], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see[5]). Further, using the results of Miller and Mocanu [9] and Bulboacă [4] many researchers [3,11] have obtained sufficient conditions on normalized analytic functions f by means of differential subordination and superordinations.

Recently, Wanas and Joudah [15] obtained sufficient condition for normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{I_{\alpha,\beta}^{m+1}(f * \Phi)(\mathrm{z})}{I_{\alpha,\beta}^{m}(f * \Psi)(\mathrm{z})} \prec q_2(z),$$

where q_1 *and* q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f * \Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f * \Psi)(\mathrm{z})}{(\mu + \gamma)\mathrm{z}}\right)^{\delta} \prec q_2(z),$$

and

$$q_1(z) \prec \left(\frac{I_{\alpha,\beta}^{m} f(\mathrm{z})}{z}\right)^{\delta} \left(\frac{\mathrm{z}}{I_{\alpha,\beta}^{m+1} f(z)}\right)^{\lambda} \prec q_2(z),$$

where q_1 *and* q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$ and $\Phi(z) = z + \sum_{n=2}^{\infty} t_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n$ are analytic functions in U with $t_n \geq 0,\ \sigma_n \geq 0$ and $t_n \geq \sigma_n$.

2. Preliminaries

In order to prove our main results, we need the following definition and lemmas.

Definition 2.1 [10]: Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$\mathrm{E}(f) = \left\{ \zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty \right\} \text{ and are such that } f'(\zeta) \neq 0 \text{ for } \zeta \in \partial U \setminus E(f).$$

Lemma 2.1 [10]: Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(1) $Q(z)$ is starlike univalent in U (2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$, for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.2 [4]: Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(1) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$, (2) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.

If $p \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q \prec p$ and q is the best subordinant of (2.2).

3. Subordination Results

Theorem 3.1: Let $\Phi, \Psi \in \mathcal{A}, v, \xi, \eta, \mu, \gamma, \delta, k \in \mathbb{C}$ such that $\eta, \delta \neq 0$ and $\mu + \gamma \neq 0$,let q be convex univalent in U with $q(0) = 1,\ q(z) \neq 0\ (z \in U)$. Suppose that $z(q(z))^{k-1}q'(z)$ is starlike univalent in U and q satisfies

$$Re\left\{1 + \frac{vk}{\eta} + \frac{\xi(k+1)}{\eta}q(z) + (k-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.1}$$

If $f \in \mathcal{A}$ satisfies

$$\Omega_1(f, \Phi, \Psi, v, \xi, \eta, \mu, \gamma, \delta, k, \alpha, \beta, m; z) \prec (v + \xi q(z))(q(z))^k + \eta z(q(z))^{k-1}q'(z), \tag{3.2}$$

where

$$\Omega_1(f, \Phi, \Psi, v, \xi, \eta, \mu, \gamma, \delta, k, \alpha, \beta, m; z)$$

$$\begin{aligned} &= v\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}{(\mu+\gamma)\mathrm{z}}\right)^{\delta k} + \xi\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}{(\mu+\gamma)\mathrm{z}}\right)^{\delta(k+1)} \\ &\quad + \frac{\delta\eta(\alpha+\beta)}{\beta}\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}{(\mu+\gamma)\mathrm{z}}\right)^{\delta k} \\ &\quad \times\left(\frac{\mu\,[I_{\alpha,\beta}^{m+2}(f*\Phi)(\mathrm{z}) - I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z})] + \gamma[I_{\alpha,\beta}^{m+1}(f*\Psi)(\mathrm{z}) - I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})]}{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}\right), \end{aligned} \tag{3.3}$$

then

$$\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}{(\mu+\gamma)\mathrm{z}}\right)^{\delta} \prec q(z)$$

and q is the best dominant of (3.2).

Proof: Let the function p be defined by

$$p(z) = \left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(\mathrm{z}) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(\mathrm{z})}{(\mu+\gamma)\mathrm{z}}\right)^{\delta}, \qquad z \in U. \tag{3.4}$$

Then the function p is analytic in U and $p(0) = 1$, differentiating (3.4) logarithmically with respect to z and using the identity (1.5) , we get

$$\frac{zp'(z)}{p(z)} = \frac{\delta(\alpha+\beta)}{\beta}\left(\frac{\mu\,[I_{\alpha,\beta}^{m+2}(f*\Phi)(z) - I_{\alpha,\beta}^{m+1}(f*\Phi)(z)] + \gamma[I_{\alpha,\beta}^{m+1}(f*\Psi)(z) - I_{\alpha,\beta}^{m}(f*\Psi)(z)]}{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(z) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}\right), \quad (3.5)$$

Therefore, by making use of (3.5), we obtain

$$\big(v+\xi p(z)\big)\big(p(z)\big)^{k} + \eta z\big(p(z)\big)^{k-1}p'(z) = \Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z), \quad (3.6)$$

where $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ is given by (3.3). From (3.2) and (3.6), we have

$$\big(v+\xi p(z)\big)\big(p(z)\big)^{k} + \eta z\big(p(z)\big)^{k-1}p'(z) \prec \big(v+\xi q(z)\big)\big(q(z)\big)^{k} + \eta z\big(q(z)\big)^{k-1}q'(z).$$

By setting $\theta(w) = (v+\xi w)w^{k}$ and $\phi(w) = \eta w^{k-1}$,
we see that $\theta(w)$ and $\phi(w)$are analytic in $\mathbb{C}/\{0\}$ and $\phi(w) \neq 0,\ w \in \mathbb{C}/\{0\}$.
Also, we get

$$Q(z) = zq'(z)\phi\big(q(z)\big) = \eta z\big(q(z)\big)^{k-1}q'(z)$$

and

$$h(z) = \theta\big(q(z)\big) + Q(z) = \big(v+\xi q(z)\big)\big(q(z)\big)^{k} + \eta z\big(q(z)\big)^{k-1}q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{vk}{\eta} + \frac{\xi(k+1)}{\eta}q(z) + (k-1)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

Thus, by applying Lemma (2.1) our proof of Theorem (3.1) is completed.

By taking $\alpha = 1-\beta$ and $\beta > 0$ in Theorem (3.1), we obtain the following corollary:
Corollary 3.1. Let $\Phi,\Psi \in \mathcal{A}, v,\xi,\eta,\mu,\gamma,\delta,k \in \mathbb{C}$ such that $\eta,\delta \neq 0$ and $\mu+\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1,\ q(z) \neq 0\ (z \in U)$. Suppose that $z\big(q(z)\big)^{k-1}q'(z)$ is starlike univalent in U and (3.1) holds true. If $f \in \mathcal{A}$ satisfies

$$\Omega_2(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\beta,m;z) \prec \big(v+\xi q(z)\big)\big(q(z)\big)^{k} + \eta z\big(q(z)\big)^{k-1}q'(z), \quad (3.7)$$

where

$$\Omega_2(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\beta,m;z) = v\left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta k}$$
$$+\xi\left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta(k+1)} + \frac{\delta\eta}{\beta}\left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta k}$$
$$\times\left(\frac{\mu\,[D_{\beta}^{m+2}(f*\Phi)(z) - D_{\beta}^{m+1}(f*\Phi)(z)] + \gamma[D_{\beta}^{m+1}(f*\Psi)(z) - D_{\beta}^{m}(f*\Psi)(z)]}{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}\right), \quad (3.8)$$

then

$$\left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta} \prec q(z)$$

and q is the best dominant of (3.7).

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem (3.1), we obtain the following corollary:
Corollary 3.2. Let $v,\xi,\eta,\mu,\gamma,\delta,k \in \mathbb{C}$ such that $\eta,\delta \neq 0$ and $\mu+\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1,\ q(z) \neq 0\ (z \in U)$. Suppose that $z\big(q(z)\big)^{k-1}q'(z)$ is starlike univalent in U and (3.1) holds true. If $f \in \mathcal{A}$ satisfies

$$\Omega_3(f,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z) \prec \big(v+\xi q(z)\big)\big(q(z)\big)^{k} + \eta z\big(q(z)\big)^{k-1}q'(z), \quad (3.9)$$

where

$$\Omega_3(f,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z) = v\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}f(z) + \gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta k} + \xi\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}f(z) + \gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta(k+1)}$$

$$+\frac{\delta\eta(\alpha+\beta)}{\beta}\left(\frac{\mu I_{\alpha,\beta}^{m+1}f(z)+\gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta k}\times\left(\frac{\mu\,[I_{\alpha,\beta}^{m+2}f(z)-I_{\alpha,\beta}^{m+1}f(z)]+\gamma[I_{\alpha,\beta}^{m+1}f(z)-I_{\alpha,\beta}^{m}f(z)]}{\mu I_{\alpha,\beta}^{m+1}f(z)+\gamma I_{\alpha,\beta}^{m}f(z)}\right), \quad (3.10)$$

then

$$\left(\frac{\mu I_{\alpha,\beta}^{m+1}f(z)+\gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta}\prec q(z)$$

and q is the best dominant of (3.9).

Theorem 3.2. Let $\Phi,\Psi\in\mathcal{A},u,v,\xi,\eta,\delta,\lambda\in\mathbb{C}$ such that $\eta\neq 0$, let q be convex univalent with $q(0)=1,\ q(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike in U, and assume that

$$Re\left\{1+\frac{v}{\eta}q(z)+\frac{2\xi}{\eta}[q(z)]^2-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0. \quad (3.11)$$

If $f\in\mathcal{A}$ satisfies

$$\Omega_4(f,\Phi,\Psi,u,v,\xi,\eta,\delta,\lambda,\alpha,\beta,m;z)\prec u+v\,q(z)+\xi[q(z)]^2+\eta\frac{zq'(z)}{q(z)}, \quad (3.12)$$

where

$$\Omega_4(f,\Phi,\Psi,u,v,\xi,\eta,\delta,\lambda,\alpha,\beta,m;z)$$

$$=u+v\left(\frac{I_{\alpha,\beta}^{m}(f*\Phi)(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}(f*\Psi)(z)}\right)^{\lambda}+\xi\left(\frac{I_{\alpha,\beta}^{m}(f*\Phi)(z)}{z}\right)^{2\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}(f*\Psi)(z)}\right)^{2\lambda}$$

$$+\frac{\delta(\alpha+\beta)\eta}{\beta}\left[\frac{I_{\alpha,\beta}^{m+1}(f*\Phi)(z)}{I_{\alpha,\beta}^{m}(f*\Phi)(z)}-1\right]+\frac{\lambda(\alpha+\beta)\eta}{\beta}\left[1-\frac{I_{\alpha,\beta}^{m+2}(f*\Psi)(z)}{I_{\alpha,\beta}^{m+1}(f*\Psi)(z)}\right], \quad (3.13)$$

then

$$\left(\frac{I_{\alpha,\beta}^{m}(f*\Phi)(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}(f*\Psi)(z)}\right)^{\lambda}\prec q(z)$$

and q is the best dominant of (3.12).

4. Superordination Results

Theorem 4.1: Let $\Phi,\Psi\in\mathcal{A},v,\xi,\eta,\mu,\gamma,\delta,k\in\mathbb{C}$ such that $\eta,\delta\neq 0$ and $\mu+\gamma\neq 0$ and q be convex univalent in U with $q(0)=1,\ q(z)\neq 0\ (z\in U)$. Suppose that $z(q(z))^{k-1}q'(z)$ is starlike univalent in U and q satisfies

$$Re\left\{\frac{vk}{\eta}q'(z)+\frac{\xi(k+1)}{\eta}q(z)\,q'(z)\right\}>0. \quad (4.1)$$

Let $f\in\mathcal{A}$, $\left(\frac{\mu I_{\alpha,\beta}^{m+1}(f*\Phi)(z)+\gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q$
and $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ be univalent in U,
where $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ is given by (3.3).
If

$$\big(v+\xi q(z)\big)\big(q(z)\big)^k+\eta z\big(q(z)\big)^{k-1}q'(z)\prec\ \Omega_1(f,\Phi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z), \quad (4.2)$$

then

$$q(z)\prec\left(\frac{\mu I_{\alpha,\beta}^{m+1}(f*\Phi)(z)+\gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta} \quad (4.3)$$

and q is the best subordinant of (4.2).

Proof. Let the function p be defined by

$$p(z) = \left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(z) + \gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta}, (z \in U). \quad (4.4)$$

By setting $\theta(w) = (v + \xi w)w^k$ and $\phi(w) = \eta w^{k-1}$, it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \eta z(q(z))^{k-1}q'(z),$$

we find that $Q(z)$ is starlike univalent in U and that

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{vk}{\eta}q'(z) + \frac{\xi(k+1)}{\eta}q(z)\,q'(z)\right\} > 0.$$

By a straightforward computation, we obtain

$$\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z) = (v+\xi p(z))(p(z))^k + \eta z(p(z))^{k-1}p'(z), \quad (4.5)$$

where $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ is given by (3.3).
By using (4.5) in (4.2), we have

$$(v+\xi q(z))(q(z))^k + \eta z(q(z))^{k-1}q'(z) \prec (v+\xi p(z))(p(z))^k + \eta z(p(z))^{k-1}p'(z).$$

The assertion (4.3) follows by an application of Lemma (2.2).

When $\alpha = 1-\beta$ and $\beta > 0$ in Theorem (4.1), we derive the following corollary:
Corollary 4.1: Let $\Phi,\Psi \in \mathcal{A}, v,\xi,\eta,\mu,\gamma,\delta,k \in \mathbb{C}$ such that $\eta,\delta \neq 0$ and $\mu+\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$ $(z \in U)$. Suppose that $z(q(z))^{k-1}q'(z)$ is starlike univalent in U and (4.1) holds true.
Let $f \in \mathcal{A}$, $\left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z)+\gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$
and $\Omega_2(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\beta,m;z)$ be univalent in U,
where $\Omega_2(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\beta,m;z)$ is given by (3.8).
If

$$(v+\xi q(z))(q(z))^k + \eta z(q(z))^{k-1}q'(z) \prec \Omega_2(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\beta,m;z), \quad (4.6)$$

then

$$q(z) \prec \left(\frac{\mu\, D_{\beta}^{m+1}(f*\Phi)(z) + \gamma D_{\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta}$$

and q is the best subordinant of (4.6).

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem (4.1), we obtain the following corollary:
Corollary 4.2: Let $v,\xi,\eta,\mu,\gamma,\delta,k \in \mathbb{C}$ such that $\eta,\delta \neq 0$ and $\mu+\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$ $(z \in U)$. Suppose that $z(q(z))^{k-1}q'(z)$ is starlike univalent in U and (4.1) holds true.
Let $f \in \mathcal{A}$, $\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}f(z)+\gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ and $\Omega_3(f,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ be univalent in U, where $\Omega_3(f,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ is given by (3.10).
If

$$(v+\xi q(z))(q(z))^k + \eta z(q(z))^{k-1}q'(z) \prec \Omega_3(f,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z), \quad (4.7)$$

then

$$q(z) \prec \left(\frac{\mu\, I_{\alpha,\beta}^{m+1}f(z) + \gamma I_{\alpha,\beta}^{m}f(z)}{(\mu+\gamma)z}\right)^{\delta}$$

and q is the best subordinant of (4.7).

Theorem 4.2. Let $\Phi,\Psi \in \mathcal{A},, u,v,\xi,\eta,\delta,\lambda \in \mathbb{C}, \eta \neq 0$, let q be convex univalent with $q(0) = 1$,

$q(z) \neq 0$ and $\frac{z\,q'(z)}{q(z)}$ is starlike in U, and assume that

$$Re\left\{(v + 2\xi q(z))\frac{q(z)q'(z)}{\eta}\right\} > 0, \quad (4.8)$$

If $f \in \mathcal{A}$, $\left(\frac{I_{\alpha,\beta}^{m}(f*\Phi)(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}(f*\Psi)}\right)^{\lambda} \in H[q(0),1] \cap Q$. Let $\Omega_4(f,\Phi,\Psi,u,v,\xi,\delta,\lambda,\alpha,\beta,m;z)$ be univalent in U and

$$u + vq(z) + \xi[q(z)]^2 + \eta\frac{z\,q'(z)}{q(z)} \prec \Omega_4(f,\Phi,\Psi,u,v,\xi,\delta,\lambda,\alpha,\beta,m;z), \tag{4.9}$$

where $\Omega_4(f,\Phi,\Psi,u,v,\xi,\delta,\lambda,\alpha,\beta,m;z)$ is given by (3.13), then

$$q(z) \prec \left(\frac{I_{\alpha,\beta}^{m}(f*\Phi)(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}(f*\Psi)}\right)^{\lambda}$$

and $q(z)$ is the best subordinant of (4.9).

5. Sandwich Results

By combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem:

Theorem 5.1: Let q_1 and q_2 be two convex univalent functions in U with $q_i(0) = 1, q_i(z) \neq 0$ and $z\big(q_i(z)\big)^{k-1}q_i'(z)$ $(i = 1,2)$ is starlike in U, let $v,\xi,\eta,\mu,\gamma,\delta,k \in \mathbb{C}$ such that $\eta,\delta \neq 0$ and $\mu+\gamma \neq 0$. Suppose q_2 satisfies (3.1) and q_1 satisfies (4.1). For $f,\Phi,\Psi \in \mathcal{A}$, let $\left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(z)+\gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$ and $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ be univalent in U, where $\Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z)$ is given by (3.3). If

$$\big(v+\xi q_1(z)\big)\big(q_1(z)\big)^{k} + \eta z\big(q_1(z)\big)^{k-1}q_1'(z) \prec \Omega_1(f,\Phi,\Psi,v,\xi,\eta,\mu,\gamma,\delta,k,\alpha,\beta,m;z) \prec \big(v+\xi q_2(z)\big)\big(q_2(z)\big)^{k} + \eta z\big(q_2(z)\big)^{k-1}q_2'(z)\,,$$

then

$$q_1(z) \prec \left(\frac{\mu\, I_{\alpha,\beta}^{m+1}(f*\Phi)(z)+\gamma I_{\alpha,\beta}^{m}(f*\Psi)(z)}{(\mu+\gamma)z}\right)^{\delta} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 5.2. Let q_1 and q_2 be two convex univalent functions in U such that $q_i(0) = 1, q_i(z) \neq 0$ and $\frac{zq_i'(z)}{q_i(z)}$ $(i = 1,2)$ is starlike in U, let $u,v,\xi,\eta,\delta,\lambda \in \mathbb{C}$ such that $\eta \neq 0$. Further assume that q_2 satisfies (3.11) and q_1 satisfies (4.8). For $f,\Phi,\Psi \in \mathcal{A}$, let $\left(\frac{I_{\alpha,\beta}^{m}f(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}f(z)}\right)^{\lambda} \in H[1,1] \cap Q$ and $\Omega_4(f,\Phi,\Psi,u,v,\xi,\eta,\delta,\lambda,\alpha,\beta,m;z)$ defined by (3.13) be univalent in U satisfying

$$u + vq_1(z) + \xi[q_1(z)]^2 + \eta\frac{z\,q_1'(z)}{q_1(z)} \prec \Omega_4(f,\Phi,\Psi,u,v,\xi,\eta,\delta,\lambda,\alpha,\beta,m;z) \prec u + vq_2(z) + \xi[q_2(z)]^2 + \eta\frac{z\,q_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left(\frac{I_{\alpha,\beta}^{m}f(z)}{z}\right)^{\delta}\left(\frac{z}{I_{\alpha,\beta}^{m+1}f(z)}\right)^{\lambda} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

Remark 5.1.

1) Putting $k = v = \gamma = 1$, $\mu = \xi = 0$, $\eta = \frac{\lambda}{\delta}$ $(\delta > 0, \lambda \in \mathbb{C}\backslash\{0\})$ and $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Corollaries (3.1),(4.1), we get the results obtained by Răducanu and Nechita [12, Theorem 3.1, Theorem 3.6].
2) By setting $k = \xi = \mu = \alpha = m = 0, v = \gamma = \beta = \delta = 1, q(z) = \frac{1}{(1-z)^{2b}}$ $(b \in \mathbb{C}\backslash\{0\})$ and $\eta = \frac{1}{b}$ in Corollary (3.2), we get the result obtained by Srivastava and Lashin [13, Theorem 3].
3) Selecting $k = \xi = \mu = \alpha = m = 0$, $v = \gamma = \beta = 1$, $\eta = \frac{e^{i\rho}}{ab\cos\rho}\left(a,b \in \mathbb{C},\ |\rho| < \frac{\pi}{2}\right)$, $\delta = a$ and $q(z) = (1-z)^{-2ab\cos\rho e^{-i\rho}}$ in Corollary (3.2), we obtain the result of Aouf et al. [2, Theorem 1].

4) For $k = v = \gamma = \beta = 1$, and $\mu = \xi = \alpha = m = 0$ and $\eta = \frac{\lambda}{\delta}$ $(\delta > 0, \lambda \in \mathbb{C}\backslash\{0\})$ in Corollary (3.2), we have the result obtained by Murugusundaramoorthy and Magesh [11, Corollary 3.3].

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