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Some geometric topics of Jackson 's (p,q)- derivative convoluted with a Subclass of convex function with negative coefficients

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ABSTRACT

In the present work, a new subclass $\wp^{\delta,k}_{p,q}(\tau,\eta)$ of convex functions with negative coefficients by derivative operator $\Upsilon^{\delta,k}_{\tau,p,q}$, considered coefficient inequalities, growth and distortion theorem, closure theorem, and some properties of sundry functions pertinence in class considered. So get radii of close-to-convexity for function pertinence in to class $\wp^{\delta,k}_{p,q}(\tau,\eta)$. Moreover, an integrated way inequality resolve for functions pertinence in class $\wp^{\delta,k}_{p,q}(\tau,\eta)$.

Keywords:

Coefficient bounds, Analytic functions ; Starlike , convex ; and Close-to-convex functions ;Integral means.

MSC

Mathematics Subject Classification: 30C45

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1. Introduction and definition

Let $\mathcal{A}$ denote the class of functions of the form

$$\varphi(z) = z + \sum_{v=2}^{\infty} c_v z^v \quad (z \in \Omega). \qquad (1)$$

which are analytic in the open unit disk $\mho = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by $\varphi(0) = 0$ and $\acute{\varphi}(0) = 1$. Let S denote the class of all functions $\varphi \in \mathcal{A}$ which are univalent in Ω . Also, q-calculus plays a rule in the theory of hypergeometric series, quantum physics and various branches of mathematics as for example, in the areas of ordinary fractional calculus, optimal control problems. The application of q- calculus was initiated by Jackson [12].

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The quantum calculus is of the paramount tools that you use to research subclass of analytic functions. Kanas and Raducanu [13] employ the fractional q-calculus operators in considerations of confirmed classes of functions which is analytical in Ʊ. A universal study on applications of q-calculus in operator theory may be found in [8,4,17].

The q-calculus it is lean on one coefficient, the circular of q-calculus is the post-quantum calculus. may be obtained by substituting p = 1 in (p , q)-calculus . For details on q-calculus one can refer [8,11,12,15,19]. We let during this paper that 0 < p < q ≤ 1. For the means of comfort, several basic definitions and registrations of (p , q) -calculus are aforesaid below:

With ($0 < p < q \leq 1$) the Jackson's (p, q) - derivative for function $\varphi \in \mathcal{A}$, as defined , presented accordingly [12]

$$D_{p,q}\ \varphi(z) = \begin{cases} \frac{\varphi(pz) - \varphi(qz)}{(p-q)z} & for\ z \neq 0 \\ \varphi'(0) & for\ z = 0 \end{cases} \tag{1.2}$$

By (1.2) we get

$$D_{p,q}\ \varphi(z) = 1 + \sum_{v=2}^{\infty} [v]_{p,q}\ c_v\ z^{v-1}, \tag{1.3}$$

Where

$$[v]_{p,q} = p^{v-1} + p^{v-2}q + p^{v-3}q^2 + \ldots + p\,q^{v-2} + q^{v-1} = \frac{p^v - q^v}{p-q}, \tag{1.4}$$

Is called (p, q) - bow or no . Basic twin observe that if $p = 1$, the no . Basic twin is a natural circularization of the q-number , this is

$$[v]_{p,q} = \frac{1-q^v}{1-q} = [v]_q \quad q \neq 1$$

observe so that if $p = 1$, the Jackson' s (p, q) – derivative,[12]

Its obviously to verify for $\psi(z) = z^v$, we have $D_{p,q}\ \psi(z) = D_{p,q}\ z^v = \frac{p^v - q^v}{p-q} z^{v-1} = [v]_{p,q}\ z^{v-1}$

with $\varphi \in \mathcal{A}$, The Salagean (p, q) – different operator defined as it comes:

$$\psi^0_{p,q}\varphi(z) = \varphi(z)$$

$$\psi^1_{p,q}\varphi(z) = z\,D_{p,q}\ \varphi(z),$$

$$\vdots$$

$$\psi^k_{p,q}\varphi(z) = \psi^1_{p,q}\left(\psi^{k-1}_{p,q}\varphi(z)\right)$$

$$= z + \sum_{v-2}^{\infty}[v]^k_{p,q}\ c_v\ z^v \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \Omega) \tag{1.5}$$

note that whether $p = 1$ and $\lim_q \to 1^-$, we get the familiar Salagean derivative [16]:

$$\psi^k\varphi(z) = z + \sum_{v-2}^{\infty} v^k c_v\ z^v \quad (k \in \mathbb{N}_0 : z \in \Omega) \tag{1.6}$$

Now let

$$\Upsilon^{0,k}_{\tau,p,q}\varphi(z) = \psi^k_{,p,q}\varphi(z),$$

$$\Upsilon^{1,k}_{\tau,p,q}\varphi(z) = (1-\tau)\psi^{\delta,k}_{\tau,p,q}\varphi(z) + \tau z(\psi^{k}_{\tau,p,q}\varphi(z))' \quad (1.7)$$

$$= z + \sum_{v-2}^{\infty} [v]^{k}_{p,q} [1 + (v-1)] c_v z^v,$$

$$\Upsilon^{2,k}_{\tau,p,q}\varphi(z) = (1-\tau)\Upsilon^{1,k}_{\tau,p,q}\varphi(z) + \tau z(\Upsilon^{1,k}_{\tau,p,q}\varphi(z))'$$

$$= \mathrm{z} + \sum_{v=2}^{\infty} [v]^{\kappa}_{p,q} [1 + (v-1)\tau]^2 c_v z^v) \quad (1.8)$$

In general, we have

$$\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z) = (1-\tau)z(\Upsilon^{\delta-1,k}_{\tau,p,q}\varphi(z)) + \tau z(\Upsilon^{\delta-1,k}_{\tau,p,q}\varphi(z))'$$

$$= \mathrm{z} + \sum_{v=2}^{\infty} [v]^{\kappa}_{p,q} \left[1 + (v-1)\tau\right]^{\delta} c_v z^v\big) \quad (\tau \geq 0; k, \xi \in \mathbb{N}_0)$$

Clearly, we have $\Upsilon^{0,0}_{\tau,p,q}\varphi(z) = \varphi(z)$ and $\Upsilon^{1,0}_{1,p,q}\varphi(z) = z\,\varphi\,'(z)$.

We note that when p = 1, we get the differential operator $\Upsilon^{\delta,k}_{\tau,,q}\varphi(z)$ knowledge and study by Frasin and Murugusundaramoorthy [9] . As will, you note that where p = 1 also $\lim_q \to 1^-$, we have the differential operator :

$$\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z) = z + \sum_{v=2}^{\infty} v^k \; [1 + (v-1)\tau \; c_v z^v]^{\delta} \quad (\tau \geq 0; k, \delta \in \mathbb{N}_0)$$

We note that when k = 0 , we get the differentiation operator Υ^{δ}_{τ} defined by Al − Oboudi [3], while if $\delta = 0$, we obtain Salagean differentiation operator Υ^{δ} [16] .

Together with the help of the aid of the differential operator $\Upsilon^{\delta,k}_{\tau,q,p}$ we say that the a function $\varphi(z)$ belonging to $\mathcal{A}$ is in the class $Q^{\delta,k}_{,q,p}(\tau,\eta)$ if and only if

$$\mathrm{R}\left\{\frac{\begin{array}{c}(1-\tau)z(\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z))''+\tau\\ z(\Upsilon^{\delta,k+1}_{\tau,p,q}\varphi(z))''\end{array}}{(1-\tau)\,(\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z))'+\tau\,(\Upsilon^{\delta,k+1}_{\tau,p,q}\varphi(z))'} + 1\right\} > \eta\;(K,\delta \in \mathbb{N}_0), \quad (1.9)$$

For some $\eta\,(0 \leq \eta < 1)$ and $\tau(\tau \geq 0)$, and for all $z \in \mho$.

Suppose $\mathcal{T}$ denote the subclass of $\mathcal{A}$ composed function given by:

$$\varphi(z) = z - \sum_{v-2}^{\infty} c_v z^v \quad (c_v \geq 0, z \in \mho). \quad (1.10)$$

We also define the class $\mathcal{P}^{\delta,k}_{,q,p}(\tau,\eta)$ by

$$\mathcal{P}^{\delta,k}_{q,p}(\tau,\eta) = Q^{\delta,k}_{,q,p}(\tau,\eta) \cap \mathcal{T}. \quad (1.11)$$

The present paper prove various interesting properties for functions belonging of the class $\mathcal{P}^{\delta,k}_{,q,p}(\tau,\eta)$. Some of the results and outcomes of the main results we use are also comparable to those used previously by Frasin et al. [10], Al-Hawary et al. [1,2], Aouf and Srivastava [7] and Amourah et al. [5,6].

2. Coefficient estimates

In this section we start to obtain a necessary and sufficient condition to function $\varphi(z)$ in the class $\wp^{\delta,k}_{p,q}(\tau,\eta)$.

Theorem 2.1. Let the function $\varphi(z)$ *be defined by* (1.10) . *Then* $\varphi(z) \in \wp^{\delta,k}_{p,q}(\tau,\eta)$ if and only if

$$\sum_{v=2}^{\infty}[v]_{p,q}^{\,k}\, v\,(v-\eta)\,\{1+([v]-1)\tau\}[1+(v-1)\tau]^{\delta}c_v \le 1-\eta \quad (2.1)$$

The result is true.

Proof. Suppose *that the function* $\varphi(z)$ *is in the class* $\wp_{p,q}^{\varepsilon,k}(\tau,\eta)$. *Then* we have

$$\text{Re}\left\{\frac{(1-\tau)z(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z))''+\tau z(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z))''}{(1-\tau)\,(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z))'+\tau\,(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z))'}+1\right\}=Re\left\{\frac{M}{H}+1\right\} > \eta\,(K,\xi\in\mathbb{N}_0) \quad (2.2)$$

for some $\eta(0\le \eta <1)$ *and* $\tau(\tau\ge 0)$, *and for all* $z\in \mho$. Now

$$M = (1-\tau)\left(z-\sum_{v=2}^{\infty} v\,(v-1)\,[v]_{p,q}^{\,k}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-2}\right)+$$

$$\tau z\left[-\sum_{v=2}^{\infty} v\,(v-1)[v]_{p,q}^{k+1}\,[1+(v-1)\,\tau]^{\delta}\,c_v\,z^{v-2}\right]$$

$$M=\left(-\sum_{v=2}^{\infty} v(v-1)[v]_{p,q}^{k}\;\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-1}\right)$$

$$H=(1-\tau)\left(1-\sum_{v=2}^{\infty} v[v]_{p,q}^{\kappa}\left[1+(v-1)\tau\right]^{\delta}c_v z^{v-1}\right)$$

$$+\tau\left(1-\sum_{v=2}^{\infty} v[v]_{p,q}^{\kappa+1}\left[1+(v-1)\tau\right]^{\delta}c_v z^{v-1}\right)$$

$$H=1-\sum_{v=2}^{\infty} v\,[v]_{p,q}^{\,k}\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-1}. \quad (2.4)$$

Consequently,

$$\text{Re}\left\{\frac{(1-\tau)z(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z))''+\tau z(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z))''}{(1-\tau)\,(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z))'+\tau\,(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z))'}+1\right\} \quad (2.5)$$

$$Re\left\{\frac{-\sum_{v=2}^{\infty} v\,(v-1)\,[v]_{p,q}^{\,k}\,\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-1}}{1-\sum_{v=2}^{\infty} v\,[v]_{p,q}^{\,k}\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-1}}+1\right\} > \eta$$

Letting $z \longrightarrow 1^-$ along the real axis, we can see that

$$1-\sum_{v=2}^{\infty} v\,(v-1)+v\,[v]_{p,q}^{k}\;\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v \;\ge$$

$$\eta\;\left(1-\sum_{v=2}^{\infty} v\,[v]_{p,q}^{k}\;\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\right) \quad (2.6)$$

Therefore we have the (2 .1).

Conversely, assume that (2.1) holds sharp. Then

$$\left|\frac{(1-\tau)z\left(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z)\right)''+\tau z\left(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z)\right)''}{(1-\tau)\left(\Upsilon_{\tau,p,q}^{\delta,k}\varphi(z)\right)'+\tau\left(\Upsilon_{\tau,p,q}^{\delta,k+1}\varphi(z)\right)'}\right|$$

$$=\left|\frac{\left(-\sum_{v=2}^{\infty} v\,(v-1)\,[v]_{p,q}^{k}\;\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\,\tau]^{\delta}\,c_v\,z^{v-1}\right)}{1-\sum_{v=2}^{\infty} v\,[v]_{p,q}^{k}\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v\,z^{v-1}}\right|$$

$$\le\frac{\left(\sum_{v=2}^{\infty} v(v-1)\,[v]_{p,q}^{k}\;\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\,\tau]^{\delta}\,c_v\right)}{1-\sum_{v=2}^{\infty} v\,[v]_{p,q}^{\,k}\{1+([v]_{p,q}-1)\tau\}\,[1+(v-1)\tau]^{\delta}\,c_v}\le 1-\eta$$

This indicates that the function values .

$$\varphi(z) = \frac{(1-\tau)z(\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z))'' + \tau z(\Upsilon^{\delta,k+1}_{\tau,p,q}\varphi(z))''}{(1-\tau)(\Upsilon^{\delta,k}_{\tau,p,q}\varphi(z))' + \tau(\Upsilon^{\delta,k+1}_{\tau,p,q}\varphi(z))'} \qquad (2.7)$$

Located in the circle its center $\omega = 1$ *and whose radius is* $1 - \eta$. *Hence* $\varphi(z)$ satisfies the condition (1.9)

At last , the function $\varphi(z)$ *given by*

$$\varphi(z) = z - \frac{1-\eta}{[v]^k_{p,q}\ v(v-\eta)\{1+([v]_{p,q}-1)\tau\}[1+(v-1)\tau]^{\delta}\ c_v} z^v \ (v \geq 2) \qquad (2.8)$$

Thus we complete the proof.

Corollary 2.2. Suppose the function $\varphi(z)$ defined by (1 .10), be in the class $\wp^{\delta,k}_{p,q}(\tau,\eta)$. Then

$$c_v \leq \frac{1-\eta}{[v]^k_{p,q}\ v(v-\eta)\{1+([v]_{p,q}-1)\tau\}[1+(v-1)\tau]^{\delta}\ c_v} \quad (v \geq 2) \qquad (2.9)$$

The equality in (2.9) is achieved for the $\varphi(z)$ *given by* (2.8).

3. Inclusion relations

In this part we start by explaining inclusion relation.

Theorem (3.1) : Suppose $0 \leq \eta 1 \leq \eta 2 < 1$, $0 \leq \tau \leq 1$, $0 \leq \delta < 1$and $K, \delta \epsilon \mathbb{N}_0$. *Then*

$$\wp^{\delta,k}_{p,qk}(\tau,\eta_1) \supseteq \wp^{\delta,k}_{p,q}(\tau,\eta_2) . \qquad (3.1)$$

Proof. Let the function $\varphi(z)$ defined by (1 .10) , be in the class m $\wp^{\varepsilon,k}_{p,q}(\tau,\eta_2)$

And let $\eta_1 = \eta_2 - \delta$. consequently, by theorem 2.1, we get

$$\sum_{v=2}^{\infty} [v]^k_{p,q}\ v(v-\eta_2)\ \{1+([v]-1)\tau\}[1+(v-1)\tau]^{\delta}c_v \leq 1-\eta_2 \qquad (3.2)$$

Also

$$\sum_{v=2}^{\infty}[v]^k_{p,q}\ V\{1+([v]-1)\tau\}[1+(v-1)\tau]^{\delta}c_v \leq \frac{1-\eta_2}{2-\eta_2} < 1 . \qquad (3.3)$$

Then

$$\sum_{v=2}^{\infty} [v]^k_{p,q}\ v(v-\eta_1)\{1+([v]_{p,q}-1)\tau\}[1+(v-1)\tau]^{\delta}c_v$$

$$= \sum_{v=2}^{\infty} [v]^k_{p,q}\ v(v-\eta_2)\ \{1+([v]_{p,q}-1)\tau\}[1+(v-1)\tau]^{\delta}c_v$$

$$+\,\delta\ \sum_{v=2}^{\infty} [v]^k_{p,q}\ v\ \{1+([v]_{p,q}-1)\tau\}[1+(v-1)\tau]^{\delta} \leq 1-\eta_1$$

Thus we complete the proof.

Theorem (3.2) . Suppose $0 \leq \eta < 1, 0 \leq \tau_1 \leq \tau_2 \leq 1,$ *and* $\kappa, \delta \in \mathbb{N}_0$.

Then

$$\wp^{\delta,k}_{p,q}(\tau_1,\eta) \supseteq \wp^{\delta,k}_{p,q}(\tau_2,\eta) \qquad (3.4)$$

Proof: Let the function $\varphi(z)$ defined by (1.10), be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta_2)$

Then, by Theorem (2.1) , we get

$$\sum_{v=2}^{\infty}[v]_{p,q}^{k}\ v\,(v-\eta)\{1+([v]-1)\tau_1\}\,[1+(v-1)\tau_1]^{\delta}c_v$$

$$\leq \sum_{v=2}^{\infty}[v]_{p,q}^{k}\ v\,(v-\eta)\,\{1+([v]-1)\tau_2\}\,[1+(v-1)\tau_2]^{\delta}c_v\ \leq 1-\eta$$

Thus completes the proof .

Theorem (3.3): Let $0\leq\eta<1, 0\leq\tau\leq 1, and\ \kappa,\delta\in\mathbb{N}_0\ .Then$

$$\wp_{p,q}^{\delta,k}(\tau,\eta)\supseteq\wp_{p,q}^{\delta,k+1}(\tau,\eta) \qquad (3.5)$$

And

$$\wp_{p,q}^{\delta,k}(\tau,\eta)\ \supseteq\wp_{p,q}^{\delta+1,k}(\tau,\eta) \qquad (3.6)$$

Proof . By Theorem 2.1, we get

$$\sum_{v=2}^{\infty}[v]_{p,q}^{k}\ v\,(v-\eta)\ \ \{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau\,]^{\delta}c_v$$

$$\leq\ \sum_{v=2}^{\infty}\ [v]_{p,q}^{k+1}\ v\,(v-\eta)\,\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau\,]^{\delta}c_v$$
$$\leq 1-\ \eta$$

Also

$$\sum_{v=2}^{\infty}[v]_{p,q}^{k}\ \ v\,(v-\eta)\,\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau\,]^{\delta}c_v$$

$$\leq\ \sum_{v=2}^{\infty}[v]_{p,q}^{k}\ \ v\,(v-\eta)\ \{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau\,]^{\delta+1}c_v\leq 1-\ \eta$$

4. Growth and distortion theorems

Theorem (4.1): Suppose the function $\varphi(z)$ given by (1.10), in the class $\wp_{p,q}^{\delta,k}(\tau\,,\eta\,)$ another, for $|z|$ = r <1,

$$\left|\gamma_{\tau,p,q}^{i,j}\,\varphi(z)\,\right|\ \geq\ \ r\ \ -\ \frac{1-\eta}{[2]_{\ p,q}^{\ \kappa-j}\ 2\,(\,2-\eta\,)\,\{1+\ (\,[2]_{p,q}-1\,)\tau\,\}\,(\,1+\tau\ )^{\,\delta-i}}\ \ r^2 \qquad (4.1)$$

And

$$\left|\gamma_{\tau,p,q}^{i,j}\varphi(z)\,\right|\leq r+\ \frac{1-\eta}{[2]_{p,q}^{\kappa-j}\ 2\,(2-\eta)\,\{1+(\,[v]_{p,q}-1)\ \tau\}\,(\,1+\tau\,)^{\delta-i}}\ \ r^2 \qquad (4.2)$$

$(\,0\leq i\leq\delta;\ 0\leq j\leq\kappa;\quad z\ \in\Omega\,)$.

In (4.1) and (4, 2) realized for the function $\varphi(z)$ as follows ;

$$\varphi(z)=z-\ \frac{1-\eta}{[2]_{p,q}^{\kappa}\ 2\ \ (2-\eta)\,\{1+(\,[2]_{p,q}-1)\ \ \tau\}\,(\,1+\tau\,)^{\delta}}\ z^2\ \ (z=\pm\,r) \qquad (4.3)$$

Proof. Observe that the function $\varphi(z)\in\ \ \wp_{p,q}^{\delta,k}(\tau,\eta)$ if and only if

$$\gamma^{i,j}_{\tau,p,q}\varphi(z) \in \wp^{\delta-i,k-j}_{p,q}(\tau,\eta)$$

and so on

$$\gamma^{i,j}_{\tau,p,q}\varphi(z) = z - \sum_{v=2}^{\infty}[v]^{j}_{p,q}\,[1+(v-1)\tau]^{i} c_v z^{v} \qquad (4.4)$$

. By Theorem (2.1). We get that

$$[2]^{\kappa-j}_{p,q}\,(2-\eta)\,\{1+([2]_{p,q}-1)\tau\}\,(1+\tau)^{\delta-i}\,\sum_{v=2}^{\infty}[v]^{j}_{p,q}\,(1+\tau)^{i}c_v \qquad (4.5)$$

$$\leq \sum_{v=2}^{\infty}[v]^{k}_{p,q}\; v\,(v-\eta)\left\{1+\left([v]_{p,q}-1\right)\tau\right\}[1+(v-1)\tau_2]^{\delta}c_v \;\leq 1-\eta$$

Which suggests ,

$$\sum_{v=2}^{\infty}[v]^{j}_{p,q}\,(1+\tau)^{i}c_v \leq \frac{1-\eta}{[2]^{\kappa-j}_{p,q}\,2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\,(1+\tau)^{\delta-i}} \qquad (4.6)$$

confirmation (4.1) and (4.2) of theorem 4.1 you will now track easily from (4.4) and (4.6).

lastly, we observe that the equivalence (4.1) and (4.2) are done for the function $\varphi(z)$ defined by

$$\gamma^{i,j}_{\tau,p,q}\varphi(z) = z - \frac{1-\eta}{[2]^{\kappa-j}_{p,q}\,2\,(2-\eta)\,\{1+\left([2]_{p,q}-1\right)\tau\}\,(1+\tau)\,[1+(v-1)\tau]^{\delta}}\,z^2 \qquad (4.7)$$

Which completes the proof.

Taking i = j = 0 in Theorem 4.1, directly we have the following corollary.

Corollary 4.2. Suppose the function $\varphi(z)$. *Defined* by (1.10) be in the class $\wp^{\delta,k}_{p,q}(\tau,\eta)$

Therefore, for $|z| = r < 1$

$$|\varphi(z)| \geq r - \frac{1-\eta}{[2]^{\kappa}_{p,q}2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\,(1+\tau)[1+(v-1)\tau]^{\delta}}\,r^2 \qquad (4.8)$$

And

$$|\varphi(z)| \leq r + \frac{1-\eta}{[2]^{\kappa}_{p,q}2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\,(1+\tau)^{\delta}}\,r^2 \quad \text{where, } z \in \mho \qquad (4.9)$$

The parity in (4.8) also (4.9) are fulfilled to the function $\varphi(z)$ define by (4.3).

putting i = $\tau = 1$ while j = 0 in Theorem 4.1 , and benefit from the definition (1.7) we have this corollary.

Corollary (4.3): Suppose a function $\varphi(z)$, knowledge by (1.10), be in the class $\wp^{\delta,k}_{p,q}(\tau,\eta)$.

Then, for $|z| = r < 1$

$$|\varphi'(z)| \geq 1 - \frac{1-\eta}{[2]^{\kappa+1}_{p,q}\,2\,(2-\eta)\,(2)^{\delta-1}}\,r \qquad (4.10)$$

$$|\varphi'(z)| \leq 1 + \frac{1-\eta}{[2]^{\kappa+1}_{p,q}\,2\,(2-\eta)\,(2)^{\delta-1}}\,r \quad \text{where, } z \in \mho \qquad (4.11)$$

With (4.10) also (4.11) are fulfilled to a function $\varphi(z)$ as follows:

$$\varphi(z) = z - \frac{1-\eta}{[2]_{p,q}^{\kappa+1}\, 2\ (2-\eta)\ (2)^{\xi}}\ z^2 \quad (z = \pm r) \qquad (4.12)$$

5. Closure theorems

We shall prove in this section that the class $\wp_{p,q}^{\delta,k}(\tau, \eta)$ is closed beneath convex linear combinations.

Theorem (5.1): The class $\wp_{p,q}^{\delta,k}(\tau,\eta)$ is convex set.

Proof. Suppose the function

$$\varphi_{\alpha}(z) = z - \sum_{v=2}^{\infty} c_{\alpha,v}\, z^v\ (c_{\alpha,v} \geq 0; \alpha = 1,2; z \in \Omega). \qquad (5.1)$$

Be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. It is sufficient to show that the function $\psi(z)$ defined by

$$\psi(z) = \mu\varphi_{1\,(z)} + (1-\mu)\varphi_2\,(z)(0 \leq \mu \leq 1) \qquad (5.2)$$

and in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Because , for $0 \leq \mu \leq 1$,

$$\psi(z) = z - \sum_{v=2}^{\infty}\{\mu c_{1,v} + (1-\mu)c_{2,v}\}z^v \qquad (5.3)$$

By Theorem (2.1), we get

$$\sum_{v=2}^{\infty}[v]_{p,q}^{k}\ v\,(v-\eta)\left\{1 + \left([v]_{p,q} - 1\right)\tau\right\}[1 + (v-1)\tau]^{\delta}\{\mu c_{1,v} + (1-\mu)c_{2,v}\} \leq 1-\eta \quad (5.4)$$

Which suggests that $\psi(z) \in \wp_{p,q}^{\delta,k}(\tau,\eta)$. Therefore $\wp_{p,q}^{\delta,k}(\tau,\eta)$ is convex set.

Theorem (5.2): Let $\varphi_1(z) = z$ also

$$\varphi_v(z) = z - \frac{1-\eta}{[v]_{p,q}^{k}\ v\,(v-\eta)\ \{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\,\tau]^{\delta}}\ z^v \quad (v \geq 2; \kappa, \delta \in \mathbb{N}0) \quad (5.5)$$

For $0 \leq \eta < 1$ and $0 \leq \tau \leq 1$. Then the function $\varphi(z)$ is in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$ if and only if can be expressed in the form:

$$\varphi(z) = \sum_{v=1}^{\infty} \mu_v\, \varphi_v(z) \qquad (5.6)$$

wherever

$$\mu_v \geq 0 (v \geq 1) \text{ and } \sum_{v=1}^{\infty} \mu_v = 1 \qquad (5.7)$$

Proof . Suppose that

$$\varphi(z) = \sum_{v=1}^{\infty} \mu_v\, \varphi_v(z) \qquad (5.8)$$

$$= Z - \frac{1-\eta}{[v]_{p,q}^{k}\ v\,(v-\eta)\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\,\tau]^{\delta}}\ \mu_v\, z^v$$

Then follow it

$$\sum_{v=2}^{\infty} \frac{[v]_{p,q}^{k}\ v\ (v-\eta)\ \{1+([v]_{p,q}-1)\,\tau\}\ [1+(v-1)\,\tau]^{\delta}}{1-\eta} \cdot \frac{1-\eta}{[v]_{p,q}^{k}\ v\,(v-\eta)\left\{1 + \left([v]_{p,q} - 1\right)\tau\right\}[1+(v-1)\tau]^{\delta}}\ \mu_v$$

$$= \sum_{v=2}^{\infty} \mu_v = 1 - \mu_1 \leq 1$$

Hence, by Theorem 2.1 $\varphi(z) \in \wp_{p,q}^{\delta,k}(\tau,\eta)$.

Conversely, suppose that the function $\varphi(z)$ defined by (1.10) belongs to the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Then

$$c_V \leq \frac{1-\eta}{[v]_{p,q}^{k} \ v\,(v-\eta)\,\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau\,]^{\delta}} \quad (v \geq 2, \delta, \kappa \in \mathbb{N}_0)$$

putting

$$\mu_v = \frac{[v]_{p,q}^{k} \ v\,(v-\eta)\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau\,]^{\delta}}{1-\eta} c_v \text{ where, } v \geq 2, \delta, \kappa \in \mathbb{N}_0$$

And

$$\mu_{1=1-\sum_{v=2}^{\infty}\mu_v,}$$

for function $\varphi(z)$ given by (5.6).That is completes proof

6. Radii of Starlikenss, convexity and close - to- convexity ,

We must in this section, determine the radii of starlikeness , convexity and close – to- convexity, , by the functions pertinence to the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$.

Theorem 6.1. Let the function $\varphi(z)$, define by (1.10), be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Then $\varphi(z)$ is close –to-convex of order p ($0 \leq p < 1$) in $|z| < r_1$, where

$$r_1 := inf \left(\frac{(1-p)v^{-1}\,[v]_{p,q}^{k} \ v\,(v-\eta)\,\{1+(\,[v]_{p,q}-1)\,\tau\,\}\,[1+(v-1)\,\tau\,]^{\delta}}{1-\eta}\right)^{1/(v-1)} \quad (v \geq 2). \qquad (6.1)$$

By the extreme function $\varphi(z)$ define by (2.8). The result is true.

Proof:- We must to prove the following:

$$|\,\varphi'(z) - 1\,| \leq 1 - \rho \text{ where } |z| < r_1 ,$$

wherever r_1 is given by (6.1). In fact , definition by (1.10) it means the following:

$$|\,\varphi'(z) - 1| \leq \sum_{v=2}^{\infty} v\,c_v\,|z|^{\,v-1}$$

So thus,

$$|\varphi'(z) - 1| \leq 1 - p$$

Whether,

$$\sum_{v=2}^{\infty} \left(\frac{v}{1-p}\right) c_v\,|z|^{\,v-1} \leq 1. \qquad (6.2)$$

however, through Theory (2.1), (6.2) that is true if

$$\left(\frac{v}{1-p}\right)|z|^{v-1} \leq \frac{[v]_{p,q}^{k}\ v\,(v-\eta)\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau]^{\delta}}{1-\eta}\ , \qquad (6.3)$$

So, if

$$|z| \leq \left(\frac{(1-p)v^{-1}\,[v]_{p,q}^{k}\ v\,(v-\eta)\,\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau]^{\delta}}{1-\eta}\right)^{1/(v-1)} \quad (v \geq 2) \quad (6.4)$$

Theorem (6.1) that is easily by (6.4).

Theorem (6.2): Let the function k φ(z), defined by (1,10), be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Then φ(z) is starlike of order p ($0 \leq p < 1$) in $|z| < r_2$, where

$$r_2 := \inf\left(\frac{(1-p)\,[v]_{p,q}^{k}\ v\,(v-\eta)\,\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau]^{\delta}}{(v-p)\,(1-\eta)}\right)^{1/(v-1)} \quad (v \geq 2) \qquad (6.5)$$

The effect is true, by the external function φ(z) by (2.8).

Proof: We need to prove this is

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)} - 1\right| \leq 1 - p \text{ for } |z| < r_2 .$$

Where r_2 is given by (6.5). In fact definition (1.10) it means this is

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)} - 1\right| \leq \frac{\sum_{v=2}^{\infty} (v-1)\, c_v\, |z|^{v-1}}{\mathbf{1} - \sum_{v=2}^{\infty} c_v |z|^{v-1}}$$

Therefore,

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)} - 1\right| \leq 1 - p,$$

Whether,

$$\sum_{v=2}^{\infty} \left(\frac{v-p}{1-p}\right) c_v |z|^{v-1} \leq 1 \qquad (6.6)$$

however, by Theorem 2.1,(6.6) considered true if

$$\left(\frac{v-p}{1-p}\right) |z|^{v-1} \leq \frac{[v]_{p,q}^{k}\ v\,(v-\eta)\,\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau]^{\delta}}{1-\eta} \qquad (6.7)$$

That is , if

$$|z| \leq \left(\frac{(1-p)\,[v]_{p,q}^{k}\ v\,(v-\eta)\{1+\left([v]_{p,q}-1\right)\tau\}\,[1+(v-1)\tau]^{\delta}}{(v-p)\,(1-\eta)}\right)^{1/(v-1)} \quad (v \geq 2) \qquad (6.8)$$

Theorem 6.2 follows easily from (6.8).

Corollary 6.3. Let the function φ(z), defined by (1,10), be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Then φ(z) is convex of order ρ ($0 \leq p < 1$) in $|z| < r_3$, where

$$R_3 = inf\left(\frac{(1-p)v^{-1}[v]_{p,q}^{k}\ \ v\,(v-\eta)\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau\,]^{\delta}}{(v-p)\,(1-\eta)}\right)^{1/(v-1)} \quad (v \geq 2) \qquad (6.9)$$

Proof: It is enough to show that

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \leq 1-p \text{ ,for } |z| < R_3 .$$

Therefore,

$$\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| = \left|\frac{-\sum_{v=2}^{\infty} v(v-1)a_n z^{v-1}}{1-\sum_{V=2}^{\infty} vc_v z^{v-1}}\right| \leq \frac{\sum_{v=2}^{\infty} v(v-1)c_v|z|^{v-1}}{1-\sum_{v=2}^{\infty} vc_v|z|^{v-1}} \ .$$

The last inequality must bounded by 1- p if

$$\sum_{v=2}^{\infty}\frac{v(v-p)}{1-p}c_v|z|^{v-1} \leq 1. \qquad (6.10)$$

then, by theorem (2.1) , is true if

$$\frac{v(v-p)}{1-p}|z|^{v-1} \leq \frac{1}{c_v} \leq \frac{[v]_{p,q}^{k}\ \ v\,(v-\eta)\,\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau\,]^{\delta}}{1-\eta}$$

which implies that

$$|z| \leq \left(\frac{(1-p)v^{-1}[v]_{p,q}^{k}\ \ (v-\eta)\{1+([v]_{p,q}-1)\,\tau\}\,[1+(v-1)\tau\,]^{\delta}}{(v-p)\,(1-\eta)}\right)^{1/(v-1)} ,v \geq 2. \qquad (6.11)$$

Thus we get the result

7. Integral means inequality

For any two functions φ and ψ analytic in Ʊ, φ is said to be subordinate to ψ in Ʊ,

written φ(z) ≺ $\psi(z)$, if there exists a Schwarz function $\omega(z)$,analytic in Ʊ, with

$$\omega(0) = 0 \text{and} |\,\omega(z)| < 1 \text{ for all } z \in \mho .$$

Such that φ(z) = $\psi\left(\omega\,(z)\right)$ for each z ∈ Ʊ . Moreover, if the function ψ is univalent in ψ, and we get the following equivalence [18]:

$$\varphi(z) \prec \psi(z) \Leftrightarrow \varphi(0) = \psi(0) \text{and } \varphi\,(\mho) \subset \psi\,(\Omega).$$

Orderly to proof integration means inequality of the functions pertinence to the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$ we want the subordination solution for to Littlewood [14].

Lemma(7.1): Whether the functions φ while ψ are analytic in Ʊ together φ(z) ≺ $\psi(z)$, another for $\gamma > 0$ also

$$Z = r\,e^{i\theta} \quad \text{where} \quad 0 < r < 1$$

$$\int_0^{2\pi} |\varphi\,(z)\,|^{\gamma} d\theta \leq \int_0^{2\pi} |\psi(z)|^{\gamma} d\theta\,. \qquad (7.1)$$

Implementation Theorem (2.1) by the external function also Lemma (7.1) . We realize this theory

Theorem (7.2): Suppose $\{ [v]_{p,q}^{k} \; v\,(v-\eta)\{1+([v]_{p,q}-1)\,\tau\}[1+(v-1)\tau\,]^{\,\delta}\}_{v=2}^{\infty}$

be a non decreasing series . If $\varphi \in \wp_{p,q}^{\delta,k}(\tau,\eta)$, then

$$\int_0^{2\pi} |\varphi(r\,e^{i\theta})|^{\gamma} d\theta \leq \int_0^{2\pi} |\,\varphi_{\star}(r\,e^{i\theta})|^{\gamma} d\theta \,.\; (0 < r < 1; \gamma > 0) \qquad (7.2)$$

Wherever

$$\varphi_{\star}(z) = z - \frac{1-\eta}{[2]_{p,q}^{\kappa} 2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\;(1+\tau)^{\delta}}\, z^2 \qquad (7.3)$$

Proof. Let the function $\varphi(z)$, defined by (1,10), be in the class $\wp_{p,q}^{\delta,k}(\tau,\eta)$. Then we want to show that $\int_0^{2\pi}|1-\sum_{v=2}^{\infty} c_v\, z^{v-1}|^{\gamma}\, d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1-\eta}{[2]_{p,q}^{\kappa} 2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\,(1+\tau)^{\delta}}\, z \right|^{\gamma} d\theta \qquad (7.4)$

by stratify Lemma (7.1) , it enough to proof this is

$$1-\sum_{v=2}^{\infty} z^{v-1} < 1 - \frac{1-\eta}{[2]_{p,q}^{\kappa} 2\,(2-\eta)\{1+([2]_{p,q}-1)\,\tau\}\;(1+\tau)^{\delta}}\, z. \qquad (7.5)$$

If the subordination (7.5) holds true, then there exists an analytic function ω with $\omega(0)=0$ and

$|\,\omega(z)| < 1$ such that

$$1-\sum_{v=2}^{\infty} z^{v-1} < 1 - \frac{1-\eta}{[2]_{p,q}^{\kappa} 2\,(2-\eta)\,\{1+([2]_{p,q}-1)\,\tau\}\;(1+\tau)^{\delta}}\, \omega(z). \qquad (7.6)$$

By Theorem 2.1, we have

$$|\,\omega(z)| = \left|\sum_{v=2}^{\infty} \frac{[2]_{p,q}^{\kappa} 2\,(2-\eta)\,\{1+([2l]_{p,q}-1)\,\tau\}\;(1+\tau)^{\delta}}{1-\eta}\, c_v \;\; z^{v-1}\right|$$

$$\leq |z| \;\sum_{v=2}^{\infty} \frac{[v]_{p,q}^{k} \;\; v\,(v-\eta)\,\{\,1+(\,[v]_{p,q}-1\,)\,\tau\}\,[1+(v-1)\,\tau]^{\delta}}{1-\eta}\, c_v \leq |z| < 1$$

That is completes the proof of theorem .

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