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Numerical Solution of Stochastic Heat Equation

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ABSTRACT

This paper will use the numerical methods to solve the stochastic heat equation driven by (Brownian motion, Brownian bridge, and Reflected Brownian motion). The numerical schema depends on the impersonation of the equation solution and includes a stochastic portion of the noise and a partial differential equation. We will apply our methods using daily temperatures in Iraq for the year 2018.

1 . Introduction

In this paper, we want to take a quick look at the numerical solution of stochastic partial differential equations. Working on the numerical solution of stochastic partial differential equations faces many difficulties. On the one hand, we should think of the known problem of numerical solution of inevitable differential equations. On the other hand, we face the challenge of solving the random term.

The SPDEs used as a model in differenced applications. Motivated the mathematics field in particularly through the need to study characterize stochastic the phenomena in normal sciences such as biology, alchemy, physics, and used in statistics and finance science.[2]

Consider the SPDE with the white noise [10]

$$\frac{\partial u}{\partial t}(x,t)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\sigma B(x,t)\,,(x,t)\in[0,T]\times[0,1] \quad \ldots(1)$$

Where $T>0$, and $B(x,t)$ is white noise. The initial condition is getted through continuous function $u_0:[0,1]\to R$ and we consider Dirichlet boundary conditions.

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$$u(0,x) = u_0(x)\ , x \in [0,1]$$
$$u(t,0) = u(t,1) = 0, t \in [0,T]$$

The real-value stochastic domain solution to equation (1) denoted by $\{u(t,x),(t,x) \in [0,T] \times [0,t]\}$. $\sigma > 0$ is constant. We suppose that $\{ B(t,x),(t,x) \in [0,T] \times [0,1]\}$ is a Brownian motion.
Numerical experiments indicating that there is a convergence between the approximate difference in the solution of (1). The aim of the present sheet is to investigate the best possible outcome.

2. Stochastic Integral With Brownian Motion

Definition 1 Brownian motion[1] [2] is a stochastic process $\{B_t : t \geq 0\}$, satisfying

I. $B_0 = 0$,
II. B_t has independent increments.
III. $B_t$is continuous in t.
IV. For any $s = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = t$ the increments $\Delta B_t = B_t - B_s$, $0 \leq s \leq t$ are normal distribution with $\mu = 0$ and $\sigma = t - s$, $B_t - B_s \sim N(0, t-s)$
V.

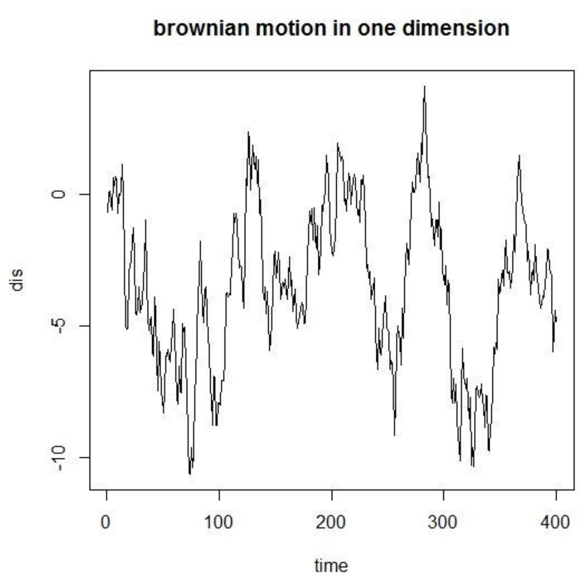

Fig.1 Brownian motion

Definition 2 Brownian Bridge.[3][6]

The process $X_t = B_t - tB_1$ is called the Brownian bridge where $X_t \in (0,1)$ [see 3,p 59]

i.e.,

$$X_t = B_t - tB_t - tB_1 + tB_t$$

$$= (1-t)(B_t - B_0) - t(B_1 - B_t)$$

Because of the increments $B_t - B_0$ and $B_1 - B_t$ are independent and normally distribution with $B_t - B_0 \sim N(0,t), Then\ B_1 - B_t \sim N(0,1-t)$

And the expected and variance are follows [1]

$$E[X_t] = (1-t)E(B_t - B_0) - tE(B_1 - B_t) = 0$$

$$Var[X_t] = (1-t)^2 var[(Bt - B0)] + t^2 var[(B_1 - B_t)]$$

$$= (1-t)^2(t) + t^2(1-t)$$

$$= t(1-t)$$

This can too be stated by saying that the Brownian bridge attached at 0 and 1 is a Gaussian process with drift 0 and variation $t(1-t)$, i.e., $X_t \sim N\big(0, t(1-t)\big)$.

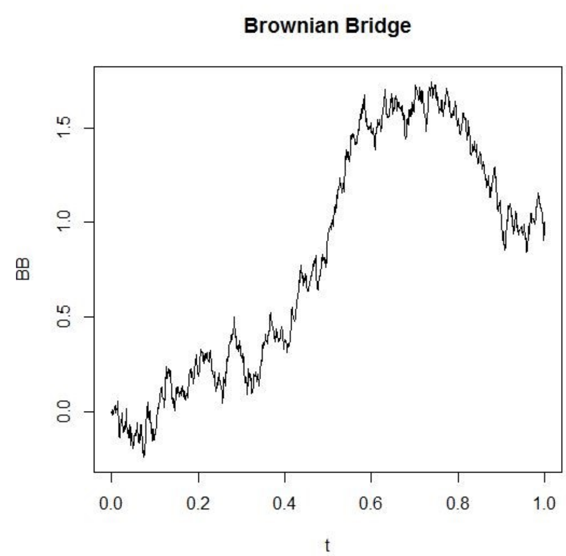

Fig.2 Brownian bridge

Definition 3(The Itô process) [2] [7][10]

A stochastic process X_t , $0 \le t \le T$ is called an Itô process with respect to $\{B_t, P, F_t\}$, where F_t is adapted to B_t, relative to $f(s), g(s)$ if

$$X_t = X_0 + \int_0^t f(s)ds + \int_0^t g(s)dB_s \quad , 0 \le t \le T \ldots (2)$$

Definition 4.[2][7][8]

denote by $L^2(\Omega \times [s,t])$ the space of all adapted stochastic process f_t such that

$$E\left[\int_s^t f_t^2 dt\right] < \infty \ \ for\ s < t \quad \ldots (3)$$

And let $\Delta t = \{s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ and $\Delta B_i = B_{ti} - B_{ti-1}$. we define the Itô integral of B . when $B \in H^2(\Omega \times [s,t])$ with respect to Brownian motion

$$I(f) = \int_s^t f_t dB_t = \lim_{||\Delta n|| \to 0} \sum_{i=1}^{n} f_{t_{i-1}} \Delta_i B \quad \ldots (4)$$

Whenever the limit in probability exists.

3. Stochastic Partial Differential Equations Solutions Types. [1][4][5]

The SPDE of following

$$dX_t = [AX_t + F(X_t)]dt + \sigma dB_{(x,t)} \quad \ldots (5)$$

Have several notions from, solution as we will see below

Definition (5): *G(A)*- valued expected process X_t , $t \in [0, T]$ is named a strong analytical solution of the eq.(3)

$$X_t = \int_0^t [AX_s + F(X_s)]ds + \int_0^t \sigma dB_{(x,s)} \quad \ldots (6)$$

Where $A = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$, F is Laplace location and $B_{(x,s)}$ Brownian motion.

In special, the integration has to be well-define, [3]

Definition (6): F-valued expected process X_t , $t \in [0, T]$ is named a weak analytical solution of the eq.(3) if

$$\langle X_t , \delta\rangle = \int_0^t \left[\langle X_s, A'\delta\rangle + \langle F(X_s), \delta\rangle\right]ds + \int_0^t \langle \delta, \sigma\, dB_s\rangle \quad \ldots (7)$$

For each$\delta \in G(A')$, the integral has to be well-define [4]

Definition (7): F-valued expected process X_t , $t \in [0, T]$ is named a mild analytical solution of the eq.(3) if

$$X_t = \int_0^t \left[e^{A(t-s)}F(X_s)\right] ds + \int_0^t e^{A(t-s)}\sigma dB_s \quad \ldots (8)$$

the integral has to be well-defined, [13]

4. Numerical Solution of Stochastic Heat Equation. [2][5]

The stochastic heat equation given by a space-time white noise

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \sigma B_{(x,t)} \quad , 0 < x < b , t > 0 \ldots (8)$$

Subject to the initial and boundary condition.

$$u(0, t) = u(b, t) = g(t) \quad , t > 0$$

$$u(0,0) = f(x) \qquad 0 < x < b$$

Where $B_{(x,t)}$ is two-dimensional white noise, by using Taylor Theorem we can find sa follows :

$$u_{i,j+1} = u_{i,j} + r[\, u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + k\sigma B_{i,j}$$

When $i = 1,2,3, \ldots , n-1$ and $j = 1,2,3, \ldots , m-1$ and $r = \frac{k}{?^2}$

5. Application of the Temperatures. [2]

In this part, we used temperatures on "SPDEs", some Brownian motion types with additional noise.
We first discussed the "SPDEs" for a reference on the quality of the work already submitted. We look at the local white noise with stochastic heat equation in the field [0,1] during the period [0, T] when T = 1.

Let the "SPDE"

$$dX_t = [A\Delta X_t + F(X_t)]dt + \sigma dB_{(x,t)} \quad \dots (9)$$

for $X_0(x) = 0 \; and \; X_t(0) = X_t(1) = 0 \; for \; x \in (0,1), t \in [0,T)$.

$F(X_t) = 0, \sigma = \sqrt{q}$, $q = j$� , where the $B_{(x,t)}$ here is the white noise.

Then the SPDE

$$dX_t = [A\Delta X_t]dt + \sigma dB_{(x,t)} \quad \dots (10)$$

has singular mild solution $X: [0,T] \times \Omega \to H_B$

example (Example to illustrate the applied side of a paper).

we attempt to discover the numeral solution for SPDE with additional noise, by taking the temperature for 2018.

$$\frac{\partial u}{\partial t}(x,t) = (\partial^2 u/\partial x^2) + \sigma B(x,t)\,, (x,t) \in [0,T] \times [0,1] \quad \dots (11.)$$

With

$$u(0,x) = u_0(x)\,, x \in [0,1]$$

$$u(t,0) = u(t,1) = 0, t \in [0,T]$$

$B(x,t)$ is white noise and $\sigma = \sqrt{q}$.

By employing the equation (2). We find

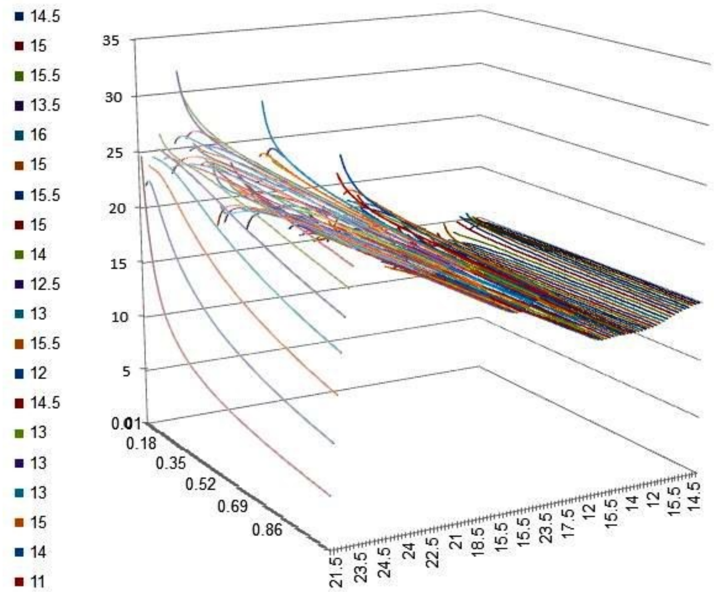

Fig.3 solution of SPDE with Brownian motion

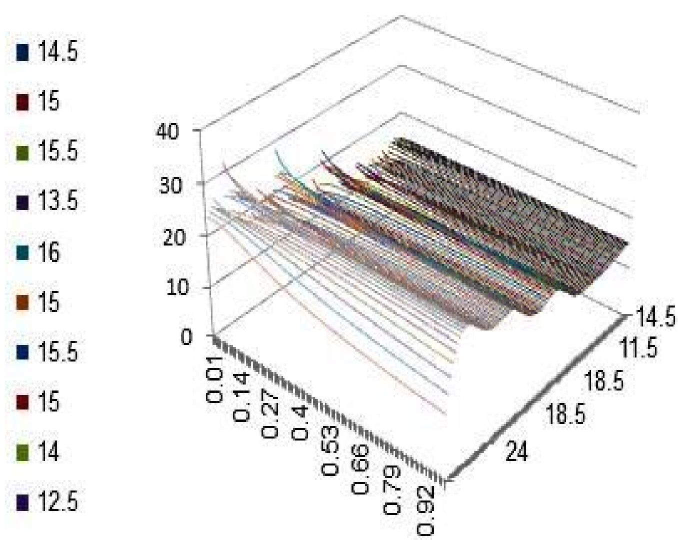

Fig.3 solution of SPDE with Brownian bridge

Table(1)mean square errors

Number	City	MSE	
		Brownian motion	Brownian Bridg
1	Qadisiyah	4.024871066	35.15392897
2	Basrah	4.372799879	30.62404944
3	Erbil	4.574865516	31.24863633

6. Conclusions

Applying a technique for solve the stochasti partial differential equations. And find numerical solutions using heat equation with the added Brownian motion, it was noted that this technique has benefits in general applications, and we find that the random factors have an effect on the temperature of the area under study, through the results shown in the application example.

The main problem we encountered during the application of numerical solutions to random differential equations is the process of noise generation, such as (Brownian motion or Brownian bridge). Typically, noise values should be distributed with an average of zero and a variation of dt, for example, N (0, dt). To achieving this value should be minimal and minimal change to get the most benefits over the specified period, and these problems affect the shape and distribution of noise and show their effect on the final solutions.

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