Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.11 (4) 2019 ,pp Math .74–80

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

GENERALIZED HYERS-ULAM STABILITY OF MIXED TYPE ADDITIVE QUADRATIC FUNCTIONAL EQUATION OF RANDOM DERIVATION IN RANDOM NORMED ALGEBRAS

Shaymaa Alshybani

Department of Mathematics, College of Science, University of Al-Qadisiyah, Al-Diwaniyah, Iraq.

Email: *Shaymaa.farhan@qu.edu.iq , shmhader@gmail.com.*

ARTICLE INFO

Article history:
Received: 18 /08/2019
Rrevised form: 11 /10/2019
Accepted : 21 /10/2019
Available online: 03 /01/2020

Keywords:
generalized Hyers-Ulam stability(H-U stability); additive mapping;
quadratic mapping;
random normed spaces(RN-spaces);
random normed algebras (RN-algebras);
fixed point.

ABSTRACT

By the fixed point method, I have proved the generalized stability of the random derivation of the following additive–quadratic functional equation

$$f(x+2y)+f(x+y)+f(y-x) = f(2x+y)-f(x)+4f(y)+f(-y)$$

in random normed algebras.

MSC-2010: 39B82, 39S40, 54E70.

DOI : 10.29304/jqcm.2019.11.4.647

1. INTRODUCTION

In some cases we cannot associate with the right value as a norm of a vector and random norm is a fit substitute in these issues . The meaning of a RN- space, in which the values of the norms are probability distribution functions instead of numbers, was found by Sherstnev in (Sherstnev, 1963) and developed by (Alsina, Schweizer, & Sklar, 1993).

The stability problem of functional equations which was made by (Ulam, 1960) in 1940. In 1941, Hyers (Hyers, 1941) finished the partial solution of the problem of Ulam. A lot of stability problems for the sorts of functional equations have been found out by huge mathematicians, and there are lots of gorgeous discovers associate with this problem [see, e.g., (Bae & Park, 2015) (Gavruta, 1994) (Alshybani, SH , Mansour Vaezpour, S , & Saadati, R, 2018)]. Also by the other way (fixed point method), the stability problems of the kinds of functional equations have been extensively found out by some seekers . [(see, e.g., (Cadariul & Radu, 2009) (Radu, 2003)]. In this sheet

∗Corresponding author Shaymaa Alshybani

Email addresses:shmhader@gmail.com
Communicated by Qusuay Hatim Egaar

paper we introduce the generalized H-U stability of the random derivation of the next mixed type additive-quadratic functional equation

$$\boldsymbol{f(x+2y)+f(x+y)+f(y-x)=f(2x+y)-f(x)+4f(y)+f(-y)} \qquad (\mathbf{1.1})$$

in RN- algebras by fixed point method.

2. PRELIMINARIES

"Before giving the main result, we present some basic facts related to random normed spaces and some preliminary results. We say $\boldsymbol{f}:\mathbb{R}\to[\mathbf{0},\mathbf{1}]$ is a distribution function if and only if it is a monotone, nondecreasing, left continuous, $\mathbf{inf}_{x\in\mathbb{R}}\boldsymbol{f(x)}=\mathbf{0}$ and $\boldsymbol{sup}_{x\in\mathbb{R}}\boldsymbol{f(x)}=\mathbf{1}$. By Δ^+ we denote a collection of all distribution functions and by $\boldsymbol{D}^+$ is a subset of Δ^+ consisting of all functions $\boldsymbol{f}\in\Delta^+$ for which $\boldsymbol{\mathcal{L}^-f}(+\infty)=\mathbf{1}$, where $\boldsymbol{\mathcal{L}^-f(x)}$ denotes the left limit of the function f at the point x, that is, $\boldsymbol{\mathcal{L}^-f(x)}=\mathbf{lim}_{t\to x^-}\boldsymbol{f(t)}$. The space Δ^+ is partially ordered by the usual point wise ordering of functions ,i,e., $\boldsymbol{F}\le\boldsymbol{G}$ if and only if $\boldsymbol{F(t)}\le\boldsymbol{G(t)}$ for all $\boldsymbol{t}\in\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function $\boldsymbol{H_0}$ given by

$$\boldsymbol{H_0(t)}:=\begin{cases}\mathbf{0}\ \textbf{if}\ \boldsymbol{t}\le\mathbf{0}\\ \mathbf{1}\ \textbf{if}\ \boldsymbol{t}>\mathbf{0}\end{cases}.$$

It is obvious that $\boldsymbol{H_0}\ge\boldsymbol{f}$ **for all** $\boldsymbol{f}\in\boldsymbol{D}^+$."

Definition 2.1. (Cho, Rassias, & Saadati, 2013). "A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions :

(1) T is commutative and associative;
(2) T is continuous;
(3) $T(a,1)=a\ \ \forall\, a\in[0,1]$;
(4) $T(a,b)\ge T(c,d)$ whenewer $a\le c$ and $b\le d$.

Typical examples of continuous t-norms are $T_p(a,b)=ab$, $T_M(a,b)=\min(a,b)$ and $T_L(a,b)=\max(a+b-1,0)$ (the Lukasiewicz t-norm) .

Recall [see (Hadzic & Pap, 2001)] that if T is a t-norm and x_n is a given sequence of numbers in [0,1], $T_{i=1}^{n}\,x_i$ is defined recurrently by $T_{i=1}^{1}x_i=x_1$ and $T_{i=1}^{n}x_i=T(T_{i=1}^{n-1}x_i\,,x_n)$ for $n\ge2$.

It is known [(Hadzic & Pap, 2001)] that for the Lukasiewicz t-norm the following implication holds" :

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}\,x_{n+i}=1\Leftrightarrow\sum_{n=1}^{\infty}(1-x_n)<\infty\,.$$

Definition 2.2. (Sherstnev, 1963) If X is vector space, T is a continuous t-norm, $\mu:X\longrightarrow D^+$, must check the following:

(1) $\mu_x(t)=H_0(t)\ \forall\, t>0$ iff $x=0$;

(2) $\mu_{\alpha x}(t)=\mu_x\left(\frac{t}{|\alpha|}\right)\ \forall x\in X,t>0,\alpha\neq0$;

(3) $\mu_{x+y}(t+s)\ge T\left(\mu_x(t),\mu_y(s)\right)\ \forall\, x,y,z\in X,t,s\ge0$.

Then $(X,\mu\,,T)$ is called random normed space (briefly RN-space).

Definition2.3. (Mihet, Saadati, & Vaezpour, 2010)" Let (X,μ,T) be a $RN-$ space.
(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, $\forall\,\varepsilon>0$ and $\lambda>0$, $\exists$ a positive integer N such that $\mu_{x_n-x}(\varepsilon)>1-\lambda$ whenever $n\ge N$.

(2) A sequence $\{x_n\}$ in X is called Cauchy sequence if, $\forall\,\varepsilon>0$ and $\lambda>0$, $\exists$

a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN − space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X ."

Theorem 2.4. (Schweizer & Sklar, 1983) If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ a.e .

Definition 2.5. A RN- algebra is a RN- space with algebraic structure such that

(4) $\mu_{xy}(ts) \geq \mu_x(t)\mu_y(s) \;\forall\; x, y, z \in X$ and $t, s \geq 0$.

Definition 2.7. Let (X, μ, T_M) and (Y, μ, T_M) *be* RN − algebras . An $\mathbb{R}$ − linear mapping $f: X \to Y$ is called a random derivation (r.d) if

$$f(xy) = f(x)y + xf(y) \qquad \forall\, x, y \in X.$$

Definition 2.8. (Cho, Rassias, & Saadati, 2013) " Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory."

Theorem 2.9 (Diaz & Margolis, 1968)" Let (X, d) be a complete generalized metric spaces (c.g.m.s) and let

$J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either

$$d(j^n x, j^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) $y^*$is the unique fixed ponit *of* J in the set $Y = \{y \in X \setminus d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy) \;\forall\; y \in Y$."

3. GENERALIZED H-U STABILITY OF RANDOM DERIVATION IN RN- ALGBRAS.

We prove generalized H-U stability of the mixed type functional equation (1.1) of random derivation in RN-algebras.

Lemma 3.1. (Kim, 2017) " Let $f : X \to Y$ be a mapping satisfying the functional equation (1.1). Then we have the following

1. If f is an odd mapping , then f is an additive mapping, and
2. If f is an even mapping, then f is an quadratic mapping."

Now we will mention our result

Theorem 3.2. Let (X,μ,T_M) be a RN- algebra and (Y,μ,T_M) be a complete RN- algebra and let $\varphi : X^2 \to [0,\infty)$ a function s.t ∃ a constant

$0 < L < \frac{1}{2}, with\ \ \varphi(x,y) \leq \varphi(2x,2y)$ for all $x,y \in X$.

Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{f(2ry+rx)+f(rx+ry)+f(ry-rx)-r[f(2x+y)+f(x)-3f(y)]}(t) \geq \frac{t}{t+\varphi(x,y)} \quad (3.1)$$

$\forall x,y \in X\ \ ,t > 0.$

$$\mu_{f(xy)-yf(x)-xf(y)}(t) \geq \frac{t}{t+\varphi(x,y)} \quad (3.2)$$

$\forall\ r \in \mathbb{R}$ all $x,y \in X, t > 0.$ Then

$$H(x) \coloneqq \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

exist $\forall\ x \in X$ and defines a r.d $H : X \to Y$ s.t

$$\mu_{f(x)-H(x)}(t) \geq \frac{(2-2L)t}{(2-2L)t+L\varphi(x,0)} \quad (3.3)$$

$\forall\ x \in X, \forall t > 0.$

Proof. Assume that y = 0 and r = 1 in (3.1); we obtain

$$\mu_{2f(x)-f(2x)}(t) \geq \frac{t}{t+\varphi(x,0)}\ \ ,$$

$\forall x \in X,\ t > 0.$ So

$$\mu_{f(x)-\frac{f(2x)}{2}}\left(\frac{t}{2}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}\ , \quad (3.4)$$

$$\mu_{2f\left(\frac{x}{2}\right)-f(x)}(2t) \geq \frac{2t}{2t+\varphi\left(\frac{x}{2},0\right)} \geq \frac{t}{t+\frac{L}{2}\varphi(x,0)}\ , \quad (3.5)$$

$\forall x \in X$ and $t > 0.$ Consider the set

$$E \coloneqq \{g: X \to Y: g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$d_G(g,h) = inf\left\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \frac{t}{t+\varphi(x,0)}\right\},$

$\forall\ x \in X, t > 0.$ Then (E,d_G) is a c.g.m.s (see the proof of [(Mihet, Saadati, & Vaezpour, 2010) [lemma 2.1]) . Now, Suppose the linear mapping $J : E \times E$ defined by

$Jg(x) = 2g\left(\frac{x}{2}\right).$

$\forall x \in X.$ Let $g,h \in E$ be the mappings s.t $d_G(h,g) = \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \frac{t}{t+\varphi(x,0)},$$

$\forall\ x \in X, t\ > 0$ and hence

$$\mu_{Jg(x)-Jh(x)}(L\epsilon t) = \mu_{2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)}(\epsilon Lt)$$

$$= \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L\epsilon t}{2}\right)$$

$$\geq \frac{Lt}{Lt+2\varphi\left(\frac{x}{2},0\right)}$$

$$\geq \frac{t}{t+\varphi(x,0)},$$

$\forall\ x \in X,\ t > 0,$ So

$$d_G(g,h) = \epsilon \Longrightarrow d_G(Jg,Jh) \leq L\epsilon.$$

This means that $d_G(Jg,Jh) \leq Ld_G(g,h)\ \forall\ g,h \in E$. It follows from (3.4) that

$$\mu_{2f\left(\frac{x}{2}\right)-f(x)}(t) \geq \frac{t}{t+\frac{L}{2}\varphi(x,0)},$$

Then

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}\left(\frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,0)},$$

$\forall\ x \in X,\ \forall\, t > 0.$ So $d_G(f,Jf) \leq \frac{L}{2}$, By theorem (2.9) , ∃ a mapping

$H : X \to Y$ satisfying the following :

(1) H is a fixed point of J, i.e.,

$$H\left(\frac{x}{2}\right) = \frac{1}{2}H(x) \qquad (3.6)$$

$\forall\, x \in X$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in E : d(f,g) < \infty\}.$$

this leads to, H is a unique mapping satisfying (3.6) s.t $\exists\ v \in (0,\infty)$

satisfying

$$\mu_{f(x)-H(x)}(vt) \geq \frac{t}{t+\varphi(x,0)}$$

$$\forall x \in X, \forall\, t > 0\,.$$

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. this leads to the equality

$$\lim 2^n f\left(\frac{x}{2^n}\right) = H(x)$$

$$\forall x \in X.$$

(3) $d(f,H) \leq \frac{1}{1-L}d(f,Jf)$, which implies inequality

$$d(f,H) \leq \frac{L}{2-2L}.$$

this leads to the inequality (3.3) holds .

Let $r = 1$ in (3.1) . By (3.1)

$$\mu_{2^n f\left(\frac{2y+x}{2^n}\right)+2^n f\left(\frac{x+y}{2^n}\right)+2^n f\left(\frac{y-x}{2^n}\right)-2^n f\left(\frac{2x+y}{2^n}\right)+2^n f\left(\frac{x}{2^n}\right)-32^n f\left(\frac{y}{2^n}\right)}(2^n t) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

$\forall x,y \in X$, $t > 0$.

$$\mu_{2^n f\left(\frac{2y+x}{2^n}\right)+2^n f\left(\frac{x+y}{2^n}\right)+2^n f\left(\frac{y-x}{2^n}\right)-2^n f\left(\frac{2x+y}{2^n}\right)+2^n f\left(\frac{x}{2^n}\right)-32^n f\left(\frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}$$

$\forall\, x,y \in X$, $t > 0$, $\forall\, n \in N$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$$

$\forall x,y \in X, \forall\ t > 0$, then

$\mu_{H(2y+x)+H(x+y)+H(y-x)-H(2x+y)+H(x)-3H(x)}(t) = 1$

$\forall\, x,y \in X$, $t > 0$. Thus the mapping $H : X \to Y$ is additive. Let

$x = 0$ in (3.1). By (3.1)

$$\mu_{2^n f\left(\frac{2ry}{2^n}\right)+22^n f\left(\frac{ry}{2^n}\right)-4rn^2 f\left(\frac{y}{2^n}\right)}(2^n t) \geq \frac{t}{t+\varphi\left(0,\frac{y}{2^n}\right)}$$

Since

$$\lim_{n\to\infty} \frac{\frac{4t}{2^n}}{\frac{4t}{2^n}+\frac{L^n}{2^n}\varphi(0,y)} = 1$$

and H is additive, then

$$\mu_{H(ry)-rH(y)}(t) = 1$$

$\forall\, r \in \mathbb{R}$, all $y \in X, \forall\, t > 0$. Then the additive mapping $H : X \to Y$ is $\mathbb{R}$-linear .

$$\mu_{4^n f\left(\frac{xy}{2^n 2^n}\right)-2^n f\left(\frac{x}{2^n}\right)y-2^n f\left(\frac{y}{2^n}\right)x}(t) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(0,y)}$$

$\forall\, r \in \mathbb{R}$, all $y \in X, \forall t > 0$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{2^n}\varphi(0,y)} = 1$$

$$\forall x \in X, \forall\ t > 0,$$

$$\mu_{H(xy)-H(x)y-H(y)x}(t) = 1$$

$\forall x, y \in X, \forall\ t > 0$, Thus the mapping $H : X \to Y$ is multiplicative .

Therefore, $\exists!$ r.d $H : X \to Y$ satisfying (3.3). ■

References

[1] Alshybani, SH., Mansour Vaezpour, S., & Saadati, R. (2018). Generalized Hyers-Ulam stability of sextic functional equation in random normed spaces. *J. Computational analysis and applications.*, pp. 172-196.

[2] Alsina, C., Schweizer, A., & Sklar, A. (1993). On thedefinition of a probabilistic normed space. *Aequationes Math*, pp. 91-98.

[3] Bae, J.-H., & Park, W.-G. (2015). On the Ulam stability of the Cauchy-Jensen equation and the additive-quadratic equation. (pp. 710-718). J. Nonlinear Sci. Appl.

[4] Cadariul, L., & Radu, V. (2009). Fixed points and generalized stability for functional equations in abstract spaces. *J. Math. Inequal.*, pp. 463-473.

[5] Diaz, J., & Margolis, B. (1968). A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc.*, pp. 305-309.

[6] Gavruta, P. (1994). A generalization of the Hyers- Ulam- Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.*, pp. 431-436.

[7] Kim, C. (2017). Fuzzy stability of additive-quadratic functional equations . *International journal of mathematical analysis* , pp. 119-131.

[8] Radu, V. (2003). The fixed point alternative and the stability of functional equations. *Sem. Fixed point Theory.*, pp. 91-96.

[9] Sherstnev, A. (1963). On the notion of the a random normed spaces. *Dokl. Akad. Nauk. SSSR. (In Russian)*, pp. 280-283.

[10] Ulam, S. (1960). *A collection of mathematical problems.* New York: Interscience Publishers.