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# Complex of Lascoux in the case of skew-partition (6, 5, 3)/(ȶ , 0, 0) ; where ȶ=1,2

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ABSTRACT

In this paper we study explicitly the complex of Lascaux in the case of skew-partition (6, 5, 3)/ (ȶ, 0, 0) ; where ȶ=1,2 as a diagram and find their terms by using Capelli identities, divided power of the place polarization $\partial_{j}^{(\kappa)}$ and idea of mapping Cone.

MSC

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## 1. Introduction

Let Ϝ be free Ʀ-module over a commutative ring Ʀ with identity and $Ð\,Ϝ = \sum_{i\geq 0} Ð_{i}\,Ϝ$. The divided power algebra can be defined as the graded commutative algebra generated by $\chi^{(i)}$ in degree $2_{i}$ , where $\chi \in Ϝ$, $i \geq 0$.

The complex of characteristic zero in the case of partitions (2,2,2), (3,3,3) and (4,4,3) are studied by authors in [1], [2] and [3], while the authors in [4], [5], [6], [7] and [8] display the complex of characteristic zero as a diagram in the case of partitions (3,3,2), (6,6,3), (6,5,3), (7,6,3) and (8,7,3). By using mapping Cone the authors in [9] and [10] find the resolution of Weyl module for characteristic zero in the case of partition (8,7,3).

In this paper we study the terms of complex of Lascoux and the complex of Lascoux as a diagram by using mapping Cone, [11] in the case of skew-partition (6,5,3)/(1,0,0) in section two, while in section three we find the same as in section two but for the case of skew-partition (6,5,3)/(2,0,0).

The map $\partial_{i\jmath}^{(\kappa)}$ which mean the divided power of the place polarization $\partial_{i\jmath}$ where $\jmath$ must be less than $i$ with its Capelli identities [12], throughout this paper we only used the following identities

$$\partial_{21}^{(\kappa)}\,\partial_{32}^{(\ell)} = \sum_{i\geq 0}(-1)^{\breve{\alpha}}\,\partial_{32}^{(\ell-\breve{\alpha})}\,\partial_{21}^{(\kappa-\breve{\alpha})}\,\partial_{31}^{(\breve{\alpha})} \qquad 1.1$$

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$\partial_{32}^{(\ell)}\,\partial_{21}^{(\kappa)} = \sum_{i\geq 0}(-1)^{\check{\alpha}}\,\partial_{32}^{(\ell-\check{\alpha})}\,\partial_{21}^{(\kappa-\check{\alpha})}\,\partial_{31}^{(\check{\alpha})}$ 1.2

$\partial_{21}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{21}^{(1)}$ 1.3

and

$\partial_{32}^{(1)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{31}^{(1)}\,\partial_{32}^{(1)}$ 1.4

Where $\partial_{ij}$ is the place polarization.

## 2. Complex of Lascoux in the case of skew-partition (6,5,3)/ (1,0,0)

### 2.1 The terms of complex of Lascoux in the case of skew-partition (6,5,3)/(1,0,0)

The terms of the complex of Lascoux in general case $(Þ, q, r; t_1, t_2)$

$$0 \longrightarrow \big((Þ+|t|+2)\big)(q)(r-|t|-2)\big) \xrightarrow{\partial_3} \begin{matrix}\big((Þ+|t|+2)(q-t_1-1)(r-t_2-1)\big)\\ \oplus \\ ((Þ+t_1+1)(q+t_2+1)((r-|t|-2))\end{matrix} \xrightarrow{\partial_2}$$

$$\begin{matrix}\big((Þ)(q+t_2+1)(r-t_2-1)\big)\\ \oplus \\ \big((Þ+t_1+1)(q-t_1-1)(r)\big)\end{matrix} \xrightarrow{\partial_1} \big((Þ)(q)(r)\big) \qquad (2.1)$$

where $|t| = t_1 + t_2$.

In our case i.e (5, 5, 3; 1, 0) the complex of Lascoux has terms as follow

$$Đ_8F \otimes Đ_5F \otimes Đ_0F \longrightarrow \begin{matrix} Đ_8F \otimes Đ_3F \otimes Đ_2F \\ \oplus \\ Đ_7F \otimes Đ_6F \otimes Đ_0F \end{matrix} \longrightarrow \begin{matrix} Đ_7F \otimes Đ_3F \otimes Đ_3F \\ \oplus \\ Đ_5F \otimes Đ_6F \otimes Đ_2F \end{matrix} \longrightarrow Đ_5F \otimes Đ_3F \otimes Đ_3F$$

### 2.2 The complex of Lascoux as a diagram

Consider the following diagram:

$$\begin{array}{ccccc}
Đ_8F \otimes Đ_5F \otimes Đ_0F & \xrightarrow{Þ_1} & Đ_7F \otimes Đ_6F \otimes Đ_0F & \xrightarrow{Þ_2} & Đ_7F \otimes Đ_3F \otimes Đ_3F \\
\Big\downarrow k_1 & \mathfrak{Q} & \Big\downarrow k_2 & \mathfrak{D} & \Big\downarrow k_3 \\
Đ_8F \otimes Đ_3F \otimes Đ_2F & \xrightarrow[q_1]{} & Đ_5F \otimes Đ_6F \otimes Đ_2F & \xrightarrow[q_2]{} & Đ_5F \otimes Đ_5F \otimes Đ_3F
\end{array}$$

So if we define

$Þ_1 : Đ_8F \otimes Đ_5F \otimes Đ_0F \longrightarrow Đ_7F \otimes Đ_6F \otimes Đ_0F$

as $Þ_1(b) = \partial_{21}(b)$ ; wher $b \in Đ_8F \otimes Đ_5F \otimes Đ_0F$

$K_1 : Đ_8F \otimes Đ_5F \otimes Đ_0F \longrightarrow Đ_8F \otimes Đ_3F \otimes Đ_2F$

as $K_1(b) = \partial_{32}^{(2)}(b)$ ; wher $b \in Đ_8F \otimes Đ_5F \otimes Đ_0F$

and

$q_1 : Đ_8F \otimes Đ_3F \otimes Đ_2F \longrightarrow Đ_5F \otimes Đ_6F \otimes Đ_2F$

as $q_1(b) = \partial_{21}^{(3)}(b)$ ; wher $b \in Đ_8F \otimes Đ_3F \otimes Đ_2F$

Now, we have to define the map

$K_2 : Đ_7F \otimes Đ_6F \otimes Đ_0F \longrightarrow Đ_5F \otimes Đ_6F \otimes Đ_2F$ by $\kappa_2 = \frac{1}{3}\,\partial_{32}^{(2)}\,\partial_{21}^{(2)} - \frac{1}{2}\,\partial_{32}\,\partial_{21}\,\partial_{31} + \partial_{31}^{(2)}$

**Remark 2.1:**

The diagram $\mathfrak{Q}$ is commute.

**Proof:**

We have to prove that $q_1 \circ \kappa_1 = \partial_{21}^{(3)} \circ \partial_{32}^{(2)}$

Now if we use Capelli identities we have

$q_1 \circ \kappa_1 =$

$\partial_{32}^{(2)}\,\partial_{21}^{(3)} - \partial_{32}^{(1)}\,\partial_{21}^{(2)}\,\partial_{31} + \partial_{21}\,\partial_{31}^{(2)}$

$\frac{1}{3}\,\partial_{32}^{(2)}\,\partial_{21}^{(2)}\,\partial_{21} - \frac{1}{2}\,\partial_{32}\,\partial_{21}\,\partial_{21}\,\partial_{31} + \partial_{31}^{(2)}\partial_{21}$

$\left[\frac{1}{3}\,\partial_{32}^{(2)}\,\partial_{21}^{(2)} - \frac{1}{2}\,\partial_{32}\,\partial_{21}\,\partial_{31} + \partial_{31}^{(2)}\right] \circ \partial_{21} = \kappa_2 \circ Þ_1$ □

Now we define

$Þ_2 : Đ_7F \otimes Đ_6F \otimes Đ_0F \longrightarrow Đ_7F \otimes Đ_3F \otimes Đ_3F$

as $Þ_2(\mathscr{b}) = \partial_{32}^{(3)}(\mathscr{b})$ ; wher $\mathscr{b} \in Đ_7F \otimes Đ_6F \otimes Đ_0F$

and

$Ҡ_3 : Đ_7F \otimes Đ_3F \otimes Đ_3F \longrightarrow Đ_5F \otimes Đ_5F \otimes Đ_3F$

as $\kappa_3(\mathscr{b}) = \partial_{21}^{(2)}(\mathscr{b})$ ; wher $\mathscr{b} \in Đ_7F \otimes Đ_3F \otimes Đ_3F$

and

$զ_2 : Đ_5F \otimes Đ_6F \otimes Đ_2F \longrightarrow Đ_5F \otimes Đ_5F \otimes Đ_3F$

as $զ_2(\mathscr{b}) = \partial_{32}(\mathscr{b})$ ; wher $\mathscr{b} \in Đ_5F \otimes Đ_6F \otimes Đ_2F$

**Remark 2.2:**

The diagram $\mathfrak{D}$ is commute.

**Proof:**

To prove that $\kappa_3 \circ Þ_2 = զ_2 \circ \kappa_2$

$\kappa_3 \circ Þ_2 = \partial_{32}^{(3)} \partial_{21}^{(2)}$

Now if we use capelli identities we have

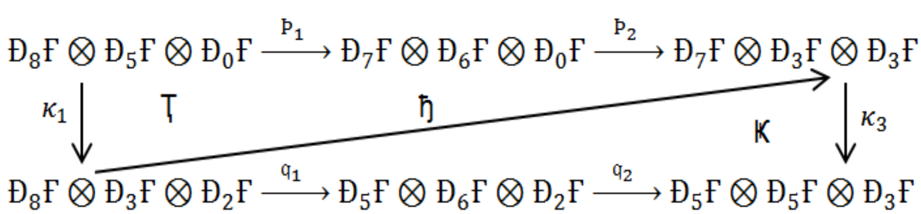
$\partial_{32}^{(3)} \partial_{21}^{(2)} - \partial_{32}^{(2)} \partial_{21} \partial_{31} + \partial_{21} \partial_{31}^{(2)}$

$\frac{1}{3} \partial_{32}^{(2)} \partial_{32} \partial_{21}^{(2)} - \frac{1}{2} \partial_{32} \partial_{32} \partial_{21} \partial_{31} + \partial_{31}^{(2)} \partial_{32}$

$\left[\frac{1}{3} \partial_{32}^{(2)} \partial_{21}^{(2)} - \frac{1}{2} \partial_{32} \partial_{21} \partial_{31} + \partial_{31}^{(2)}\right] \partial_{32}$

$= զ_2 \circ \kappa_2$ □

Now consider the following diagram




Now we define $ђ : Đ_8F \otimes Đ_3F \otimes Đ_2F \longrightarrow Đ_7F \otimes Đ_3F \otimes Đ_3F$

By $ђ = \left[\frac{1}{3} \partial_{21}^{(1)} \partial_{32} + \partial_{31}\right] \circ \partial_{32}^{(2)}$

**Remark: 2.3**

The diagram Ţ is commute:

**Proof:**

To prove Ţ is commute, we need to prove $Þ_2 \circ Þ_1 = ђ \circ \kappa_1$

$ђ \circ \kappa_1 = \left[\frac{1}{3} \partial_{21}^{(1)} \partial_{32} + \partial_{31}\right] \circ \partial_{32}^{(2)}$

$= \partial_{21}^{(1)} \partial_{32}^{(3)} + \partial_{32}^{(2)} \partial_{31}$

a gain by using capelli identities we have.

$ђ \circ \kappa_1 = \partial_{32}^{(3)} \partial_{21}^{(1)} - \partial_{32}^{(2)} \partial_{31} + \partial_{32}^{(2)} \partial_{31}$

$= \partial_{32}^{(3)} \partial_{21}^{(1)}$

$= Þ_2 \circ Þ_1$ □

**Remark:2.4**

The diagram Ҡ is commute:

**Proof:**

$զ_2 \circ զ_1 = \partial_{21}^{(3)} \partial_{32}$

From Capelli identities we have.

$զ_2 \circ զ_1 = \partial_{32} \partial_{21}^{(3)} - \partial_{21}^{(2)} \partial_{31}$

$$= \frac{1}{3}\, \partial_{32}\, \partial_{21}^{(2)}\, \partial_{21} - \partial_{21}^{(2)}\, \partial_{31}$$
$$= \left[\frac{1}{3}\, \partial_{31}\partial_{21} - \partial_{31}\right] \partial_{21}^{(2)}$$
$$= \kappa_3 \circ \mathrm{h} \quad \square$$

Finally, we can define the maps $\sigma_1$, $\sigma_2$ and $\sigma_3$ where;

$$\sigma_3 : Đ_8Ƒ \otimes Đ_5Ƒ \otimes Đ_0Ƒ \longrightarrow \begin{array}{c} Đ_7Ƒ \otimes Đ_6Ƒ \otimes Đ_0Ƒ \\ \oplus \\ Đ_8Ƒ \otimes Đ_3Ƒ \otimes Đ_2Ƒ \end{array} \quad \text{by}$$

- $\sigma_3\,(\chi) = \big(Þ_1\,(\chi), \kappa_1\,(\chi)\big)$ ; *where* $\chi \in Đ_8Ƒ \otimes Đ_5Ƒ \otimes Đ_0Ƒ$

$$\sigma_2 : \begin{array}{c} Đ_7Ƒ \otimes Đ_6Ƒ \otimes Đ_0Ƒ \\ \oplus \\ Đ_8Ƒ \otimes Đ_3Ƒ \otimes Đ_2Ƒ \end{array} \longrightarrow \begin{array}{c} Đ_7Ƒ \otimes Đ_3Ƒ \otimes Đ_3Ƒ \\ \oplus \\ Đ_5Ƒ \otimes Đ_6Ƒ \otimes Đ_2Ƒ \end{array} \quad \text{by}$$

- $\sigma_2\,\big((\chi_1, \chi_2)\big) = \big(Þ_2\,(\chi_1) - ђ(\chi_2)\big), \kappa_1(\chi_2) - \kappa_2\,(\chi_1)$ ; *where* $(\chi_1, \chi_2) \in \begin{array}{c} Đ_7Ƒ \otimes Đ_6Ƒ \otimes Đ_0Ƒ \\ \oplus \\ Đ_8Ƒ \otimes Đ_3Ƒ \otimes Đ_2Ƒ \end{array}$

and

$$\sigma_1 : \begin{array}{c} Đ_7Ƒ \otimes Đ_3Ƒ \otimes Đ_3Ƒ \\ \oplus \\ Đ_5Ƒ \otimes Đ_6Ƒ \otimes Đ_2Ƒ \end{array} \longrightarrow Đ_5Ƒ \otimes Đ_5Ƒ \otimes Đ_3Ƒ$$

by

- $\sigma_1\big((\chi_1, \chi_2)\big) = \big(\kappa_3(\chi_1) + ɋ_2(\chi_2)\big)$ ; where $(\chi_1, \chi_2) \in \begin{array}{c} Đ_7Ƒ \otimes Đ_3Ƒ \otimes Đ_3Ƒ \\ \oplus \\ Đ_5Ƒ \otimes Đ_6Ƒ \otimes Đ_2Ƒ \end{array}$

## Proposition: 2.5

$$0 \longrightarrow Đ_8Ƒ \otimes Đ_5Ƒ \otimes Đ_0Ƒ \xrightarrow{\sigma_3} \begin{array}{c} Đ_7Ƒ \otimes Đ_6Ƒ \otimes Đ_0Ƒ \\ \oplus \\ Đ_8Ƒ \otimes Đ_3Ƒ \otimes Đ_2Ƒ \end{array}$$
$$\xrightarrow{\sigma_2} \begin{array}{c} Đ_7Ƒ \otimes Đ_3Ƒ \otimes Đ_3Ƒ \\ \oplus \\ Đ_5Ƒ \otimes Đ_6Ƒ \otimes Đ_2Ƒ \end{array} \xrightarrow{\sigma_2} Đ_5Ƒ \otimes Đ_5Ƒ \otimes Đ_3Ƒ$$

is complex

**proof:**

Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective from it is definition (see [12]), then we get $\sigma_3$ is injective

$$\sigma_2 \circ \sigma_3(\chi) = \sigma_2\big(Þ_1(\chi), \kappa_1\,(\chi)\big)$$
$$= \sigma_2\,\left(\,\partial_{21}(\chi), \partial_{32}^{(2)}\,(\chi)\right)$$
$$= \left(Þ_2\,\left(\partial_{21}\,(\chi) - ђ\,\left(\partial_{32}^{(2)}\,(\chi)\right)\right), ɋ_1\,\left(\partial_{32}^{(2)}(\chi)\right) - \kappa_2(\partial_{21}\,(\chi)\right)$$

Now

$$Þ_2\,(\partial_{21}\,(\chi)) - \mathrm{h}\left(\partial_{32}^{(2)}\,(\chi)\right) = \left(\,\partial_{32}^{(2)}\,\partial_{21}\right)(\chi) - \left(\tfrac{1}{3}\,\partial_{21}^{(1)}\,\partial_{32} + \partial_{31}\right)\partial_{32}^{(2)}$$
$$= \partial_{32}^{(3)}\,\partial_{21} - \frac{1}{3}\,\partial_{21}\partial_{32}\,\partial_{32}^{(2)} - \partial_{31}\,\partial_{32}^{(2)}$$
$$= \partial_{32}^{(3)}\,\partial_{21} - \partial_{21}\partial_{32}^{(3)} - \partial_{32}^{(2)}\,\partial_{31}$$
$$= \partial_{32}^{(3)}\,\partial_{21} - \partial_{32}^{(3)}\,\partial_{21} + \partial_{32}^{(2)}\partial_{31} - \partial_{32}^{(2)}\partial_{31}$$
$$= 0$$

$$ɋ_1\left(\partial_{32}^{(2)}\,(\chi)\right) - \kappa_2\big(\partial_{21}\,(\chi)\big)$$
$$= \partial_{21}^{(3)}\partial_{32}^{(2)} - \frac{1}{3}\,\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{21} - \frac{1}{2}\,\partial_{32}\partial_{21}\partial_{31}\partial_{21} + \partial_{31}^{(2)}\partial_{21}$$
$$= \partial_{21}^{(3)}\partial_{32}^{(2)} - \partial_{32}^{(2)}\partial_{21}^{(3)} - \partial_{32}\partial_{21}^{(2)}\partial_{31} + \partial_{21}\partial_{31}^{(2)}$$
$$= \partial_{21}^{(3)}\partial_{32}^{(2)} - \partial_{21}^{(3)}\partial_{32}^{(2)} - \partial_{21}^{(2)}\partial_{32}\partial_{31} - \partial_{32}\partial_{21}^{(2)}\partial_{31} + \partial_{21}\partial_{31}^{(2)}$$
$$= \partial_{21}^{(2)}\partial_{32}\partial_{31} - \partial_{21}^{(2)}\partial_{32}\partial_{31} - \partial_{21}\partial_{31}^{(2)} + \partial_{21}\partial_{31}^{(2)} = 0$$

Hence $(\sigma_2 \circ \sigma_3)(\chi) = 0$

$\sigma_1 \circ \sigma_2 = \sigma_1\big(Þ_2(\chi_1) - ђ(\chi_2), զ_1(\chi_2) - \kappa_2(\chi_1)\big)$
$= \sigma_1\Big(\Big(\partial_{32}^{(3)} - \frac{1}{3}\,\partial_{21}\,\partial_{32} - \partial_{31}\Big)(\chi_2), \partial_{21}^{(3)}(\chi_2) - \frac{1}{3}\,\partial_{32}^{(2)}\partial_{21}^{(2)} + \frac{1}{2}\,\partial_{32}\,\partial_{21}\partial_{31} + \partial_{31}^{(2)}\Big)$
$= \Big(\partial_{21}^{(2)}(\chi_1) + \partial_{32}(\chi_2)\Big)\,(\partial_{32}^{(3)} - \frac{1}{3}\,\partial_{21}\partial_{32} - \partial_{31}(\chi_2), \partial_{21}^{(3)}(\chi_2) - \frac{1}{3}\,\partial_{32}^{(2)}\partial_{21}^{(2)}$
$+\frac{1}{2}\,\partial_{32}\partial_{21}\partial_{31} + \partial_{31}^{(2)}\Big)$
$= \partial_{21}^{(2)}\partial_{32}^{(3)} - \partial_{21}^{(3)}\partial_{32} - \partial_{21}^{(2)}\partial_{31} + \partial_{32}\,\partial_{21}^{(3)} - \partial_{32}^{(3)}\,\partial_{21}^{(2)} + \partial_{32}^{(2)}\,\partial_{21}\,\partial_{31}\,\partial_{32}\,\partial_{31}^{(2)}$
$= \partial_{32}^{(3)}\,\partial_{21}^{(2)} - \partial_{32}^{(2)}\,\partial_{21}\,\partial_{31} - \partial_{32}\,\partial_{31}^{(2)} - \partial_{32}\,\partial_{21}^{(3)} + \partial_{21}^{(2)}\partial_{31} - \partial_{21}^{(2)}\,\partial_{31} + \partial_{32}\,\partial_{21}^{(3)} - \partial_{32}^{(3)}$
$\partial_{21}^{(2)} + \partial_{32}^{(2)}\,\partial_{21}\,\partial_{31} + \partial_{32}\,\partial_{31}^{(2)}$
$= 0$ □

## 3. Complex of Lascoux in the case of skew-partition (6, 5, 3)/ (2, 0, 0)

### 3.1 The terms of Lascoux complex in the case of skew-partition (6, 5, 3)/(2, 0, 0).

By applying the sequence (2.1) for our case the Lascaux complex has terms as follows

$$Đ_8F \otimes Đ_2F \otimes Đ_2F \longrightarrow \begin{matrix} Đ_7F \otimes Đ_2F \otimes Đ_3F \\ \oplus \\ Đ_4F \otimes Đ_6F \otimes Đ_2F \end{matrix} \longrightarrow Đ_4F \otimes Đ_5F \otimes Đ_3F$$

### 3.2 The complex of Lascaux as a diagram

$$\begin{array}{ccc} Đ_8F \otimes Đ_2F \otimes Đ_2F & \xrightarrow{\Gamma_1} & Đ_7F \otimes Đ_2F \otimes Đ_3F \\ \psi_1 \downarrow & ß & \downarrow \psi_2 \\ Đ_4F \otimes Đ_6F \otimes Đ_2F & \xrightarrow[\Gamma_2]{} & Đ_4F \otimes Đ_5F \otimes Đ_3F \end{array}$$

And we define the diagram maps as follows

$\psi_1$: $Đ_8F \otimes Đ_2F \otimes Đ_2F \longrightarrow Đ_4F \otimes Đ_6F \otimes Đ_2F$
$as\ \psi_1(b) = \partial_{21}^{(4)}(b)$ ; $where\ b \in Đ_8F \otimes Đ_2F \otimes Đ_2F$
$\psi_2$: $Đ_7F \otimes Đ_2F \otimes Đ_3F \longrightarrow Đ_4F \otimes Đ_5F \otimes Đ_3F$
$as\ \psi_2(b) = \partial_{21}^{(3)}(b)$ ; $where\ b \in Đ_7F \otimes Đ_2F \otimes Đ_3F$
$\Gamma_2$: $Đ_4F \otimes Đ_6F \otimes Đ_2F \longrightarrow Đ_4F \otimes Đ_5F \otimes Đ_3F$
$as\ \psi_2(b) = \partial_{32}(b)$ ; $where\ b \in Đ_4F \otimes Đ_6F \otimes Đ_2F$
$\Gamma_1$: $Đ_8F \otimes Đ_2F \otimes Đ_2F \longrightarrow Đ_7F \otimes Đ_2F \otimes Đ_3F$
$\Gamma_1(b) = \Big(\frac{1}{4}\,\partial_{21}\,\partial_{32} + \partial_{31}\Big)\ (b)$ ; $where\ b \in Đ_8F \otimes Đ_2F \otimes Đ_2F$

**Remark 3.1**

The diagram (ß) is commute

**Proof: -** To prove ß is commute, we need to prove $\psi_2 \circ \Gamma_1 = \Gamma_2 \circ \psi_1$ *so*
$\psi_2 \circ \Gamma_1\,(b) =$
$\partial_{21}^{(3)} \circ \Big(\frac{1}{4}\,\partial_{21}\,\partial_{32} + \partial_{31}\Big)$
$\frac{1}{4}\,\partial_{21}^{(4)}\,\partial_{23} + \partial_{21}^{(3)}\,\partial_{31}$
$\partial_{32}\,\partial_{21}^{(4)} - \partial_{21}^{(3)}\,\partial_{31} + \partial_{21}^{(3)}\,\partial_{31}$
$= \partial_{32}\,\partial_{21}^{(4)}$
$= \Gamma_2 \circ \psi_1\,(b)$. □

Finally by using the mapping cone we can define the maps $\sigma_1, \sigma_2$

$$\sigma_2:\ Đ_8F \otimes Đ_2F \otimes Đ_2F \longrightarrow \begin{matrix} Đ_7F \otimes Đ_2F \otimes Đ_3F \\ \oplus \\ Đ_4F \otimes Đ_6F \otimes Đ_2F \end{matrix}$$

$\sigma_1$: $Đ_7F \otimes Đ_2F \otimes Đ_3F$
$\oplus$ $\longrightarrow$ $Đ_4F \otimes Đ_5F \otimes Đ_3F$
$Đ_4F \otimes Đ_6F \otimes Đ_2F$

**Proposition 3.2**

$$0 \longrightarrow Đ_8F \otimes Đ_2F \otimes Đ_2F \xrightarrow{\sigma_2} \begin{matrix} Đ_7F \otimes Đ_2F \otimes Đ_3F \\ \oplus \\ Đ_4F \otimes Đ_6F \otimes Đ_2F \end{matrix} \xrightarrow{\sigma_1} Đ_4F \otimes Đ_5F \otimes Đ_3F$$

is complex, where

$$\sigma_2\ (\chi, ɏ) = (\psi_2(\chi) + \Gamma_2(ɏ)), \forall\ (\chi, ɏ) \in \begin{matrix} Đ_7F \otimes Đ_2F \otimes Đ_3F \\ \oplus \\ Đ_4F \otimes Đ_6F \otimes Đ_2F \end{matrix}$$

$\sigma_1(\chi) = (\ \Gamma_1\ (\chi), \psi_1(\chi)\ )\ ;\ \forall\, \chi\ \epsilon\ Đ_8F \otimes Đ_2F \otimes Đ_2F$

**Proof**

Since $\partial_{32}^{(1)}$ and $\partial_{21}^{(1)}$ are injective from it is definition see [1], then we have

$\sigma_1 \circ \sigma_1\ (\chi) = \sigma_1\ (\ \Gamma_1\ (\chi), \psi_1(\chi)\ )$

$= \sigma_1 \left( -\left( \frac{1}{4}\ \partial_{21} + \partial_{32} + \partial_{31} \right)(\chi)\ , \partial_{21}^{(4)}\ (\chi) \right)$

$= \left( \partial_{21}^{(3)}\ (\chi) + \partial_{32}(ɏ) \right) \circ \left( \frac{1}{4}\ \partial_{21}\partial_{32} - \partial_{31} \right)(\chi) + \ \partial_{32}\ \partial_{21}^{(4)}\ (ɏ)$

$-\frac{1}{4}\ \partial_{21}^{(3)}\ \partial_{21}\partial_{32} - \partial_{21}^{(3)}\ \partial_{31} + \partial_{32}\ \partial_{21}^{(4)}$

$-\frac{1}{4}\ 4\ \partial_{21}^{(4)}\ \partial_{32} - \partial_{21}^{(3)}\ \partial_{31} + \partial_{21}^{(4)}\ \partial_{32} + \partial_{21}^{(3)}\ \partial_{31} = 0$

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