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On Derivations of Period 2 On Near–Rings

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Abstract

In this paper , we introduce the notion of mapping of period 2 on near-ring N. Also we investigate the existence and properties of derivations and generalized derivations of period 2 on near – rings.

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1. Introduction

A right near – ring (resp.left near ring) is a set N together with two binary operations (+) and (.) such that (i) (N,+) is a group (not necessarily abelian). (ii) (N,.) is a semi group. (iii) For all a,b,c ϵ N ; we have (a+b).c = a.c + b.c (resp. a.(b+c) = a.b + b.c. Through this paper, N will be a zero symmetric left near – ring (i.e., a left near-ring N satisfying the property 0.x=0 for all x∈ N). we will denote the product of any two elements x and y in N ,i.e.; x.y by xy. The symbol Z will denote the multiplicative centre of N, that is Z=$\{x \in N \mid xy = yx \; for \; all \; y \in N\}$. N is called a prime near-ring if xNy = {0} implies either x = 0 or y = 0. It is called semiprime if xNx={0} implies x=0. Near-ring N is called n-torsion free if nx = 0 implies x=0. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal) if NU⊆U (resp.UN⊆U), If U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. A normal subgroup (I,+) of (N, +) is called a right ideal (resp. left ideal) of N if (x + i)y – xy ∈ I for all x,y ∈ N and i ∈ I (resp. xi ∈ I for all i∈I and x∈N). I is called ideal of N if it is both a left ideal as well as a right ideal of N. For terminologies concerning near-rings, we refer to Pilz [6].

The concept of mapping of period 2 has already introduced in ring by Bell . H.E and Daif. M. N [2], in the present paper, motivated by this concept we define a mapping of period 2 on a near-ring N. A mapping of the near ring N into itself is of period 2 on N if $f^2(x) = x$ for all x∈ N. Let U be a non empty subset of N, a mapping f:N → N is of period 2 on U if $f^2(x) = x$ for all x∈ U**.**

An additive mapping d:N ⟶ N is said to be derivation of N if d(xy) = xd(y) + d(x)y , or equivalently, as noted in ([1], lemma 4) that d(xy) = d(x)y + xd(y) for all x,y ϵ N. The concept of derivation has been generalized in several ways by various authors. the notion of generalized derivation has been already introduced and study in near-rings by Öznur Gölbasi [5]**.**

An additive mapping $f:N \to N$ is called a right generalized derivation with associated derivation d if $f(xy) = f(x)y + xd(y)$, for all $x,y \in N$ and f is called a left generalized derivation with associated derivation d if $f(xy) = d(x)y + xf(y)$, for all $x,y \in N$. f is called a generalized derivation with associated derivation d if it is both a right as well as a left generalized derivation with associated derivation d.

Many authors studied the relationship between structure of near – ring N and the behaviour of special mapping on N (see [1],[3],[4],[5],[7] for reference where further references can be found). In the year 2014 Bell. H. E and Daif. M. N. in [2] studied the existence and properties of derivations and generalized derivations of period 2 on prime and semiprime rings. Motivated by these works, we have extended these results in the setting of derivations and generalized derivations of period 2 on certain subset of prime and semiprime near-ring.

2. Preliminaries

The following lemmas are essential for developing the proofs of our main results. .

Lemma 2.1.[7] Let N be near-ring and d be arbitrary derivation of N, then
(i) $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$;
(ii) $(d(x)y + xd(y))z = d(x)yz + xd(y)z$; for all $x , y , z \in N$.

Lemma 2.2.[5] Let N be prime near-ring admitting a derivation d and $a \in N$, if $ad(N) = 0$, then $a =0$.

Lemma 2.3.[1] Let N be a prime near-ring. If U is non-zero semigroup right ideal (resp, semigroup left ideal) and x is an element of N such that $Ux = \{0\}$(resp, $xU = \{0\}$), then $x = 0$.

Lemma 2.4.[5] Let N be a 2–torsion free prime near-ring and d is derivation of N if $d^2= 0$, then $d = 0$.

Lemma 2.5.[7] Let N be a near-ring**.** If N admits a derivation d, then $d(Z)\subseteq Z$.

Lemma 2.6.[5] Let N be a near-ring
(i) Let f be a right generalized derivation of N with associated derivation d. Then $f(xy) = xd(y) + f(x)y$ for all $x , y \in N$.
(ii) Let f be a left generalized derivation of N with associated derivation d. Then $f(xy) = xf(y) + d(x)y$ for all $x , y \in N$.

Lemma 2.7.[5] Let N be a near-ring**.**
(i) Let f be a right generalized derivation of N with associated derivation d. Then $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ for all $x,y,z \in N$.
(ii) Let f be a left generalized derivation of N with associated derivation d. Then $(d(x)y + xf(y))z = f(x)yz + xd(y)z$ for all $x,y,z \in N$.

Lemma 2.8. Let N be near–ring admitting a derivation d of N then
$$2(d(x)y + xd(y))z = 2d(x)yz + 2xd(y)z .$$

Proof. By Lemma 2.1(ii) we have
$$\begin{aligned} 2(d(x)y + xd(y))z &= 2(d(x)yz + xd(y)z) \\ &= d(x)yz + xd(y)z + d(x)yz + xd(y)z \\ &= d(x)yz + (xd(y) + d(x)y)z + xd(y)z \\ &= d(x)yz + (d(x)y + xd(y))z + xd(y)z \\ &= d(x)yz + d(x)yz +xd(y)z + xd(y)z \\ &= 2d(x)yz + 2xd(y)z . \end{aligned}$$

Lemma 2.9. Let N be near – ring admitting a generalized derivation f with associated derivation d then $f(Z) \subseteq Z$.

Proof. Let $z \in Z$, $x \in N$, then $f(zx) = f(xz)$; hence $zd(x) + f(z)x = d(x)z + xf(z)$, therefore $f(z)x = xf(z)$ for all $x \in N$, we conclude that $f(Z) \subseteq Z$.

3. Main Result

In this section we study derivation and generalized derivation when they are of period two.

If we consider f is a derivation of period 2, its clear that f is bijective , so there exists no $x \neq 0$ such that $d(x) = 0$, it follows that the near-ring with 1 admits no derivation which is of period 2 on N. We will show that a semiprime near-ring N admits no derivation of period 2 on N.

Theorem 3.1. Let N be semiprime near ring and U is anon zero semigroup left ideal of N. Then N admits no derivation such that d is of period 2 on U.

Proof. Suppose that there exist derivation d on N such that $d^2(x) = x$ for all $x \in U$. For all $x , y \in U$, $d(x)y \in U$ and the condition $d(x)y = d^2(d(x)y) = d(d^2(x)y + d(x)d(y)) = d(xy + d(x)d(y)) = d(xy) + d(d(x)d(y)) = d(x)y + xd(y) + d^2(x)d(y) + d(x)d^2(y) = d(x)y + 2xd(y) + d(x)y$ yeilds

$d(x)y + 2xd(y) = 0$ for all $x , y \in U$. (1)

Since $xy = d^2(xy) = d(d(xy)) = d(d(x)y + xd(y)) = d^2(x)y + d(x)d(y) + d(x)d(y) + xd^2(y) = xy + d(x)d(y) + d(x)d(y) + xy$, so we get

$xy + 2d(x)d(y) = 0$ for all $x , y \in U$ (2)

Replacing x by rx, where $r \in N$, in (2) we obtain $0 = rxy + 2d(rx)d(y) = rxy + 2(rd(x) + d(r)x)d(y)$. Using Lemma 2.8 in previous relation we get

$rxy + 2rd(x)d(y) + 2d(r)xd(y) = 0$. i.e.;

$0 = rxy + rd(x)d(y) + rd(x)d(y) + 2d(r)xd(y) = r(xy + 2d(x)d(y)) + 2d(r)xd(y)$

Using (2) in previous relation implies

$2d(r)xd(y) = 0$ for all $x , y \in U$ and $r \in N$. (3)

Substituting yr for r in (3), we get $0 = 2d(yr)xd(y) = 2(d(y)r + yd(r))xd(y)$, by lemma 2.8 we get $2d(y)rxd(y) + 2yd(r)xd(y) = 0$, using (3) in previous relation implies $2d(y)rxd(y) = 0$, hence $2xd(y)rxd(y) = 0$. Semiprimeness of N yields $2xd(y) = 0$ for all $x , y \in U$, then relation (1) can be reduced to $d(x)y = 0$ for all $x , y \in U$. Therefore $d(d(x)y) = d^2(x)y + d(x)d(y) = xy + d(x)d(y) = 0$ for all $x , y \in U$. Which together with (2) yields $xy = 0$ for all $x , y \in U$, therefore $xU=0$ for all $x \in U$, by Lemma 2.3 we obtain $x = 0$ for all $x \in U$, so we get a contradiction.

Corollary 3.1. A semiprime near – ring N admits no derivation of period 2 on N.

Any near- ring admits a right generalized derivation of period 2 namely, the identity map and its negative. Moreover, if N has 1 and $c \in N$ such that $c^2=1$ then $f(x) = cx$ define a right generalized derivation of period 2 on N. Now we show that in many prime near-rings there are no other possibilities.

Theorem 3.2. Let N be a 2-torsion free prime near-ring. Let f be a right generalized derivation on N with associated derivation d. If f is of period 2, then $d(Z) = \{0\}$.

Proof. For all x,y ϵN, we have
$xy = f^2(xy) = f(f(x)y + xd(y)) = f(f(x)y) + f(xd(y)) = f^2(x)y + f(x)d(y) + f(x)d(y) + xd^2(y) = xy + 2f(x)d(y) + xd^2(y)$. i.e.;

$2f(x)d(y) + xd^2(y) = 0$ for all x, y ϵN (4)
Replacing x by f(x) in (4) yields
$2xd(y) + f(x)d^2(y) = 0$ for all x , y ϵ N (5)
Letting z ϵ Z and x ϵN and replacing x by xz in (5), we get $2xzd(y) + f(xz)d^2(y) = 0$, by Lemma 2.7, proceeding relation takes the form
$2xzd(y) + f(x)zd^2(y) + xd(z)d^2(y) = 0$. But z ϵ Z, hence we get
$0 = 2zxd(y) + zf(x)d^2(y) + xd(z)d^2(y) = z(2xd(y) + f(x)d^2(y)) + xd(z)d^2(y)$, using (5) in previous relation we get $x\ d(z)d^2(y) = 0$, by Lemma 2.5 we conclude that $d(z)xd^2(y) = 0$ for all x, y ϵ N and z ϵ Z. i.e.; $d(z)\ N\ d^2(y) = \{0\}$, primness of N yields that either $d(z) = 0$ or $d^2(y) = 0$, if $d(z) \neq 0$ then $d^2(y) = 0$ then by Lemma 2.4 we conclude that d = 0. Thus d(Z) = {0} .

Theorem 3.3. let N be a 2-torsion free prime near-ring with 1. If a right generalized derivation f associated with d given by f(x) = x + d(x) (resp f(x) = -x + d(x)) for all x ϵ N, of period 2 on N then f is the identity map (resp , the negative of the identity map) on N.

Proof. Consider the case f(x) = x + d(x) for all x ϵ N. If f is of period 2, we have
$x = f^2(x) = f(f(x) = f(x + d(x)) = x + d(x) + d(x) + d^2(x)$. i.e.;
$2d(x) + d^2(x) = 0$ for all x ϵ N (6)
replacing x by xy in(6), we get

$$\begin{aligned} 0 &= 2d(xy) + d^2(xy) \\ &= 2(d(x)y + xd(y)) + d(xd(y) + d(x)y) \\ &= 2d(x)y + 2xd(y) + xd^2(y) + d(x)d(y) + d(x)d(y) + d^2(x)y \\ &= 2d(x)y + x(2d(y) + d^2(y)) + 2d(x)d(y) + d^2(x)y \end{aligned}$$

for all x , y ϵ N. Using (6) in previous relation implies
$2d(x)y + 2d(x)d(y) + d^2(x)y = 0$ (7)
since 1 ϵ N, hence 1 ϵ Z and by Theorem 2.3 we conclude that d(1) = 0, by lemma 2.7(i) we have (f(x)y + xd(y)z = f(x)yz + xd(y)z (8)
Take x = 1 in (8) we get (f(1)y + d(y))z = f(1)yz + d(y)z; hence we get the partial distributive law (y + d(y))z = yz + d(y)z (9)
replacing y by d(y) in (9), we obtain
$(d(y) + d^2(y))z = d(y)z + d^2(y)z$ (10)
Frome relation (7) we obtain $d(x)y + 2d(x)d(y) = -d(x)y - d^2(x)y$, using (10) in previous relation implies $d(x)y + 2d(x)d(y) = -(d(x) + d^2(x))y$, using (6) in previous relation, we get d(x)y + 2d(x)d(y) = d(x)y. i.e.; 2d(x)d(y) = 0 for all x , y ϵ N, since N is 2-torsion free then d(x)d(y) = 0 for all x , y ϵ N **,** then d(x) d(N) = 0 for all x ϵ N. By lemma 2.2, we conclude that d(x) = 0 for all x ϵ N. Thus f is the identity map on N
A similar argument works if f(x) = -x + d(x) $\forall$ x ϵ N.

Theorem 3.4 Let N be a 2-torsion free prime near-ring with 1. N has no nonzero divisors. If f is a right generalized derivation on N of period 2, then f is the identity map or its negative.

Proof . Note that f(x) = f(1.x) = f(1)x + d(x) for all x ϵ N. (11)
For all x,y ϵN, we have
$xy = f^2(xy) = f(f(x)y + xd(y)) = f(f(x)y) + f(xd(y)) = f^2(x)y + f(x)d(y) + f(x)d(y) + xd^2(y) = xy + 2f(x)d(y) + xd^2(y)$. i.e.;
$2f(x)d(y) + xd^2(y) = 0$ for all x, y ϵN

(12)

Replacing x by f(x) in (4) yields

$2xd(y) + f(x)d^2(y) = 0$ for all x , y ϵ N (13)

Taking x = 1 in (12) and (13). we have

$2f(1)d(y) + d^2(y) = 0$ and $2d(y)+f(1)\, d^2(y)=0$ for all x ϵ N. It follows that $2d(y) - 2d(y)f(1)^2 = 0$, that is $2d(y)(1 - f(1)^2) = 0$, since N is 2-torsion free then $d(y)(1 - f(1)^2) = 0$. But N has no nonzero divisor if d ≠ 0 , we obtain $0 = 1 - f(1)^2 = 1-f(1)+f(1) - f(1)^2 = (1-f(1)) + f(1)(1-f(1)) = (1-f(1)) + (1-f(1))f(1) = (1-f(1))(1+f(1))$, since N has no nonzero divisor then we get either f(1) = 1 or f(1) = -1, from (11) we have f(x) = x + d(x) or f(x) = -x + d(x). So by Theorem 3.3 this would imply d = 0, thus we get

f(x) = f(1)x for all xϵ N.

(14)

Since f is of period 2 then $x = f^2(1)x = x\, f^2(1)$ and $x(1- f^2(1)) = 0$ for all xϵ N. Thus we conclude that f(1) = 1 or f(1) = -1, by (12) we find that f is the identity map or its negative .

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في هذا البحث قدمنا 2 تعريفا للدوال ذات الدورة على الحلقات المقتربة وقمنا بدراسة وجود وخواص الاشتقاقات والاشتقاقات المعممة عندما تكون من الدوال ذات الدورة ٢.