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Fuzzy Neighborhood Systems in Fuzzy Topological Ring

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ABSTRACT

In this paper, we have established a necessary and sufficient condition for a family of fuzzy sets in the form $\{r + r.E\}$ in a ring R that dependent of two binary operations $\{+ , .\}$ to be the family of neighborhoods of an element r of a fuzzy topological ring.

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1 . Introduction

In 1965 [9], Zadeh L. A. gave the definition of fuzziness. After three years C. Chang [2] gave the notion of fuzzy topology. In 1990[1], Ahsanullah and Ganguli, depended on the convergent in fuzzy topological space in the sense of Lowen[7, 8] to introduce the concept of fuzzy nbhd rings which gives the necessary and sufficient condition for a prefilter basis to be fuzzy nbhd prefilter of 0 in fuzzy topological ring. Also they are study the notions of right and left bounded fuzzy set and precompact fuzzy set fuzzy nbhd rings.
In 2009, Deb Ray, A. and Chettri, P [3] introduced fuzzy topology on a ring. Also in [4] they introduced fuzzy continuous function and studied left fuzzy topological ring
We introduce a new study of fuzzy nbhds system in the form $N_0 = r.E$ of 0 and using this family to constructed fuzzy nbhds system in the form $N_r = r + r.E$ of fuzzy point $r \in R$. We using two binary operations $\{+ , .\}$ to constructed a family of neighborhoods of an element r of a fuzzy topological ring. Also we obtain for this family of fuzzy nbhds there exists a unique fuzzy topological ring space.

For rich the paper, some basic concept of fuzzy set and fuzzy topology are given below. The symbol I will denote the closed interval [0,1]. Let R be a non-empty set:

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Definition [9] 1.1

A fuzzy set in R is a map $\partial: R \to I$ and, that is, belonging to I^R(the set of all fuzzy set of R) . Let $E \in I^R$, for every $r \in R$, we expressed by $E(r)$ of the degree of membership of r in R . If $E(r)$ be an element of $\{0, 1\}$, then E is said a crisp set.

Definition [2] 1.2

A class $\mu \in I^R$ of fuzzy set is called a fuzzy topology for R if the following are satisfied

1) $\emptyset, R \in \mu$
2) $\forall E , H \in \mu \to E \wedge H \in \mu$
3) $\forall (E_j)_{j\in J} \in \mu \to \vee_{j\in J} E_j \in \mu$

(R, μ) is called fuzzy topological space. if $A \in \mu$ Then A is fuzzy open and A^c (complement of A)is a fuzzy closed set.

Definition [1, 3 and 4] 1.3

A pair (R , μ) , where R a ring and μ a fuzzy topology on R, is called fuzzy topological ring if the following functions are fuzzy continuous:

1) $R \times R \to R , (r , k) \to r + k$.
2) $R \to R , r \to -r$
3) $R \times R \to R , (r , k) \to r.k$

Definition [4]1.4

A family B of fuzzy nbhds of r_α, for $0 < \alpha \leq 1$, is called a fund. system of fuzzy nbhds of r_α iff for any fuzzy nbhd V of r_α, there is $U \in B$ such that $r_\alpha \leq U \leq V$

Definition [4]1.5

Let R be a ring and μ a FZT on R. Let U and V are fuzzy sets in R. We define $U + V$, $-V$ and $U.V$ as follows

$(U + V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$

$-V(k) = V(-k)$

$(U.V)(k) = \sup_{k=k_1+k_2} min\{U(k_1), V(k_2)\}$

Theorem [4]1.6

If R is a fuzzy topological ring then there is a fundamental system of fuzzy nbhds B of 0 $(0 < \alpha \leq 1)$, such that conditions:

(i)$\forall U \in B$, then $-U \in B$

(ii)$\forall U \in B$, then U is symmetric

(iii)$\forall U, V \in B$, then $U \wedge V \in B$

(iv) $\forall U \in B$, there is $V \in B$ such that $V + V \leq U$

(v) $\forall U \in B$, there is $V \in B$ such that $V.V \leq U$

(vi) $\forall r \in R, \forall U \in B$, there is $V \in B$ such that a $r.V \leq U$ and $V.r \leq U$.

Definition [7] 1.7

(R, μ) is fully stratified fuzzy topology on R if the fuzzy topology μ on R contain all constant fuzzy set

2. Fuzzy Neighborhood Systems

Proposition 2.1

Let R be a ring with identity element 1and E is a fuzzy ring in a ring R and $r \in R$ s.t $E(r) = \max\{E(k), \forall k \in R\}$. Then (1) $r + E = E$ (2) $r.E = E$

Proof

(1) Since $(E,+,.)$ is fuzzy ring implies $(E,+)$ is a commutative group. By theorem 1.7 [5], we get $r+E=E$.

(2) Since $E(r) = \max\{E(k), \forall k \in r\}$ implies

$$
\begin{aligned}
(r.E)(k) &= \sup_{k=k_1.k_2} min\{r(k_1), E(k_2)\} \\
&= min\left\{\sup_{k=k_1.k_2} r(k_1), \sup_{k=k_1.k_2} E(k_2)\right\} \\
&= \min\{r(r), E(r^{-1}k)\} = E(r^{-1}k) \\
&\geq \min\{E(r), E(k)\} = E(k)
\end{aligned}
$$

Also,

$$
\begin{aligned}
E(k) &= E(1.k) = E(r.r^{-1}.k) \\
&\geq \min\{E(r), E(r^{-1}.k)\} \\
&= E(r^{-1}.k) = r.E(k) \quad ,\forall k \in R
\end{aligned}
$$

Thus $r.E=E$

Corollary 2.2

If E is a fuzzy ring in a ring R, then $r+r.E=r.E$, where $(r.E)(r) = \max\{E(k), \forall k \in r\}$.

Proof

By using proposition 2.1, we have $r+r.E=r.E$

Proposition 2.3

Let E, F be a fuzzy open fuzzy closed subset (respectively), and U any fuzzy subset of a fuzzy topological ring R. Suppose $k \in \{r : R(r) = \max\{R(h)\}, \forall h \in R\}$. Then $h+E, E+U, h.E, E.U$, are fuzzy open and $h+F, h.F$ are fuzzy closed.

Proof

Since $(E,+,.)$ is fuzzy topological ring implies $(E,+)$ is a fuzzy topological commutative group. Let $f: R \to R$ defined by $f(r)=h+r$, and then f is fuzzy homeomorphism. By theorem 1.8 [5], we get $f(E)=h+E$ is fuzzy open. Similarly we may prove the remaining parts

Corollary 2.4

Let E be a fuzzy open (fuzzy closed) of a fuzzy topological ring R. Suppose $r \in \{r : E(r) = \max\{E(k)\}, \forall k \in R\}$. Then $r+r.E$ is fuzzy open (fuzzy closed).

Proof

Since E is fuzzy open (fuzzy closed) implies $r.E$ is fuzzy open(fuzzy closed). Then by Proposition 2.2, we have $r+r.E$ is fuzzy open (fuzzy closed).

Proposition 2.5

Let R be a ring with identity 1 and μ be a fully stratified fuzzy topological ring on R. let $r \in R$ be an invertible fuzzy point s.t $r \in \{k: R(k) = \max\{R(h)\}, \forall h \in R$. Then there is $\{r.E: E(0) > 0, E \in \mu\}$ a fundamental system of 0 has the following

(1)$\forall \alpha \in I \backslash \{0\}$, then $r.E_\alpha \in \{r.E\}$.

(2) $\forall r.E \in \{r.E\}$, then $-(r.E) \in \{r.E\}$.

(3) $\forall r.E \in \{r.E\}$, then $r.E$ is symmetric.

(4) If $r.E_1, r.E_2 \in \{r.E\}$, then $r.E_1 \wedge r.E_2 \in \{r.E\}$.

(5) If $r.E_1, r.E_2 \in \{r.E\}$, then $r.E_1 + r.E_2 \in \{r.E\}$.

(6) If $r.E_1, r.E_2 \in \{r.E\}$, then $(r.E_1).(r.E_2) \in \{r.E\}$.

Proof

(1)

Let $\alpha \in I\backslash\{0\}$, then $E_\alpha(k) = \alpha$, $\forall k \in R$.

$$E_\alpha(r^{-1}.k) = \alpha > 0 \quad , \qquad \forall k \in R$$

Therefore

$$r.E_\alpha(0) = \alpha > 0$$

Thus $r.E_\alpha \in \{r.E\}$

(2)

Since $E \in \mu$ then $-E \in \mu$ and $(r.(-E)) \in \{r.E\}$,

Now

$$-(r.E)(k) = (r.E)(-k) = E(r^{-1}(-k))$$
$$= E(-r^{-1}k) = (-E)(r^{-1}.k)$$
$$= (r.(-E))(k)$$

Thus $-(r.E) \in \{r.E\}$

(3)

Let $r.E \in \{r.E\}$, then by (2) we have $-(r.E) \in \{r.E\}$. Let $r.V = r.E \wedge -(r.E)$ then $r.V \in \mu$, $-(r.V) \in \{r.E\}$ and $-(r.V) = r.V \leq r.U$.

Now

$$(r.V)(0) = \min\{r.U(0), -(r.U)(0)\} = \min\{r.U(0), (r.U)(-0)\}$$
$$= \min\{r.U(0), (r.U)(0)\} = r.U(0) > 0$$

Therefor $r.U$ is symmetric

(4)

let $E_1, E_2 \in \{r.E\}$, then $E_1 \wedge E_2 \in \{r.E\}$and $r.(E_1 \wedge E_2) \in \{r.E\}$ … … . (1)

$$(r.E_1 \wedge r.E_2)(k) = \min\{r.E_1(k), r.E_2(k)\} = \min\{E_1(r^{-1}.k), E_2(r^{-1}.k)\} = (E_1 \wedge E_2)(r^{-1}.k)$$
$$= r.(E_1 \wedge E_2)(k) \qquad \ldots \ldots .(2)$$

From (1) and (2) we have $r.E_1 \wedge r.E_2 \in \{r.E\}$

(5)

let $E_1, E_2 \in \{r.E\}$, then $E_1 + E_2 \in \{r.E\}$and $r.(E_1 + E_2) \in \{r.E\}$ … … . (1)

$$(r.E_1 + r.E_2)(k) = \sup_{k=k_1+k_2} \min\{r.E_1(k_1), r.E_2(k_2)\}$$
$$= \sup_{k=k_1+k_2} \min\{E_1(r^{-1}k_1), E_2(r^{-1}k_2)\}$$
$$= (E_1 + E_2)(r^{-1}.k) = r.(E_1 + E_2)(k) \qquad \ldots \ldots .(2)$$

From (1) and (2) we have $r.E_1 + r.E_2 \in \{r.E\}$

(6)

Let $E_1, E_2 \in \{r.E\}$, then $E_1.E_2 \in \{r.E\}$ and $r.(E_1.E_2) \in \{r.E\}$ … … . (1)

$$(r.E_1).(r.E_2)(k) = \sup_{k=k_1.k_2} \min\{r.E_1(k_1), r.E_2(k_2)\}$$
$$= \sup_{k=k_1+k_2} \min\{E_1(r^{-1}k_1), E_2(r^{-1}k_2)\}$$
$$= (E_1.E_2)(r^{-1}.k) = r.(E_1.E_2)(k) \qquad \ldots \ldots .(2)$$

From (1) and (2) we have $r.E_1 + r.E_2 \in \{r.E\}$

Proposition 2.6

Let R be a ring with identity 1 and $r \in R$ be an invertible fuzzy point s.t $r \in \{k: R(k) = \max\{R(h)\}, \ \forall h \in R$. Let $\{r.E: E(0) > 0, E \in \mu\}$ a fundamental system of 0 satisfied the condition 1-6 of theorem 2.5. Then $\{r + r.E\}$be a fundamental system of r.

Proof

(1)

Let $\alpha \in I \backslash \{0\}$, then $r.E_\alpha(k) = \alpha$, $\forall k \in R$.

$$(r + r.E_\alpha)(r) = r.E_\alpha(r - r)$$
$$= r.E_\alpha(0) = \alpha > 0 \quad , \qquad \forall r \in R$$

$(r + r.E_\alpha)(0) = \alpha > 0$

Thus $(r + r.E_\alpha) \in \{r + r.E\}$

(2)

Let $(r + r.E) \in \{r + r.E\}$ then $r.E \in \{r.E\}$ implies $-(r.E) \in \{r.E\}$ and by corollary 2.4 we have $(r - (r.E) \in \{r + r.E\}$,

$$-(r + r.E)(r) = (r + r.E)(-r) = \sup min\,\{r(-r), (r.E)(-r)\} = supmin\{r(r), -(r.E)(r)\} = (r - (r.E)(r)$$

Also

$$-(r + r.E)(r) = (r + r.E)(-r) = (r.E)(r - r) = (r.E)(0) > 0$$

Thus $-(r + r.E) \in \{r + r.E\}$

(3)

Let $(r + r.E) \in \{r + r.E\}$ then by (2) $-(r + r.E) \in \{r + r.E\}$ and
$V = -(r + r.E) \wedge (r + r.E) \in \{r + r.E\}$, then we have $-V \in \mu$ also

$$V(r) = \min\{-(r + r.E)(r), (r + r.E)(r)\}$$
$$= \min\{(r + r.E)(-r), (r + r.E)(r)\}$$
$$= \min\{(r + r.E)(r), (r + r.E)(r)\}$$
$$= (r + r.E)(r)$$

Thus $(r + r.E)$ is symmetric

(4)

Let $r.E_1, r.E_2 \in \{r.E\}$, then $r.E_1 \wedge r.E_2 \in \{r.E\}$ and
$r + (r.E_1 \wedge r.E_2) \in \{r + r.E\}$ $\quad \dots \dots .(1)$,

$$\big((r + r.E_1) \wedge (r + r.E_2)\big)(k) = \min\{(r + r.E_1)(k), (r + r.E_2)(k)\} = \min\{r.E_1((k - r), r.E_2((k - r)\}$$
$$= (r.E_1 \wedge r.E_2)(k - r) = r + (r.E_1 \wedge r.E_2)(k) \quad \dots \dots .(2)$$

From (1) and (2) we have $(r + r.E_1) \wedge (r + r.E_2) \in \{r + r.E\}$

(5)

Let $r.E_1, r.E_2 \in \{r.E\}$, then $r.E_1 + r.E_2 \in \{r.E\}$ and
$(r + r) + (r.E_1 + r.E_2) \in \{r + r.E\}$ $\quad \dots \dots \dots \dots \dots \dots \dots \dots \dots ..(1)$

$$\big((r + r.E_1) + (r + r.E_2)\big)(k) = \sup_{k = k_1 + k_2} min\{(r + r.E_1)(k_1), (r + r.E_2)(k_2)\}$$
$$= \sup_{k = k_1 + k_2} min\{(r.E_1)(k_1 - r), (r.E_2)(k_2 - r)\} = (r.E_1 + r.E_2)((k_1 - r) + (k_2 - r))$$
$$= (r.E_1 + r.E_2)((k_1 + k_2 - (r + r))$$
$$= (r + r) + (r.E_1 + r.E_2)(k_1 + k_2)$$
$$= (r + r) + (r.E_1 + r.E_2)(k) \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .(2)$$

From (1) and (2) we have$(r + r.E_1) + (r + r.E_2) \in \{r + r.E\}$

(6)

let $r.E_1, r.E_2 \in \{r.E\}$, then$(r.E_1).(r.E_2) \in \{r.E\}$ and $r + ((r.E_1).(r.E_2)) \in \{r + r.E\}$,

$$\big((r + r.E_1).(r + r.E_2)\big)(k) = \sup_{k=k_1+k_2} min\{(r + r.E_1)(k_1), (r + r.E_2)(k_2)\}$$
$$= \sup_{k=k_1.k_2} min\{(r.E_1)(k_1 - r), (r.E_2)(k_2 - r)\} = (r.E_1).(r.E_2)((k_1.k_2) - r)$$
$$= (r + \big((r.E_1).(r.E_2)\big)(k)$$

Thus $(r + r.E_1).(r + r.E_2) \in \{r + r.E\}$

Theorem 2.7

Let R be a ring with identity 1 and $r \in R$ be an invertible fuzzy point s.t $r \in \{k: R(k) = \max\{R(h)\}, \ \forall h \in R$. Let $N_r = \{r + r.E\}$ a fundamental system of r satisfied the conditions of theorem 2.6. Then there exists a unique fuzzy topology μ on R s.t (R, μ) is fuzzy topological ring space.

Proof

We want to prove that (R, μ) is a fuzzy topological ring. We claim the following mappings are fuzzy continuous.

(1) $g: R \times R \to R, g(r, k) = r + k$

(2) $g: R \times R \to R, g(r, k) = r.k$

(3) $g: R \to R, g(r) = -r,$

(1)

Let $U \in N_{r+k}$. There exists $(r + k).E \in \{r.E_0\}$ s.t $U = (r + k) + (r + k)E$.

In the other hand

$$g^{-1}(U)(r, k) = U\big(g(r, k)\big) = U(r + k)$$
$$= \big((r + k) + (r + k)E\big)(r + k) = ((r + k).E)(0) > 0$$

Let, $r + r.E \in \{r + r.E\}$ and $k + k.E \in \{k + k.E\}$, we have

$$(r + r.E) + (k + k.E) = r + k + r.E + k.E$$
$$= r + k + (r + k).E$$
$$= U$$

i.e. $g(r + r.E) \times (k + k.E) = (r + r.E) + (k + k.E) = U$

Thus $g: R \times R \to R, (r, k) \to r + k$ is fuzzy continuous

(2)

Let $U \in N_{r.k}$. There exists $(r.k).E \in \{r.E_0\}$ s.t $U = (r.k) + (r.k)E$.

In the other hand

$$g^{-1}(U)(r, k) = U\big(g(r, k)\big) = U(r.k)$$
$$= \big((r.k) + (r.k)E\big)(r.k) = ((r.k).E)(0) > 0$$

since $(r.k).E \in \{r.E_0\}$ then by theorem 2.6 there exists

$$(r.k).E_1 \in \{r.E_0\} \text{ s.t } ((r.k).E_1).((r.k).E_1) \le (r.k).E$$

implies,

$$(r.k) + ((r.k).E_1).((r.k).E_1) \le (r.k) + (r.k).E$$

Since

$(r.k).E_1 \in \{r.E_0\}$, then by theorem 2.6 there is $(r.k).E_2 \in \{r.E_0\}$such that

$$(r.k).E_2 + (r.k).E_2 + (r.k).E_2 \le (r.K).E_1 \text{ and}$$
$$(r + (r.E_2)) \in \{r + r.E\}, (k + (k.E_2)) \in \{k + k.E\}$$

Now

$$\big(r + (r.E_2)\big).\big(k + (k.E_2)\big) = r.k + r.(k.E_2) + (r.E_2).k + (r.E_2)(k.E_2)$$
$$= r.k + (r.k).E_2 + (r.E_2).k) + (r.k).E_2$$
$$\le r.k + (r.k).E_1 \le r.k + (r.k).E = U$$

Thus $g: R \times R \to R, (r, k) \to r.k$ is fuzzy continuous

Fuzzy continuity of $r \to -r$ follows from the symmetric condition

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