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On DG − Topological operators Associated with Digraphs

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ABSTRACT

In this paper ,we introduced new concepts of $\boldsymbol{DG}-$ topological operators such as $\boldsymbol{DG}-$ closuer ,$\boldsymbol{DG}-$ kernel ,$\boldsymbol{DG}-$ cor, and $\boldsymbol{DG}-$intiorer and we investigated certain types between the digraphs and the topology by associated new topology on a digraph named $\boldsymbol{DG}-$ topological space .

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1 . Introduction

Graph theory is an important mathematical tool in many subjects play an important role in discrete mathematics for two reasons .Firstly , the graph are mathematically elegant in theory. Although are simple relation graphs , they can be used to represent topographic space , harmonic objects , and many other mathematical graphs . The second reason many concepts will be very useful from a practical perspective when they are empirically represented by graphs . There is a relation between topology and digraphs and therefor many authors studied this relations . In 1967,J.N.Evans , etal. [4] proved very important relation which find a one to one correspondence between them .In 1968 , T.N. Bhargav and T.J.Ahlporen [7] studied and investigated some properties of topological spaces and digraphs by showed that each digraph defines a unique topology . In 1972 , R.N. Lieberman [8]defined two topologies on the set of vertices of every digraph called left $E-$ topology and the right $E-$ topology . In 2010 , C. Marijuan [2] associated each topology τ to each digraph D by constructed a subbasis of closed sets for τ such that the set of vertices adjacent to u in D ,for all vertices u from this subbasis and he associated a digraph to a topology by specialization relation between points in a topological space such that for any two points $x, y \in X, x$ is adjacent to

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y iff $x \in \overline{\{y\}}$. In 2013 , A . H .Mahdi and S . N . Al-khafaji [1] constructed a topology on finite undirected graphs and a topology on subgraph on the set of edges and discussed the connectedness of each of the graph and the topological space , that induced that finite undirected graph . In 2015 , Khalid Al'Dzhabri in [6] found the correspondence between the finite topology and the graph of finite reflexive transitive relation . In 2018 , K .A .Abdu and A . Kilicman in [5] fended new certain types of topological space which associated with digraphs called compatible and incompatible edges topologies .
In our work , we introduced and studied new concepts of topological operators such as $DG-$ closure , $DG-$ kernel , $DG-$ core and $DG-$ interior . Firstly , we introduced a relationship between topology and digraph named $DG-$ topological space induced by new open set called $DG-$ open set and the topology associated with the digraph $D=(V,E)$ denoted by τ_{DG} and $\tau_{DG}=\{A: A\ is\ DG-open\ set\}$.

A subset A of V is called $DG-$ open if for every $u \in A$ and an arc $vu \in E$ then $v \in A$. The pair (V, τ_{DG}) is called the $DG-$ topological space . In addition , we investigated some properties of these concepts .

2 . Basic definitions and facts

In this section , we recalled that some definitions and facts and update another definition by using our new concepts .

Definition2.1[3]: A digraph (directed graph) is a set V of vertices and a set $E$of order pairs of vertices such that $E \subseteq V \times V$ and denoted by $D=(V,E)$ or simply by $D(V)$ if the set $E$is fixed .

Definition2.2[3]: Let $\acute{V} \subseteq V$,the digraph $D=(\acute{V}, E \cap \acute{V} \times \acute{V})$ denoted simply by $D(\acute{V})$,is a subdigraph of the digraph $D=(V,E)$.

Definintion2.3[3]: An element of E is called an arc of the digraph $D=(V,E)$and it is denoted by uv ;and said to be an arc from u to v.

Definition2.4[3]: An arc from u_i to u_i is called a loop at u_i and denoted by $u_i u_i \in E$.

Definition2.5 [3]:A directed path (dipath) of length L from u_i to u_j is an ordered $(L+1)-$tuple of vertices of $D=(V,E)$, $u_i, u_{k_1}, u_{k_2}, u_{k_3}, \dots, u_{k_{(L-1)}}, u_j$ in which L is a positive integer and $\{u_i u_{k1}, u_{k1} u_{k2}, u_{k2} u_{k3}, u_{k(L-1)} u_j\}$ is a subset of the arc set E of $D=(V,E)$ .The vertex u_i is called the initial vertex ,the vertices $u_{k_1}, u_{k_2}, \dots, u_{k_{(L-1)}}$ is called intermediate vertices ,and u_j is called the terminal vertex of the digraph .

Definition2.6: If there exists a dipath from u_i to u_j in $D=(V,E)$,we say that u_i indegree to u_j or u_j outdegree from u_i and denoted by $\psi(i,j)$ .The ordered pair (u_i, u_j) is called an indegrees pair . If u_i is not indegree to u_j,denoted by $\tilde{\psi}(i,j)$.

Definition2.7: If both $\psi(i,j)$ and $\psi(j,i)$ that is if u_i is indegree to u_j and u_j indegree to u_i we say that u_i and u_j are symmetrically indegrable and denoted by $\psi^*(i,j)$

Remarrk2.8: Note that the relation ψ^* is an equivalence relation on the set V in $D=(V,E)$.

Definition2.9[3]: A digraph $D=(V,E)$ is called a transitive digraph if $uv \in E$ and $vw \in E$ implies that $uw \in E$.

Now by using $\psi(i,j)$ in the definition 2.6 we give the following definitions.

Definition2.10: Let $D=(V,E)$ be a digraph . Then D is called

i) $\psi-$strongly connected , if $\psi^*(i,j)$, for every u_i and u_j in V.

ii) $\psi-$unilaterally connected , if $\psi(i,j)$ or $\psi(j,i)$ for every u_i and u_j in V.

iii)ψ −weakly connected , if $D = (V, E \cup E^c)$ is ψ −strongly connected where $E^c = \{vu: uv \in E\}$.

iv) ψ −disconnected if it is not even ψ −weakly connected.

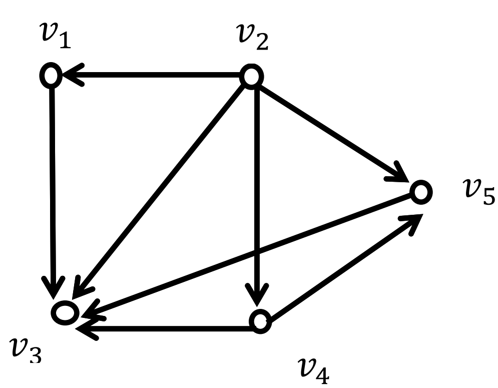
3 . On DG − Operators Topology Associated with Digraphs

In this section we introduced DG −topological space by define new concept called DG − open set . A topology may be determined on a set V defining certain subset of V to be open with respect to a digraph $D(V, E)$, and we introduced concepts DG − closure , DG −kernel , DG − core , DG − limit point , and DG − interior operators to investigate the connectedness of the digraph with these concepts and some properties we will be study in this section .

Definition3.1: Let $D = (V, E)$ be a digraph . A subset A of V is called DG − open set if for $u \in A$ and an arc $vu \in E$, then $v \in A$.

Remark 3.2: From the definition above the topology associated with the digraph $D = (V, E)$ is denoted by τ_{DG} where $\tau_{DG} = \{A: A \text{ is } DG - \text{open set}\}$ and (V, τ_{DG}) is called DG − topological space .

Example3.3: Consider the digraph $D = (V, E) where\ V = \{v_1, v_2, v_3, v_4, v_5\}$



And the topology corresponding to the above digraph $\tau_{DG} = \{\emptyset, V, \{v_2\}$, $\{v_1, v_2\}$,$\{v_2, v_4\}$,$\{v_1, v_2, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_1, v_2, v_4, v_5\}$ }

Theorem3.4: Let $D = (V, E)$ be a digraph . Then (V, τ_{DG}) is a topology on the set V associated with the digraph $D = (V, E)$.

Proof:

[O1] Notice $\emptyset$ and $V \in \tau_{DG}$

[O2] Let $\{U_\alpha\}$ be a collection of subsets of V in τ_{DG} ,and let $u \in \bigcup_\alpha U_\alpha$ and $uv \in E$. Then $\exists U_{\alpha_0} \in \{U_\alpha\}$ with $uv \in E$. This implies that $v \in U$, so $\bigcup_\alpha U_\alpha \in \tau_{DG}$.

[O3] Let $U_i \in \tau_{DG}, \forall i = 1,2,3 \dots, n$. Now let $u \in \bigcap_{i=1}^{n} U_i$ and $vu \in E$,then $u \in U_i$ for all i and $v \in U_i$ and therefore the family $\bigcap_{i=1}^{n} U_i \in \tau_{DG}$ .Hence τ_{DG} is topology on V .

Theorem 3.5: Every DG − topological space is Alexandroff space.

Proof: To prove that the arbitrary intersection of DG − open sets is DG − open set . Let U and W two DG − open sets and $u \in U \cap W$, $uv \in E$, to prove that $v \in U \cap W$ since $u \in U \cap W$,then $u \in U$, $u \in W$, and $uv \in E$, since each of them U and W are DG − open sets ,then $v \in U$ and $v \in W$, therefore $v \in U \cap W$, hence $U \cap W$ is DG − open set.

Proposition 3.6: Let $D = (V, E)$ be a digraph . A subset A of V is DG −open set if and only if $v \in A$ and $u \in A^c$, implies $uv \notin E$.

Proof : Let A be a DG −open set then $u \in A$, and $uv \in E$ and we have $v \in A$ and that is means that if $u \notin A$, then $v \in A$ and $uv \notin E$ .Now suppose that $v \in A$ and $u \in A^c$, then $uv \notin E$ and to prove that A is a DG − open set since $v \in A$ and $u \in A^c$, then $uv \notin E$ and that means if $uv \in E$ and $v \in A$ then we have $u \in A$ and hence A is a DG −open set.

Definition3.7: The complement of DG −open set is called DG −closed set.

Remark3.8:A subset A of V is DG −closed if and only if $u_i \in A$ and $u_j \in A^c$ implies that $u_i u_j \notin E$,That is a subset A of V is called DG −closed if there dose not exists an arc from A to A^c in $D = (V, E)$.

Proposition 3.9: Let $D = (V, E)$,and let u_i and u_j be fixed vertices of a set V. Then u_i is indegree to u_j if and only if for each subset $A \subseteq V$ such that $u_i \in A$,and $u_j \notin A$,there exists an arc from A to A^c.

Proof: If $u_i = u_j$, the proposition is obviously .Now suppose that u_i is indegree to u_j for $u_i \neq u_j$. Thus there exists a dipath of finite length from u_i to u_j .Let A be a subset of V such that $u_i \in A$ and $u_j \notin A$. From definition 2.5, a dipath is an ordered tuple of finite length , let u_k ,the k th vertex in this tuple , be the first vertex of this dipath which is not in A and other vertices belong to A,and also $u_k \neq u_i$. Thus $u_{k-1} \in A$ and we have the required arc namely $u_{k-1}u_k$ .And $A \subseteq V$ with $u_i \in A$ and $u_j \notin A$,and suppose that exists an arc from A to A^c .Now form the set $A_i = \{u_d : \psi(i, d)\}$ .Suppose that $u_j \notin A_i$,then by hypothesis , there exists an arc from A_i to A_i^c , say $u_r u_s$ such that $u_r \in A_i$ and $u_s \in A_i^c$ .But u_i is indegree to u_r and this means $u_i = u_r$ or there exists a finite dipath from u_i to u_r and thus there exists a finite dipath from u_i to u_s which containing the vertex u_r .Thus u_i is indegree to u_s, and therefore $u_s \in A_i$ which is a contradiction. Hence $u_j \in A_i$ i.e u_i indegree to u_j.

Proposition3.10: Let $D = (V, E)$ u_i and u_j be fixed vertices of the V.Then u_i is indegree to u_j if and only if each DG − closed set W ,such that $u_i \in W$,implies $u_j \in W$;or equivalently each DG −open set M ,such that $u_j \in M$,then $u_i \in M$.

Proof : Let u_i and u_j be two vertices of the V and suppose that u_i is indegree to u_j . Let W be an DG −closed set such that $u_i \in W$ .If $u_j \in W^c$, then by Proposition 3.9 there exists an arc from W to W^c ,which implies that W is not DG − closed set. Hence $u_j \in W$. Now suppose W that a DG − closed set such that $u_i \in W$ implies $u_j \in W$ this means there does not exist an DG − open set W^c such that $u_j \in W^c$ but $u_i \notin W^c$ .Now assume that each DG − open set M such that $u_j \in M$, then $u_i \in M$ and so each DG − open set M such that $u_j \in M$ but $u_i \notin M$ is not DG − open set .Hence M^c is not DG − closed set .Thus for each set M^c , there exists an arc from M^c to M and by Proposition 3.9 u_i is indegree to u_j .

Proposition 3.11: Let (V, τ_{DG}) be a topological space associated with the digraph $D = (V, E)$. Then $D = (V, E)$ is ψ − strongly connected if and only if τ_{DG} is an indiscrete topology .

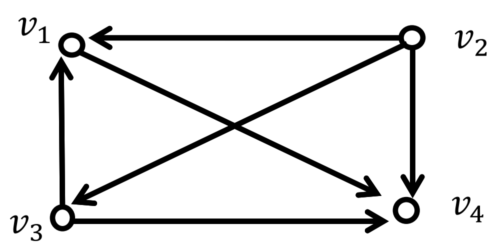
Proof : Suppose that $D = (V, E)$ is ψ −strongly connected and u_i and u_j are arbitrary vertices of V ,then $\psi(i, j)$; i.e u_i is in degree to u_j .If A is any DG − open set A such that $u_j \in A$. Then $u_i \in A$. Proposition 3.10 implies the only DG − open set in (V, τ_{DG}) is V. Hence $\tau_{DG} = \{\emptyset, V\}$. Now assume that $\tau_{DG} = \{\emptyset, V\}$. To prove that $D = (V, E)$ is ψ −strongly connected , let u_i and u_j be arbitrary vertices of V, since each DG − open set containing u_i contains u_j and each DG − open set containing u_j contains u_i and since V is the only DG − open set in (V, τ_{DG}) then by Proposition 3.10 $\psi(i, j)$ and $\psi(j, i)$. i.e $\psi^*(i, j)$. Hence $D = (V, E)$ is ψ −strongly connected .

Proposition3.12: Let (V, τ_{DG}) be a topological space associated with the digraph $D = (V, E)$. Then $D = (V, E)$ is ψ −unilaterally connected if and only if every two DG − open sets of (V, τ_{DG}) one of them containing the other .

Proof : Suppose that every two $DG-$open sets f (V,τ_{DG}) one is subset of the other. Suppose that u_i and u_j are two vertices of V.Then there does not exist an $DG-$open set A_1 such that $u_i \in A_1$ but $u_j \notin A_1$ or there does not exist an $DG-$open set A_2 such that $u_j \in A_2$ but $u_i \notin A_2$. Since either$A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, then either each $DG-$open set A_1 such that $u_j \in A_1$implies that $u_i \in A_1$ or each $DG-$open set A_2 such that $u_i \in A_2$ implies that $u_j \in A_2$ and this satisfy if and only if $\psi(i,j)$ or $\psi(j,i)$ for any two vertices u_i and u_j of V. Hence $D=(V,E)$ is ψ −unilaterally connected .

Definition 3.13: Let $D=(V,E)$ be a digraph and (V,τ_{DG}) be a $DG-$topological space .The $DG-$closure of a subset A of V denoted by $\overline{A}^{DG}$, is the intersection of all $DG-$closed subsets of V containing A .i.e $\bar{A}^{DG} = \cap \{U: U \text{ is } DG-closed, A \subseteq U \subseteq V\}$.

Example3.14: Consider the digraph $D=(V,E)$, where $V=\{v_1,v_2,v_3,v_4\}$.



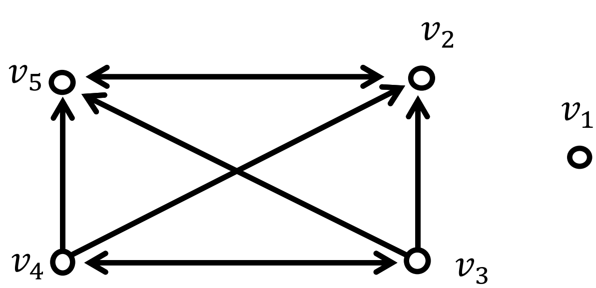
The topology associated to above digraph is

$\tau_{DG}=\{\emptyset,V,\{v_2\},\{v_2,v_3\},\{v_1,v_2,v_3\}\}$, then $\overline{\{v_1,v_3\}}^{DG}=\{v_1,v_3,v_4\}$.

Now, in the following definition, we define a new operator by using the term of the $DG-$open set namely $DG-$kernel .

Definition3.15: Let $D=(V,E)$ be a digraph and (V,τ_{DG}) be a $DG-$topological space . The $DG-$kernel of a subset $A$of V , denoted by $kl_{DG}(A)$, is defined by $kl_{DG}(A) = \cap \{U: U \text{ is } DG-\text{open}, A \subseteq U \subseteq V\}$.

Example3.16 : Consider the digraph $D=(V,E)$where $V=\{v_1,v_2,v_3,v_4,v_5\}$



And the topology associated to above digraph is $\tau_{DG}=\{\emptyset,V,\{v_1\},\{v_3,v_4\},$

$\{v_1,v_3,v_4\},\{v_2,v_3,v_4,v_5\}\}$,then $Kl_{DG}(\{v_2,v_4\})=\{v_2,v_3,v_4,v_5\}$.

Remarks 3.17:i) The $DG-$closure set is always $DG-$closed set and that a set A is $DG-$closed set if and only if $A=\bar{A}^{DG}$.

ii) $\bar{A}^{DG}$ is the set of all vertices which are reachable from A.

iii) From Theorem 3.5 the topology τ_{DG} on a digraph $D(V)$has completely additive closure i.e(The intersection of an arbitrary $DG-$ open sets is $DG-$ open set) and we see that the $DG-$ kernel of a set is $DG-$ open set and that a set A is $DG-$ open set if and only if $A = Kl_{DG}(A)$.

iv) Since τ_{DG} satisfy the completely additive closure then $\overline{(A \cup B)}^{DG} = \bar{A}^{DG} \cup \bar{B}^{DG}$ and $Kl_{DG}(A \cup B) = \mathrm{Kl}_{DG}(A) \cup \mathrm{Kl}_{DG}(B)$.

Theorem3.18: Let (V, τ_{DG}) be a $DG-$ topological space associated with the digraph $D = (V, E)$.Then any vertex u_i of $D(V)$,

i)$\overline{(u_\iota)}^{DG} = \{u_i: \psi(i,j)\}$, in another words , $\overline{(u_\iota)}^{DG}$ is the set of vertices of the digraph $D(V)$ which are out degree from u_i.

ii) $Kl_{DG}\ (u_i) = \{u_j: \psi(j,i)\}$, in another words $Kl_{DG}(u_i)$ is the set of vertices of the digraph $D(V)$ which are indegree to the vertex u_i .

Proof: i) From definition 3.13, $\overline{(u_\iota)}^{DG} = \cap \{U: U\ is\ DG-$ closed and $u_i \in U\}$ and that is, $\overline{(u_\iota)}^{DG}$ is the set of all vertices u_j such that every $DG-$ closed set U such that $u_i \in U$, then $u_j \in U$, and from Proposition 3.10 we have $\overline{(u_\iota)}^{DG} = \{u_j: \psi(i,j)\}$.

ii) similarly by using definition 3.15 and Proposition 3.10 we have that $Kl_{DG}\ (u_i) = \{u_j: \psi(j,i)\}$.

Definition3.19: Let $D = (V, E)$ be a digraph and (V, τ_{DG}) be a $DG-$ topological space a subset A of V is called $DG-$ dense if $\overline{(A)}^{DG} = V$.

Definition3.20: Let $D = (V, E)$ be a digraph and (V, τ_{DG}) be a $DG-$ topological space A_i and A_j are called $DG-$separated in (V, τ_{DG}) if $\overline{(A_\iota)}^{DG} \cap A_{j=}\emptyset$ and $A_i \cap \overline{(A_j)}^{DG} = \emptyset$

Theorem3.21: Let (V, τ_{DG}) be a topological space associated with the digraph $D = (V, E)$. Then for any $A \subseteq V$,

i) $\bar{A}^{DG} = \{u_j: \psi(i,j)$, for some $u_i \in A\}$ and,

ii) $Kl_{DG} = \{u_j: \psi(j,i)$ for some $u_i \in A\}$.

Proof: i) Since using remark 3.17(iv) and Theorem 3.18(i) we have $\overline{(A \cup B)}^{DG} = \bar{A}^{DG} \cup \bar{B}^{DG}$, $\overline{(A)}^{DG} = \overline{(\cup \{u_\iota: u_\iota \in A\})}^{DG} = \cup \{\overline{(u_\iota)}^{DG}: u_i \in A\} = \cup \{u_j: \psi(i,j): u_i \in A\} = \{u_j: \psi(i,j)$ for some $u_i \in A\}$.

ii) Since using remark 3.17(iv) and Theorem 3.18(ii) we have$Kl_{DG}(A \cup B) = \mathrm{Kl}_{DG}(A) \cup \mathrm{Kl}_{DG}(B)$,

$Kl_{DG}(A) = \mathrm{Kl}_{DG}(\cup \{u_i\}: u_i \in A) = \cup \{Kl_{DG}(u_i): u_i \in A\} = \cup (\{u_j: \psi(j,i): u_i \in A\} = \{u_j: \psi(j,i)$ for some $u_j \in A\}$.

Corollary3.22:A set A is $DG-$ dense in V of $D(V)$if and only if for each vertex u_k of V, there exists a vertex u_i of A such that is in degree to u_k .

Proof: The proof is follows from Theorems (3.18)(i) and (3.21)(i).

Corollary3.23: Let A_i and A_j be two subsets of V .Then A_i and A_j are $DG -$ separated in $D(V)$ iff there does not exist vertices $u_i \in A_i$ and $u_j \in A_j$,such that $\psi(i,j)$ or $\psi(j,i)$.

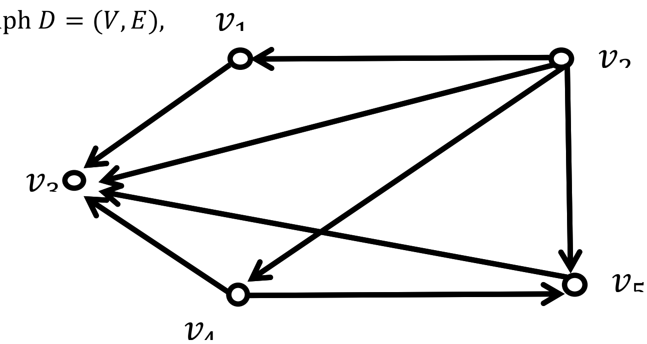
Proof:⟹) Since A_i and A_j are $DG -$ separated in $D(V)$, then $\overline{(A_\iota)}^{DG} \cap A_{j=}\emptyset$ and $A_i \cap \overline{(A_j)}^{DG} = \emptyset$ iff .From Theorems (3.18)(i) and (3.21)(i), there dose not exist vertices $u_i \in A_i$ and $u_j \in A_j$ such that $\psi(i,j)$ and there dose not exist vertices $u_r \in A_i$ and $u_s \in A_j$ such that $\psi(s,r)$.

Definition3.24: The $DG -$core of a set A , denoted by $C_{DG}(A)$, is the intersection of all subsets of V containing A which are DG −closed or $DG -$ open i.e $C_{DG}(A) = \cap\{U: U$ is $DG -$ closed or DG − open , A ⊆ U ⊆ V}.

Remark 3.25: Note that $C_{DG}(A) = \bar{A}^{DG} \cap Kl_{DG}(A)$.

Example3.26: Consider the digraph $D = (V, E)$,

where $V = \{v_1, v_2, v_3, v_4, v_5\}$



The topology associated to above digraph is $\tau_{DG} = \{\emptyset, V, \{v_2\}, \{v_1, v_2\}, \{v_2, v_4\}$

$, \{v_1, v_2, v_4\}, \{v_2, v_4, v_5\}, \{v_1, v_2, v_4, v_5\}$,then $C_{DG}(\{v_1, v_4, v_5\}) = \{v_1, v_2, v_4, v_5\}$.

Theorem3.27: For any vertex u_i of $D = (V, E)$, $C_{DG}(u_i) = \{u_j: \psi^*(i,j)\}$. In another words ,$C_{DG}(u_i)$ is the set of all vertices of the digraph which are symmetrically indegrable to u_i .

Proof: By using remark 3.25 and definition 3.24 ,we have the following :

$$C_{DG}(u_i) = \overline{(u_\iota)}^{DG} \cap Kl_{DG}(u_i) = \{u_j: \psi(i,j)\} \cap \{u_j: \psi(j,i)\} = \{u_j: \psi(i,j) \text{ and } \psi(j,i)\} = \{u_j: \psi^*(i,j)\}$$

Theorem3.28: For any set $A \subseteq V$ of $D = (V, E)$,

$$C_{DG}(A) = \{u_j: \psi(i,j) \text{for some } u_i \in A \text{ and } \psi(j,k) \text{ for some } u_k \in A\}$$

Prove: By using remark 3.25 and definition 3.24 we have the following : $C_{DG}(A) = \bar{A}^{DG} \cap Kl_{DG}(A) = \{u_j: \psi(i,j) \text{for some } u_i \in A\} \cap \{u_j: \psi(j,k) \text{for some } u_k \in A\} = \{u_j : \psi(i,j) \text{for some } u_i \in A \text{ and } \psi(j,k) \text{for some } u_k \in A\}$.

Note that $\cup\{C_{DG}(A_i): i \in I\} \subseteq \{C_{DG} \cup A_i: i \in I\}$ and thus for any $A \subseteq V$,

$\{u_j: \psi^*(i,j) \text{for some } u_i \in A\} \subseteq C_{DG}(A)$.

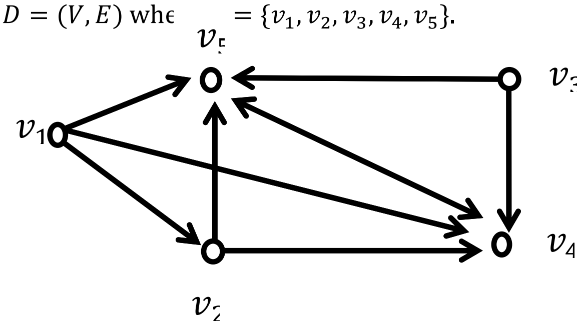
Corollary3.29: For any vertex u_i of $D = (V, E)$, the $DG -$ core of u_i is the maximum subset , say A of V containing u_i such that the subdigraph $D(A)$ is $\psi -$ strongly connected .

Proof: From Theorem 3.27,$u_k \in C_{DG}(u_i)$ iff $\psi^*(i,k)$ .Then the subdigraph $D(C_{DG}(u_i))$ is $\psi -$ strongly connected. And for any set A_i containing u_i and containing some other vertex say u_j , where $u_j \notin C_{DG}(u_i)$, $D(A_i)$ is not $\psi -$ strongly connected.

Defintion3.30: Let (V, τ_{DG})be a $DG -$ topological space associated with the digraph $D = (V, E)$ and $A \subseteq V$. The point $v \in V$ is called $DG -$ limit point of A if for every $DG -$ open set U containing v,$(U - \{v\}) \cap A \neq \emptyset$.

Remark3.31: The set of $DG -$ limit points of A is denoted by $\acute{A}^{DG}$.

Example 3.32: Consider the digraph $D = (V, E)$ whe $= \{v_1, v_2, v_3, v_4, v_5\}$.



the topology associated to above digraph is $\tau_{DG} = \{\emptyset, V, \{v_1\}, \{v_3\}, \{v_1, v_2\}$

, $\{v_1, v_3\}, \{v_1, v_2, v_3\}\}$,then $\{\acute{(v_1, v_3\}})^{DG} = \{v_2, v_4\}$.

Theorem3.33: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$ and let $A \subseteq V$. Then $A \cup \acute{A}^{DG}$ is $DG -$ closed set .

Proof: Let $v \in (A \cup \acute{A}^{DG})$ and $u \in (A \cup \acute{A}^{DG})^c$. To prove that $vu \notin E$, since $u \in (A \cup \acute{A}^{DG})^c$ then $u \notin (A \cup \acute{A}^{DG})$, so $u \notin A$ $\wedge\, u \acute{\notin} A^{DG}$. Then there exist a $DG -$ open set U containing u such that $(U - \{u\}) \cap A = \emptyset$. Then we get a $DG -$ open set U containing u but not v and $vu \notin E$. Hence $(A \cup \acute{A}^{DG})$ is $DG -$ closed .

Theorem3.34: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$. Then $\acute{A}^{DG} \subseteq A$ if and only if A is $DG -$ closed .

Proof: Suppose that $\acute{A}^{DG} \subseteq A$ and to prove that A is $DG -$ closed , let $u \in A^c$ and $v \in A$. Since $u \in A^c$ then $u \notin A$, then $u \notin \acute{A}^{DG}$ since $\acute{A}^{DG} \subseteq A$. Then there exist a $DG -$ open set U such that $u \in U$ and $(U - \{u\}) \cap A = \emptyset$. Then $vu \notin E$.Hence A is $DG -$ closed .

Now assume that A is $DG -$closed. To prove that $\acute{A}^{DG} \subseteq A$, let $u \notin A$. Then

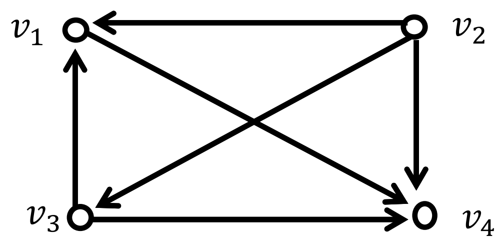
$u \in A^c$. Since A^c is a $DG -$open set containing u and $A \cap A^c = \emptyset$, then $u \notin \acute{A}^{DG}$. Hence $\acute{A}^{DG} \subseteq A$.

Definition3.35: Let $D = (V, E)$ be a digraph and (V, τ_{DG}) be a $DG -$ topological space .The $DG -$ interior of a set A denoted by $A^{\circ DG}$ is the union of all $DG -$ open sets of V contained in A. That is $A^{\circ DG} = \cup\{U : U \subseteq A , U$ is $DG -$ open set in $V\}$.

Remarks 3.36: i) For any $A \subseteq V$, $A^{\circ DG} = V - \overline{A^c}^{DG}$.

ii) $A^{\circ DG} =$ the set of all vertices which are not reachable from $V - A$.

Example3.37: Consider the digraph $D = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$



The topology associated to above digraph is $\tau_{DG} = \{\emptyset, V, \{v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$ then $\{v_2, v_4\}^{\circ DG} = \{v_2\}$.

Theorem3.38: Let (V, τ_{DG}) be a topological space associated with the digraph $D = (V, E)$. Then $(A^{\circ DG})^c = \overline{A^c}^{DG}$.

Proof: It is clear [remark 3.36(i)].

Theorem3.39: Let (V, τ_{DG}) be a $DG-$topological space associated with the digraph $D = (V, E)$ and let $A, B \subset V$. Then:

i) $A^{\circ DG}$ is the largest $DG-$open contained in A.

ii)$A^{\circ DG} = A$ if and only if A is $DG-$open.

iii) $(A^{\circ DG})^{\circ DG} = A^{\circ DG}$.

iv)$(A \cap B)^{\circ DG} = A^{\circ DG} \cap B^{\circ DG}$.

Proof: i) By definition 3.35 , $A^{\circ DG}$ is the $DG-$open contained in A .To prove $A^{\circ DG}$ is the largest $DG-$open contained in A. Suppose that U is $DG-$open and $U \subseteq A$ .To prove $U \subset A^{\circ DG}$ if (1). Suppose $v \in U$ if (2). Then $v \in \cup_i U_i$ where U_i is $DG-$open set for all i and $U \subseteq A$.

Since U is a $DG-$open set then $u \in U$ and an arc $vu \in E$ then $v \in U$, i.e $v \in U_i$ such that U_i is $DG-$open and $U_i \subseteq A_i$ for all i , then $v \in A^{\circ DG}$, then $U \subset A^{\circ DG}$.

Hence $A^{\circ DG}$ is largest $DG-$open contained in A .

ii) Suppose that $A^{\circ DG} = A$. Then A is $DG-$open ,since $A^{\circ DG}$ is $DG-$open .

Suppose that A is $DG-$open , we need to prove on $A \subset A^{\circ DG}$.

Since $A^{\circ DG} \subset A$,Since A is a $DG-$open set , then $u \in A$ and an arc $vu \in E$ then $v \in A$, i.e $v \in \cup_i U_i$, such that U_i is $DG-$open set and $U_i \subseteq A$ for all i , then $v \in A^{\circ DG}$ .Then $A \subseteq A^{\circ DG}$. Hence $A = A^{\circ DG}$

iii)Since $A^{\circ DG}$ is a $DG-$open set then $(A^{\circ DG})^{\circ DG} = A^{\circ DG}$.

iv)$(A \cap B)^{\circ DG} = ((\overline{(A \cap B)^c})^{DG})^c = (\overline{(A^c \cup B^c)}^{DG})^c = (\overline{A^c}^{DG} \cup \overline{B^c}^{DG})^c = (\overline{A^c}^{DG})^c \cap (\overline{B^c}^{DG})^c = A^{\circ DG} \cap B^{\circ DG}$.

4.On $DG-$Connected Spaces.

In this section we introduced the concept of DG-connected space and investigate some theorems which associated with digraphs.

Definition4.1: Let(V, τ_{DG}) be a $DG -$ topological space , then (V, τ_{DG}) is called a $DG -$ topologically connected if V can not be expressed as union of two disjoint non empty $DG -$ open sets and other wise (V, τ_{DG}) is called a $DG -$disconnected space **Theorem4.2:** Let (V, τ_{DG}) be a topological space associated with the digraph $D = (V, E)$.Then $D = (V, E)$ is $\psi -$ weakly connected iff (V, τ_{DG}) is a $DG -$ topologically connected .

Proof: Suppose that (V, τ_{DG}) is a $DG -$ connected space , then V cannot be expressed as the union of two nonempty disjoint $DG -$ open sets and this iff any nonempty proper subset of V is not a $DG -$ open or is not $DG -$ closed . Equivalently , by definition 3.1 , for each proper subset, say A, of V , there exists an arc from A^c to A or there exists an arc from A to A^c in the digraph $D(V)$, that is in $D = (V, E \cup E^c)$, where $E^c = \{uv: vu \in E\}$. Hence, the only $DG -$ open set in $D = (V, E \cup E^c)$ are $\emptyset$ and V . Then from Proposition 3.11 $D = (V, E \cup E^c)$ is $\psi -$strongly connected and hence $D = (V, E)$ is $\psi -$ weakly connected .

Theorem4.3: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$.Then $D = (V, E)$ is a $\psi -$ disconnected iff (V, τ_{DG}) is a $DG -$ topologically disconnected .

Proof: A proof is the contrapositive of Theorem 4.2

Theorem4.4: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$.Then $D = (V, E)$ is a $\psi -$ disconnected iff V can be expressed as the union of two $DG -$ separated subsets of (V, τ_{DG}).

Proof: The digraph $D = (V, E)$ is $\psi -$ disconnected iff, the set V can be expressed as the union of two disjoint nonempty $DG -$ open sets , say A_1 and A_1^c that is , there does not exist an arc from A_1 to A_1^c or from A_1^c to A_1. Hence $\bar{A}_1^{DG} = A_1$ and $\overline{A_1^c}^{DG} = A_1^c$. Therefore $D = (V, E)$ is $\psi -$ disconnected iff A_1 and A_1^c are $DG -$ separated in (V, τ_{DG}) .

Theorem4.5: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$.Then $D = (V, E)$ is $\psi -$ disconnected or $\psi -$ weakly connected , but not $\psi -$ unilaterally connected iff (V, τ_{DG}) contains two nonempty $DG -$separated subsets .

Proof: Suppose that $D = (V, E)$ is either $\psi -$disconnected or $\psi -$weakly connected but not $\psi -$unilaterally connected , then $D = (V, E)$ is neither $\psi -$strongly connected nor $\psi -$unilaterally connected. Then there exists vertices u_i and u_j in V such that $\widetilde{\psi}(i, j)$ and $\tilde{\psi}(j, i)$ and hence $u_i \notin \bar{u}_j^{DG}$ and $u_j \notin \bar{u}_i^{DG}$ [Theorem 3.4]. Thus $\{u_i\}$ and $\{u_j\}$ are two nonempty $DG -$ separated subsets of ($V. \tau_{DG}$) .

Now suppose that (V, τ_{DG}) contains two nonempty $DG -$ separated subsets , say A_i and A_j . By corollary 3.23, there does not exist $u_i \in A_i$ and $u_j \in A_j$, such that $\psi(i, j)$ or $\psi(j, i)$. since A_i and A_j are nonempty ,then there exist vertices say $u_r \in A_i$ and $u_s \in A_j$, such that $\tilde{\psi}(r, s)$ and $\tilde{\psi}(s, r)$. Then $D(V)$ is either $\psi -$ disconnected or $\psi -$ weakly connected but not $\psi -$ unilaterally connected .

Theorem4.6: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$.Then $D = (V, E)$ is $\psi -$ strongly connected iff each a vertex of V is $DG -$ dense in (V, τ_{DG}) .

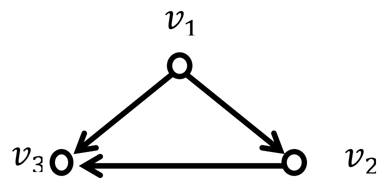
Proof : By Proposition 3.11, $D = (V, E)$ is $\psi -$ strongly connected iff $\tau_{DG} = \{\emptyset , V \}$ and this is true iff $\bar{u}_i^{DG} = A$, for each vertex $u_i \in V$

5. On $DG -$ Separation Axioms

In this section , we introduce the following two $DG -$ separation axioms $DG - T_0$ space and $DG - T_1$ spaces and we prove some theorems of a digraph in terms of the $DG -$ separation axioms satisfied by the $DG -$ topological space which is determined by that digraphs .

Definition5.1: Let $D = (V, E)$ be a digraph and (V, τ_{DG}) be an $DG-$topological space. Then (V, τ_{DG}) is a $DG-T_0$ space if for every $u_i, u_j \in V, u_i \neq u_j$ there exists a $DG-$ open set contains either u_i or u_j.

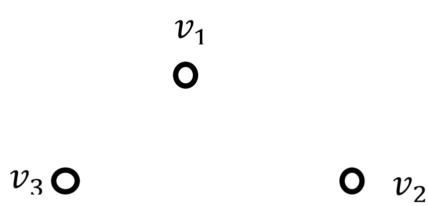
Example5.2: Cosider the digraph $D = (V, E)$, where $V = \{v_1, v_2, v_3\}$



Which has the topology associated to above digraph is $\tau_{DG} = \{\emptyset, V, \{v_1\}, \{v_1, v_2\}\}$. Since for each two different elements, there exists an $DG-$ open set containing one of them and does not contain the other , then (V, τ_{DG}) is a $DG-T_0$ space .

Definition5.3: Let $D = (V, E)$ be a digraph and (V, τ_{DG}) be an $DG-$topological space. Then (V, τ_{DG}) is a $DG-T_1$ if each set which consists of a single vertex is $DG-$closed .

Example5.4: Consider the digraph $D = (V, E)$ where $V = \{v_1, v_2, v_3\}$ which has the topology



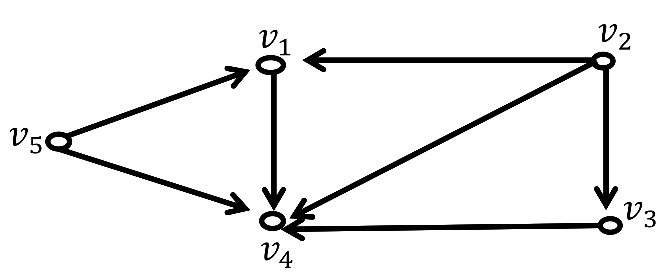
$\tau_{DG} = \{\emptyset, V, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$. Since each set, which consists of a single vertex is a $DG-$ closed set then (V, τ_{DG}) is $DG-T_1$ space .

Proposition5.5: Let (V, τ_{DG}) be a $DG-$topological space associated with the digraph $D = (V, E)$.Then every $DG-T_1$ space is $DG-T_0$ space .

Proof: Let (V, τ_{DG}) be a $DG-T_1$ space we prove that (V, τ_{DG}) is a $DG-T_0$ space, let $v_1, v_2 \in V$ such that $v_1 \neq v_2$,then there exists two $DG-$open sets $U, H \in V$such that $(v_1 \in U, v_2 \notin U)$ and $(v_1 \notin H, v_2 \in H)$, then there exist a $DG-$ open set contains one of two points and no contains the other points . Hence (V, τ_{DG}) is $DG-T_0$ space .

Remark5.6: The converse of the above proposition is not true in general from the following example .

Example5.7: Consider the digraph $D = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$



The topology associated above digraph is

$\tau_{DG} = \{\emptyset, V, \{v_2\}, \{v_5\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_5\}, \{v_2, v_3\}, \{v_2, v_3, v_5\}\}$, (V, τ_{DG}) is $DG-T_0$ space but not $DG-T_1$ space .

Now we the following theorems which investigate a digraph in term of the $DG -$ separation axioms .

Theorem5.8: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$. Then $\tilde{\psi}(i, j)$ or $\tilde{\psi}(j, i)$ iff (V, τ_{DG}) is a $DG - T_0$ space .

Proof: Let u_i and u_j denoted distinct arbitrary vertices of $D = (V, E)$. Then u_i is not indegree to u_j or u_j is not indegree to u_i iff by proposition 3.10 there exists an $DG -$ open set W such that $u_j \in W$ but $u_i \notin W$ or there exists an $DG -$ open set U, such that $u_i \in U$ but $u_j \notin U$.Hence (V, τ_{DG}) is $DG - T_0$ space .

Theorem5.9: Let (V, τ_{DG}) be a $DG -$ topological space associated with the digraph $D = (V, E)$. Then there does not exists an arc from any vertex of V to any other distinct vertex of A iff (V, τ_{DG}) is a $DG - T_1$ space or equivalently , iff (V, τ_{DG}) is a discrete space .

Proof: Let u_i, u_j be two distinct vertices of V such tat $u_i, u_j \notin E$ iff for each set consists of a single vertex is $DG -$ closed. Then (V, τ_{DG}) is $DG - T_1$ space . Since (V, τ_{DG}) is allexondoff space , then each set consists of a single vertex is $DG -$closed iff every subset of V is a $DG -$open set ,hence (V, τ_{DG}) is a discrete space .

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