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Oscillation and Asymptotic Behavior of Solution of nth Order System Nonlinear of Neutral Differential Equation

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ABSTRACT

In this paper, authors obtained sufficient conditions to ensure that every bounded solution of nth order nonlinear neutral differential system oscillates or nonoscillatory converges to zero as $t \to \infty$, some examples were given to explain the results obtained.

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1 . Introduction

In the last few years, some authors have studied the oscillation theory of higher-order neutral differential equations see references [1]-[2] and [6]-[9], a few of which have studied the oscillation of the system of neutral differential equations [3],[4]. There are many applications for system of neutral differential equations in various fields such as physics, biology, ecology, engineering, for example a neutral lotka-volterra system in the modeling of bio-dynamic, etc. In this paper we discussed the oscillation of bounded solutions of system of nth order neutral differential equations of the form

$$\begin{aligned} \frac{d^n}{dt^n}[x(t)+p_1(t)x(\tau_1(t))] &= q_1(t)f_1(y(\sigma_1(t))) \\ \frac{d^n}{dt^n}[y(t)+p_2(t)y(\tau_2(t))] &= q_2(t)f_2(x(\sigma_2(t))) \end{aligned}, t \geq t_0 . \qquad (1)$$

We assume that the following hypotheses are satisfied:

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(H1) $q_1, q_2 \in C\big[[t_0,\infty),\mathbb{R}\big], \tau_1,\tau_2,\sigma_1,\sigma_2 \in C\big[[t_0,\infty),\mathbb{R}\big], \tau_1(t) \le t, \tau_2(t) \le t, \sigma_1(t) \le t,\ \sigma_2(t) \le t, \lim_{t\to\infty}\tau_1(t) = \infty$, $\lim_{t\to\infty}\tau_2(t) = \infty, \lim_{t\to\infty}\sigma_1(t) = \infty,\ \lim_{t\to\infty}\sigma_2(t) = \infty$, σ_1, σ_2 are increasing functions.
(H2) $p_1, p_2 \in C\big[[t_0,\infty),\mathbb{R}\big], 0 \le p_1(t) \le \delta_1 < 1\ \ ,\ 0 \le p_2(t) \le \delta_2 < 1$.
(H3) $f_i \in C\big[[t_0,\infty),\mathbb{R}\big], xf_i(x) > 0, x \ne 0,\ i = 1,2.\ \lambda_1 \le \frac{f_1(y)}{y} \le \lambda_2\ ,\lambda_3 \le \frac{f_2(x)}{x} \le \lambda_4$. We consider only those solutions(x,y) of (1) which satisfy $\sup\{|x(t)|: t \ge T\} > 0, \sup\{|y(t)|: t \ge T\} > 0$. As usual, a solutions of (1) is called oscillatory if all of its components are oscillatory that is it has arbitrarily large zeros, otherwise is said to be nonoscillatory.

2. Main Results

In this section two results are presented where some sufficient conditions obtained to ensure every bounded solution of Sys.(1) either oscillate or nonoscillatory converges to zero as $t \to \infty$.

Theorem 1. Suppose that $q_1(t), q_2(t) \ge 0$ and (H1)-(H3) hold and

$$\begin{aligned} \limsup_{t\to\infty} t^{n-1}\int_{\sigma_1^{-1}(t)}^{\infty} q_1(s)\,ds &> 1, \\ \limsup_{t\to\infty} t^{n-1}\int_{\sigma_2^{-1}(t)}^{\infty} q_2(s)\,ds &> 1, \end{aligned} \tag{2}$$

Then every bounded solution of (1) either oscillates or converging to zero as $t \to \infty$.

Proof. Assume for the sake of contradiction that Sys.(1) has bounded nonoscillatory solution, (that is either eventually positive or eventually negative). From (2) we get:

$$\begin{aligned} 1 &< \limsup_{t\to\infty} t^{n-1}\int_{\sigma_1^{-1}(t)}^{\infty} q_1(s)ds \\ &\le \limsup_{t\to\infty} t^{n-1}\int_{t}^{\infty} q_1(s)\,ds \\ &\le \limsup_{t\to\infty} \int_{t}^{\infty} s^{n-1} q_1(s)ds. \end{aligned} \tag{3}$$

We claim that (2) implies to,

$$\begin{aligned} &\int_{t_1}^{\infty} s^{n-1} q_1(s)ds = \infty \ \text{ and} \\ &\int_{t_1}^{\infty} s^{n-1} q_2(s)ds = \infty, \end{aligned} \qquad t_1 \ge t_0. \tag{4}$$

Otherwise if

$$\int_{t_1}^{\infty} s^{n-1} q_1(s)ds < \infty\,. \tag{5}$$

Then there exist $t_2 \ge t_1$ such that:

$$\int_{t_2}^{\infty} s^{n-1} q_1(s)ds < 1.$$

this contradicts (3), in the same way we can show that,

$$\int_{t_1}^{\infty} s^{n-1} q_1(s)ds = \infty.$$

Setting:

$$\begin{aligned} u(t) &= x(t) + p_1(t)x\big(\tau_1(t)\big), \\ v(t) &= y(t) + p_2(t)y\big(\tau_2(t)\big), \end{aligned}$$

From the integral identity:

$$w^{(k)}(t_1)=\sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-t_1)^{i-k}}{(i-k)!}w^{(i)}(t)+\frac{(-1)^{n-k}}{(n-k-1)!}\int_{t_1}^{t}(\xi-t_1)^{n-k-1}w^{(n)}(\xi)d\xi. \quad (6)$$

Where $t>t_1$.

1. Assume that n is odd and the Sys.(1) has nonoscillatory solution, then we have the following cases:

1. **Case 1.** $x(t)>0, y(t)>0$ (the case $x(t)<0, y(t)<0$, is similar).

2. Then

$$u(t)>0,\ \ u^{(n)}(t)\geq 0,\ \ \text{and}$$
$$v(t)>0, v^{(n)}(t)\geq 0,\ \ t\geq t_1\geq t_0\,.$$

Since n is odd and $u(t)$ is bounded so we have only the following possible case:

$$u^{(n)}(t)\geq 0, u^{(n-1)}(t)<0,\dots,u'(t)>0, u(t)>0$$
$$v^{(n)}(t)\geq 0, v^{(n-1)}(t)<0,\dots,v'(t)>0, v(t)>0$$

Further, we have

$y(t)=v(t)-p_2(t)y(\tau_2(t))$, then

$$q_1(t)y(\sigma_1(t))=q_1(t)v(\sigma_1(t))-q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t))) \quad (7)$$
$$-q_1(t)p_2(\sigma_1(t))v(\tau_2(\sigma_1(t)))\leq -q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t))).$$

From the equation (6), we obtain:

$$u^{(k)}(t)=\sum_{i=k}^{n-1}(-1)^{i-k}\frac{(s-t)^{i-k}}{(i-k)!}u^{(i)}(s)+\frac{(-1)^{n-k}}{(n-k-1)!}\int_t^s(\xi-t)^{n-k-1}u^{(n)}(\xi)d\xi.$$

Let $k=1$ then:

$$u'(t)\geq\frac{1}{(n-2)!}\int_t^s(\xi-t)^{n-2}u^{(n)}(\xi)d\xi,$$
$$\geq\frac{1}{(n-2)!}\int_t^s(\xi-t)^{n-2}q_1(\xi)f_1\left(y(\sigma_1(\xi))\right)d\xi,$$
$$\geq\frac{\lambda_1}{(n-2)!}\int_t^s(\xi-t)^{n-2}q_1(\xi)y(\sigma_1(\xi))d\xi,$$
$$\geq\frac{\lambda_1}{(n-2)!}\int_t^s(\xi-t)^{n-2}(q_1(\xi)v(\sigma_1(\xi))-q_1(\xi)p_2(\sigma_1(\xi))y\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$
$$\geq\frac{\lambda_1}{(n-2)!}\int_t^s(\xi-t)^{n-2}(q_1(\xi)v(\sigma_1(\xi))-q_1(\xi)p_2(\sigma_1(\xi))v\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$
$$\geq\frac{\lambda_1}{(n-2)!}\int_t^s(\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)v\left(\tau_2(\sigma_1(\xi))\right)d\xi,$$
$$u'(t)\geq\frac{\lambda_1 v\left(\tau_2(\sigma_1(t))\right)}{(n-2)!}\int_t^s(\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)\,d\xi.$$

Integrating the last inequality from t_1 to s we get the following:

$$u(s)-u(t_1)\geq\frac{\lambda_1 v\left(\tau_2(\sigma_1(t_1))\right)}{(n-1)!}\int_{t_1}^s(\xi-t_1)^{n-1}\left(1-p_2(\sigma_1(\xi))\right)q_1(\xi)\,d\xi$$

Letting $s\to\infty$ and by using (4) we get a contradiction from the last inequality.

Case 2. $x(t)>0\,, y(t)<0$ (the case $x(t)<0\,, y(t)>0$ is similar to case 2). Then

$$u(t)>0,\qquad u^{(n)}(t)\leq 0,\qquad \text{and } v(t)<0, v^{(n)}(t)\geq 0,\qquad t\geq t_1\geq t_0.$$

We have only the following possible case:

$$u^{(n)}(t) \leq 0, u^{(n-1)}(t) > 0 \ldots, u'(t) < 0, u(t) > 0,$$
$$v^{(n)}(t) \geq 0, \ldots, v'(t) > 0, v(t) < 0, \quad t \geq t_1 \geq t_0 .$$

Further, we have

$v(t) = y(t) + p_2(t)y(\tau_2(t))$ then $v(t) \leq y(t)$.

By equation (6), we obtain:

$$u^{(k)}(T) = \sum_{i=k}^{n-1} (-1)^{i-k} \frac{(t-T)^{i-k}}{(i-k)!} u^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!} \int_T^t (\xi - T)^{n-k-1} u^{(n)}(\xi) d\xi$$

$T \geq t_1$. Let $k = 0$ then:

$$u(T) \geq -\frac{1}{(n-1)!} \int_T^t (\xi - T)^{n-1} u^{(n)}(\xi) d\xi,$$
$$\geq -\frac{1}{(n-1)!} \int_T^t (\xi - T)^{n-1} q_1(\xi) f_1\left(y(\sigma_1(\xi))\right) d\xi,$$
$$\geq -\frac{\lambda_2}{(n-1)!} \int_T^t (\xi - T)^{n-1} q_1(\xi) y(\sigma_1(\xi)) d\xi,$$

We claim that $\limsup_{t\to\infty} y(t) = 0$ otherwise $\limsup_{t\to\infty} y(t) < 0$. So there exist $c_1 < 0$ such that $\limsup_{t\to\infty} y(t) \leq c_1 < 0$.

Hence there exists $t_2 \geq t_1$, $y(t) \leq c_1$ for $t \geq t_2$ we get

$$u(t_2) \geq \frac{-\lambda_2 c_1}{(n-1)!} \int_{t_2}^t (\xi - t_2)^{n-1} q_1(\xi) d\xi.$$

Letting $t \to \infty$ in the last inequality we get a contradiction. Hence $\limsup_{t\to\infty} y(t) = 0$.

Then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = 0$. Since $v(t)$ is monotone bounded function, let $\lim_{t\to\infty} v(t) = L \leq 0$,

$$v(t_n) = y(t_n) + p_2(t_n) y(\tau_2(t_n)),$$
$$\geq y(t_n) + \delta_2 v(\tau_2(t_n)).$$

As $n \to \infty$ then we get $L \geq \delta_2 L$ or $L(1-\delta_2) \geq 0$, this is possible only when $L = 0$. This implies that: $\lim_{t\to\infty} y(t) = 0$, since $v(t) \leq y(t) < 0$.

2- Assume n is even and Sys.(1) has nonoscillatory solution then the following cases must be consider:

Case 1. $x(t) > 0, y(t) > 0$ (the case $x(t) < 0, y(t) < 0$ is similar).

Then $u(t) > 0, v(t) > 0, u^{(n)}(t) \geq 0$, $v^{(n)}(t) \geq 0$, $t \geq t_1 \geq t_0$.

In this case we have only the possible:

$$u^{(n)}(t) \geq 0, u^{(n-1)}(t) < 0 \ldots, u'(t) < 0, u(t) > 0,$$
$$v^{(n)}(t) \geq 0, v^{(n-1)}(t) < 0 \ldots, v'(t) < 0, v(t) > 0, \qquad t \geq t_1 \geq t_0 .$$

Further, we have

$u(t) = x(t) + p_1(t)x(\tau_1(t))$.

From the equation (6), we obtain:

$v^{(k)}(T) = \sum_{i=k}^{n-1} (-1)^{i-k} \frac{(t-T)^{i-k}}{(i-k)!} v^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!} \int_T^t (\xi - T)^{n-k-1} v^{(n)}(\xi) d\xi,$

where $T \geq t_1$. Let $k = 0$ then:

$$v(T) \geq \frac{1}{(n-1)!}\int_T^t (\xi - T)^{n-1}\, v^{(n)}(\xi)d\xi,$$
$$\geq \frac{1}{(n-1)!}\int_T^t (\xi - T)^{n-1}\, q_2(\xi)f_2\left(x\big(\sigma_2(\xi)\big)\right)d\xi,$$
$$\geq \frac{\lambda_3}{(n-1)!}\int_T^t (\xi - T)^{n-1}\, q_2(\xi)x\big(\sigma_2(\xi)\big)d\xi.$$

We claim that $\liminf_{t\to\infty} x(t) = 0$ otherwise $\liminf_{t\to\infty} x(t) > 0$. So there exist $c_2 > 0$ such that $\liminf_{t\to\infty} x(t) \geq c_2 > 0$. Hence there exist $t_2 \geq t_1$, large enough such that $x(t) \geq c_2$ for $t \geq t_2$.

$$v(t_1) \geq \frac{\lambda_3 c_2}{(n-1)!}\int_{t_1}^t (\xi - t_1)^{n-1}\, q_2(\xi)d\xi.$$

Letting $t \to \infty$ we get a contradiction. Thus $\liminf_{t\to\infty} x(t) = 0$, so there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = 0$.

Since $u(t)$ is monotone and bounded then $\lim_{t\to\infty} u(t) = l \geq 0$. From the equation,

$$u(t_n) = x(t_n) + p_1(t_n)x(\tau_1(t_n))$$

it follows that:

$$u(t_n) \leq x(t_n) + \delta_1 u\big(\tau_1(t_n)\big),$$

As $n \to \infty$ we get from the last inequality $l \leq \delta_1 l$ that is $l(1-\delta_1) \leq 0$, this is possible only, when $l = 0$. This implies to $\lim_{t\to\infty} x(t) = 0$, since $x(t) \leq u(t)$.

Case 2. $x(t) > 0$, $y(t) < 0$ (the proof of the case $x(t) < 0, y(t) > 0$ is similar).

Then $u(t) > 0, v(t) < 0, u^{(n)}(t) \leq 0$, $v^{(n)}(t) \geq 0, t \geq t_1 \geq t_0$, and we have only the possible case:

$$u^{(n)}(t) \leq 0, u^{(n-1)}(t) > 0 \ldots, u'(t) > 0, u(t) > 0,$$
$$v^{(n)}(t) \geq 0, \ldots, v'(t) < 0, v(t) < 0, \qquad t \geq t_1 \geq t_0.$$

Further, we have

$y(t) = v(t) - p_2(t)y\big(\tau_2(t)\big)$. Then

$$q_1(t)y(\sigma_1(t)) = q_1(t)v(\sigma_1(t)) - q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t))).$$
$$q_1(t)y\big(\sigma_1(t)\big) \geq q_1(t)v\big(\sigma_1(t)\big) - q_1(t)p_2\big(\sigma_1(t)\big)v(\tau_2(\sigma_1(t))) \geq q_1(t)p_2\big(\sigma_1(t)\big)v\left(\tau_2\big(\sigma_1(t)\big)\right).$$

From the equation (6), we obtain:

$$u^{(k)}(t) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(s-t)^{i-k}}{(i-k)!}u^{(i)}(s) + \frac{(-1)^{n-k}}{(n-k-1)!}\int_t^s (\xi - t)^{n-k-1}\, u^{(n)}(\xi)d\xi.$$

Since n is even and let $k = 1$ then:

$$u'(t) \geq -\frac{1}{(n-2)!}\int_t^s (\xi - t)^{n-2}\, u^{(n)}(\xi)d\xi.$$
$$\geq -\frac{1}{(n-2)!}\int_t^s (\xi - t)^{n-2}\, q_1(\xi)f_1\left(y\big(\sigma_1(\xi)\big)\right)d\xi,$$
$$\geq -\frac{\lambda_2}{(n-2)!}\int_t^s (\xi - t)^{n-2}\, q_1(\xi)y\big(\sigma_1(\xi)\big)d\xi,$$
$$\geq -\frac{\lambda_2}{(n-2)!}\int_t^s (\xi - t)^{n-2}(q_1(\xi)v(\sigma_1(\xi)) - q_1(\xi)p_2\big(\sigma_1(\xi)\big)y\left(\tau_2\big(\sigma_1(\xi)\big)\right))d\xi,$$

$$\geq -\frac{\lambda_2}{(n-2)!}\int_t^s (\xi-t)^{n-2}(q_1(\xi)v(\sigma_1(\xi)) - q_1(\xi)p_2(\sigma_1(\xi))v\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$

$$\geq -\frac{\lambda_2}{(n-2)!}\int_t^s (\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)v\left(\tau_2(\sigma_1(\xi))\right)d\xi,$$

$$u'(t) \geq -\frac{\lambda_2}{(n-2)!}v\left(\tau_2(\sigma_1(t))\right)\int_t^s (\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)\,d\xi.$$

Integrating the last inequality from t_1 to s.

$$u(s)-u(t_1) \geq -\frac{\lambda_2 v\left(\tau_2(\sigma_1(t_1))\right)}{(n-2)!}\int_{t_1}^s\int_r^s (\xi-r)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)\,d\xi dr.$$

$$\geq -\frac{\lambda_2 v\left(\tau_2(\sigma_1(t_1))\right)}{(n-1)!}\int_{t_1}^s (\xi-t_1)^{n-1}(1-p_2(\sigma_1(\xi)))q_1(\xi)d\xi.$$

Letting $s \to \infty$ we get a contradiction. □

Theorem 2. Suppose that $q_1(t), q_2(t) \leq 0$, and (H1)-(H3) hold and

$$\limsup_{t\to\infty} t^{n-1}\int_{\sigma_1^{-1}(t)}^{\infty}|q_1(s)|ds > 1,$$
$$\limsup_{t\to\infty} t^{n-1}\int_{\sigma_2^{-1}(t)}^{\infty}|q_2(s)|ds > 1, \quad (2')$$

Then every bounded solution of (1) either oscillates or converges to zero.

Proof. For the sake of contradiction assume that system (1) has bounded nonoscillatory solution, (either eventually positive or eventually negative). In similar way as in theorem 1 we can show that (2') implies to

$$\int_{t_1}^{\infty} s^{n-1}|q_1(s)|ds = \infty \text{ and } \int_{t_1}^{\infty} s^{n-1}|q_2(s)|ds = \infty, t_1 \geq t_0. \quad (4')$$

Setting:

$$u(t) = x(t) + p_1(t)x(\tau_1(t))$$
$$v(t) = y(t) + p_2(t)y(\tau_2(t))'$$

1- Assume that n is odd and the system (1) has nonoscillatory solution, then we have the following cases to consider:

Case 1. $x(t) > 0, y(t) > 0$ (the case $x(t) < 0, y(t) < 0$ can be treat in similar way). Then $u(t) > 0, v(t) > 0, u^{(n)}(t) \leq 0,\ v^{(n)}(t) \leq 0,\ t \geq t_1 \geq t_0$.

Since n is odd so we have only the following possible case:

$u^{(n)}(t) \leq 0, u^{(n-1)}(t) > 0 \dots, u'(t) < 0, u(t) > 0,\ v^{(n)}(t) \leq 0, \dots, v'(t) < 0, v(t) > 0,\ t \geq t_1 \geq t_0$.

From the equation (6), we obtain:

$$u^{(k)}(t_1) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-t_1)^{i-k}}{(i-k)!}u^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!}\int_{t_1}^t (\xi-t_1)^{n-k-1}u^{(n)}(\xi)d\xi.$$

Let $k = 0$ then:

$$u(t_1) \geq -\frac{1}{(n-1)!}\int_{t_1}^t (\xi-t_1)^{n-1}u^{(n)}(\xi)d\xi,$$

$$\geq \frac{-1}{(n-1)!}\int_{t_1}^t (\xi-t_1)^{n-1}|q_1(\xi)|f_1\left(y(\sigma_1(\xi))\right)d\xi$$

$$\geq \frac{-\lambda_1}{(n-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-1}\,|q_1(\xi)|y\big(\sigma_1(\xi)\big)d\xi.$$

We claim that $\liminf_{t\to\infty} y(t) = 0$ otherwise $\liminf_{t\to\infty} y(t) > 0$. So there exist $c_1 > 0$ such that $\liminf_{t\to\infty} y(t) \geq c_1 > 0$.

Hence there exist $t_2 \geq t_1$, $y(t) \geq c_1$ for $t \geq t_2$.

$$u(t_2) \geq -\frac{\lambda_1 c_1}{(n-1)!}\int_{t_2}^{t}(\xi - t_2)^{n-1}\,|q_1(\xi)|d\xi.$$

Letting $t \to \infty$ in the last inequality, we get a contradiction. Hence $\liminf_{t\to\infty} y(t) = 0$.

Then there exists a sequence $\{t_n\}$such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} y(t_n) = 0$.

Since $v(t)$ is monotone bounded function then:

$\lim_{t\to\infty} v(t) = L \geq 0$. Then by using

$$v(t_n) = y(t_n) + p_2(t_n)y\big(\tau_2(t_n)\big) \leq y(t_n) + \delta_2 v\big(\tau_2(t_n)\big).$$

As $n \to \infty$ we get $L \leq \delta_2 L$ or

$L(1-\delta_2) \leq 0$, this is possible only when $L = 0$. Which implies that:

$$\lim_{t\to\infty} y(t) = 0, \qquad \text{since } v(t) \geq y(t).$$

Case 2. $x(t) > 0$, $y(t) < 0$ (the case $x(t) < 0$, $y(t) > 0$ can be treat in similar way. Then $u(t) > 0, v(t) < 0, u^{(n)}(t) \geq 0,\ v^{(n)}(t) \leq 0,\ t \geq t_1 \geq t_0$.

We have only the following possible case:

$u^{(n)}(t) \geq 0, u^{(n-1)}(t) < 0 \ldots, u'(t) > 0,\ u(t) > 0,\ v^{(n)}(t) \leq 0, \ldots,\ v'(t) < 0,\ v(t) < 0,\ t \geq t_1 \geq t_0$.

Further, we have

$y(t) = v(t) - p_2(t)y(\tau_2(t))$, hence

$q_1(t)y\big(\sigma_1(t)\big) = q_1(t)v(\sigma_1(t)) - q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t)))$,

$$q_1(t)v\big(\sigma_1(t)\big) \geq q_1(t)y(\sigma_1(t)),$$

$$-q_1(t)v\big(\sigma_1(t)\big) \leq -q_1(t)y(\sigma_1(t)) - q_1(t)p_2\big(\sigma_1(t)\big)v(\tau_2(\sigma_1(t))) \leq -q_1(t)p_2\big(\sigma_1(t)\big)y(\tau_2(\sigma_1(t))).$$

From the equation (6), we obtain:

$$u^{(k)}(t_1) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-t_1)^{i-k}}{(i-k)!}u^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-k-1}\,u^{(n)}(\xi)d\xi.$$

Let $k = 1$ then:

$$u'(t) \geq \frac{1}{(n-2)!}\int_{t}^{s}(\xi - t)^{n-2}\,u^{(n)}(\xi)d\xi,$$

$$\geq \frac{1}{(n-2)!}\int_{t}^{s}(\xi - t)^{n-2}\,q_1(\xi)f_1\left(y\big(\sigma_1(\xi)\big)\right)d\xi,$$

$$\geq \frac{\lambda_2}{(n-2)!}\int_{t}^{s}(\xi - t)^{n-2}\,q_1(\xi)y\big(\sigma_1(\xi)\big)d\xi,$$

$$\geq \frac{\lambda_2}{(n-2)!}\int_{t}^{s}(\xi - t)^{n-2}(q_1(\xi)v(\sigma_1(\xi)) - q_1(\xi)p_2\big(\sigma_1(\xi)\big)y\left(\tau_2\big(\sigma_1(\xi)\big)\right))d\xi,$$

$$\geq \frac{\lambda_2}{(n-2)!}\int_{t}^{s}(\xi - t)^{n-2}(1 - p_2\big(\sigma_1(\xi)\big))q_1(\xi)v\left(\tau_2\big(\sigma_1(\xi)\big)\right)d\xi,$$

$$\geq \frac{\lambda_2}{(n-2)!}\int_t^s (\xi-t)^{n-2}(q_1(\xi)v(\sigma_1(\xi)) - q_1(\xi)p_2(\sigma_1(\xi))v\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$

$$u'(t) \geq \frac{\lambda_2}{(n-2)!}v\left(\tau_2(\sigma_1(t))\right)\int_t^s (\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))q_1(\xi)d\xi.$$

Integrating from t_1 to s we get the following:

$$u(s)-u(t_1) \geq \frac{\lambda_2}{(n-1)!}v\left(\tau_2(\sigma_1(t_1))\right)\int_{t_1}^s (\xi-t_1)^{n-1}(1-p_2(\sigma_1(\xi)))q_1(\xi)\,d\xi.$$

Letting $s \to \infty$ we get a contradiction from the last inequality.

2- Assume n is even and system (1) has nonoscillatory solution in the following cases:

Case 1. $x(t) > 0, y(t) > 0$ (the case $x(t) < 0, y(t) < 0$ can be treated in a similar way). Then $u(t) > 0, v(t) > 0, u^{(n)}(t) \leq 0,\ v^{(n)}(t) \leq 0,\ t \geq t_1 \geq t_0$. We have only the following possible case: $u^{(n)}(t) \leq 0, u^{(n-1)}(t) > 0 \dots, u'(t) > 0,\ u(t) > 0,\ v^{(n)}(t) \leq 0, \dots, v'(t) > 0,\ v(t) > 0,\ t \geq t_1 \geq t_0$.

Further, we have

$$y(t) = v(t) - p_2(t)y(\tau_2(t)),$$

$$q_1(t)y(\sigma_1(t)) = q_1(t)v(\sigma_1(t)) - q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t))),$$

$$q_1(t)y(\sigma_1(t)) \geq q_1(t)v(\sigma_1(t)),$$

$$-q_1(t)p_2(\sigma_1(t))y(\tau_2(\sigma_1(t))) \leq -q_1(t)p_2(\sigma_1(t))v\left(\tau_2(\sigma_1(t))\right),$$

From the equation (6), we obtain:

$$u^{(k)}(t_1) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-t_1)^{i-k}}{(i-k)!}u^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!}\int_{t_1}^t (\xi-t_1)^{n-k-1}u^{(n)}(\xi)d\xi.$$

Let $k = 1$ then:

$$u'(t) \geq -\frac{1}{(n-2)!}\int_t^s (\xi-t)^{n-2}u^{(n)}(\xi)d\xi,$$

$$\geq \frac{1}{(n-2)!}\int_t^s (\xi-t)^{n-2}\,|q_1(\xi)|f_1\left(y(\sigma_1(\xi))\right)d\xi,$$

$$\geq \frac{\lambda_1}{(n-2)!}\int_t^s (\xi-t)^{n-2}\,|q_1(\xi)|y(\sigma_1(\xi))d\xi,$$

$$\geq \frac{\lambda_1}{(n-2)!}\int_t^s (\xi-t)^{n-2}(|q_1(\xi)|v(\sigma_1(\xi)) - |q_1(\xi)|p_2(\sigma_1(\xi))y\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$

$$\geq \frac{\lambda_1}{(n-2)!}\int_t^s (\xi-t)^{n-2}(|q_1(\xi)|v(\sigma_1(\xi)) - |q_1(\xi)|p_2(\sigma_1(\xi))v\left(\tau_2(\sigma_1(\xi))\right))d\xi,$$

$$\geq \frac{\lambda_1}{(n-2)!}\int_t^s (\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))\,|q_1(\xi)|v\left(\tau_2(\sigma_1(\xi))\right)d\xi,$$

$$u'(t) \geq \frac{\lambda_1}{(n-2)!}v\left(\tau_2(\sigma_1(t))\right)\int_t^s (\xi-t)^{n-2}(1-p_2(\sigma_1(\xi)))|q_1(\xi)|\,d\xi.$$

Integrating from t_1 to s we obtain:

$$u(s)-u(t_1) \geq \frac{\lambda_1}{(n-1)!}v\left(\tau_2(\sigma_1(t_1))\right)\int_{t_1}^s (\xi-t_1)^{n-1}(1-p_2(\sigma_1(\xi)))|q_1(\xi)|\,d\xi.$$

Letting $s \to \infty$ in the last inequality we get a contradiction.

Case 2. $x(t) > 0\,, y(t) < 0$ (the case $x(t) < 0, y(t) > 0$ can be treated in a similar way).

Then $u(t) > 0, v(t) < 0, u^{(n)}(t) \geq 0, v^{(n)}(t) \leq 0,\ t \geq t_1 \geq t_0$. We have only the following possible :

$u^{(n)}(t) \geq 0, u^{(n-1)}(t) < 0 \dots, u'(t) < 0,\ u(t) > 0,\ v^{(n)}(t) \leq 0, \dots, v'(t) > 0,$

$v(t) < 0,\ t \geq t_1 \geq t_0$.

Further, we have

$u(t) = x(t) + p_1(t)x(\tau_1(t))$ Then $u(t) \geq x(t)$.

From equation (6), we obtain:

$$v^{(k)}(t_1) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-t_1)^{i-k}}{(i-k)!}v^{(i)}(t) + \frac{(-1)^{n-k}}{(n-k-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-k-1}\,v^{(n)}(\xi)d\xi.$$

Let $k = 0$ then:

$$v(t_1) \leq \frac{1}{(n-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-1}\,v^{(n)}(\xi)d\xi,$$

$$\leq \frac{1}{(n-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-1}\,q_2(\xi)f_2\left(x\left(\sigma_2(\xi)\right)\right)d\xi,$$

$$\leq \frac{\lambda_3}{(n-1)!}\int_{t_1}^{t}(\xi - t_1)^{n-1}\,q_2(\xi)x\left(\sigma_2(\xi)\right)d\xi,$$

We claim that $\liminf_{t\to\infty} x(t) = 0$ otherwise $\liminf_{t\to\infty} x(t) > 0$. So there exist $c_1 > 0$ such that $\liminf_{t\to\infty} x(t) \geq c_1 > 0$. Hence there exist $t_2 \geq t_1, x(t) \geq c_1$ for $t \geq t_2$.

$$v(t_2) \leq \frac{\lambda_3 c_1}{(n-1)!}\int_{t_2}^{t}(\xi - t_2)^{n-1}\,q_2(\xi)d\xi\ .$$

Letting $t \to \infty$ in the last inequality, we get a contradiction. Hence $\liminf_{t\to\infty} x(t) = 0$. Then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = 0$. Since $u(t)$ is monotone bounded function then: $\lim_{t\to\infty} u(t) = L \geq 0$,

$$u(t_n) = x(t_n) + p_1(t_n)x\left(\tau_1(t_n)\right) \leq x(t_n) + \delta_1 u\left(\tau_1(t_n)\right).$$

As $n \to \infty$, we get $L \leq \delta_1 L$ or $L(1-\delta_1) \leq 0$, this is possible only when $L = 0$. Which implies that:$\lim_{t\to\infty} x(t) = 0$, since $u(t) \geq x(t)$. □

3. Applications

In this section, two examples of an oscillatory solution are presented, we think that examples of a nonoscillatory solution tend to zero can be expressed more easily than an oscillatory solution.

Example 1. Consider the system of neutral differential equations

$$\begin{aligned}\left[x(t) + \left(\tfrac{1}{3} + \tfrac{1}{5}\cos(2t)\right)x(t-2\pi)\right]''' &= \left(\tfrac{4}{3} + \tfrac{1}{5}\cos(2t)\right)y(t-\pi),\\ \left[y(t) + \left(\tfrac{1}{3} + \tfrac{1}{5}\cos(2t)\right)y(t-2\pi)\right]''' &= \left(\tfrac{4}{3} + \tfrac{1}{5}\cos(2t)\right)x(t-2\pi),\end{aligned}\quad t \geq 0. \quad (3)$$

$q_1(t) = q_2(t) = \frac{4}{3} + \frac{1}{5}\cos(2t)$, Then $\frac{17}{15} \leq q_1(t), q_2(t) \leq \frac{23}{15}$,

$\tau_1(t) = t - 2\pi, \lim_{t\to\infty}\tau_1(t) = \infty,\ \tau_2(t) = t - 2\pi, \lim_{t\to\infty}\tau_2(t) = \infty,$

$\sigma_1(t) = t - \pi, \lim_{t\to\infty}\sigma_1(t) = \infty, \sigma_2(t) = t - 2\pi, \lim_{t\to\infty}\sigma_2(t) = \infty,$

$p_1(t) = \frac{1}{3} + \frac{1}{5}\cos(2t),\ \ p_2(t) = \frac{1}{3} + \frac{1}{5}\cos(2t)$, hence $\frac{2}{15} \leq p_1(t), p_2(t) \leq \frac{8}{15}$

$f_1\left(y(t)\right) = y(t), f_2\left(x(t)\right) = x(t)$

$$t^{n-1}\int_{t_1}^{t} q_1(s)\,ds = t^2\int_0^t \left(\frac{4}{3}+\frac{1}{5}\cos(2s)\right) ds \ge t^2\int_0^t \frac{17}{15}ds \ge \frac{17}{15}t^3 \,,$$

Then

$$\lim_{t\to\infty} t^{n-1}\int_{t_1}^{t} q_1(s)\,ds = \infty$$

According to theorem 1 every solution of Sys.(3) oscillates. For instance the solution

$(x(t),y(t)) = (\frac{\sin t}{\frac{4}{3}+\frac{1}{5}\cos(2t)} , \frac{\cos t}{\frac{4}{3}+\frac{1}{5}\cos(2t)})$. Is such an oscillatory solution . □

Example 2. Consider the system of neutral differential equations

$$\left[x(t)+\left(\frac{1}{2}+\frac{1}{4}\sin(2t)\right)x(t-2\pi)\right]'''' = (-\frac{3}{2}+\frac{1}{4}\sin(2t))\,y(t-\frac{3\pi}{2})$$

$$\left[y(t)+\left(\frac{1}{2}+\frac{1}{4}\sin(2t)\right)y(t-2\pi)\right]'''' = (-\frac{3}{2}+\frac{1}{4}\sin(2t))\,x\left(t-\frac{\pi}{2}\right), t\ge 0 \qquad (4)$$

$q_1(t) = q_2(t) = -\frac{3}{2}+\frac{1}{4}\sin(2t)$, then $-\frac{7}{4} \le q_1(t), q_2(t) \le -\frac{5}{4}$,

$$\tau_1(t) = t-2\pi, \quad \lim_{t\to\infty}\tau_1(t)=\infty, \tau_2(t) = t-2\pi \quad \lim_{t\to\infty}\tau_2(t)=\infty,$$

$$\sigma_1(t) = t-\frac{3\pi}{2}, \quad \lim_{t\to\infty}\sigma_1(t)=\infty, \sigma_2(t) = t-\frac{\pi}{2}, \quad \lim_{t\to\infty}\sigma_2(t)=\infty,$$

$p_1(t) = \frac{1}{2}+\frac{1}{4}\sin(2t)$, $p_2(t) = \frac{1}{2}+\frac{1}{4}\sin(2t)$, then $\frac{1}{4} \le p_1(t), p_2(t) \le \frac{3}{4}$,

$f_1(y(t)) = y(t)$, $f_2(x(t)) = x(t)$.

$$t^{n-1}\int_{t_1}^{t}|q_1(s)|\,ds = t^3\int_0^t \left|-\frac{3}{2}+\frac{1}{4}\sin(2s)\right| ds \ge t^3\int_0^t \frac{5}{4}ds = \frac{5t^4}{4}.$$

Then $\lim_{t\to\infty} t^{n-1}\int_{t_1}^{t}|q_1(s)|\,ds = \infty$.

According to theorem 2 every solution of Sys.(4) oscillates. For instance the solution

$(x(t),y(t)) = (\frac{\sin t}{\frac{3}{2}+\frac{1}{4}\sin(2t)} , \frac{\cos t}{\frac{3}{2}+\frac{1}{4}\sin(2t)})$ is such an oscillatory solution.

4. Conclusions

In the present paper, we have studied the oscillatory and asymptotic behavior solutions of an nth-order nonlinear system of neutral differential equation (1). As has been illustrated through two examples, the results obtained show that under certain conditions, every bounded solutions of Sys.(1) oscillate, or nonoscillatory tend to zero as t goes to infinity.

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